Control of trapped-ion quantum states

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- The rose city

http://qist.lanl.gov/qcomp_map.shtml

The Mid-Level Quantum Computation Roadmap: Promise Criteria

QC Approach	The DiVincenzo Criteria: Quantum Computation #1	#2	#3	#4	#5	QC Networkability #6	#7
NMR	🔴	🟠	🟠	🟢	🟠	🔴	🔴
Trapped Ion	🟠	🟢	🟠	🟢	🟢	🟢	🟢
Neutral Atom	🟠	🟢	🟠	🟠	🟠	🟠	🟠
Cavity QED	🟠	🟢	🟠	🟠	🟢	🟠	🟠
Optical	🟠	🟠	🟢	🟠	🟠	🟠	🟢
Solid State	🟠	🟠	🟠	🟠	🟠	🔴	🔴
Superconducting	🟠	🟢	🟠	🟠	🟠	🔴	🔴
Unique Qubits	This field is so diverse that it is not feasible to label the criteria with "Promise" symbols.						

Legend: 🟢 = a potentially viable approach has achieved sufficient proof of principle

🟠 = a potentially viable approach has been proposed, but there has not been sufficient proof of principle

🔴 = no viable approach is known

The column numbers correspond to the following QC criteria:

#1. A scalable physical system with well-characterized qubits.

#2. The ability to initialize the state of the qubits to a simple fiducial state.

#3. Long (relative) decoherence times, much longer than the gate-operation time.

#4. A universal set of quantum gates.

#5. A qubit-specific measurement capability.

#6. The ability to interconvert stationary and flying qubits.

#7. The ability to faithfully transmit flying qubits between specified locations.

Outline

Introduction to the physical system/model

Control of spin-half coupled to SHO

Controllability:

Eigenstate vs. finite (approx.) vs. complete controllability

What are possible (feasible) control schemes?

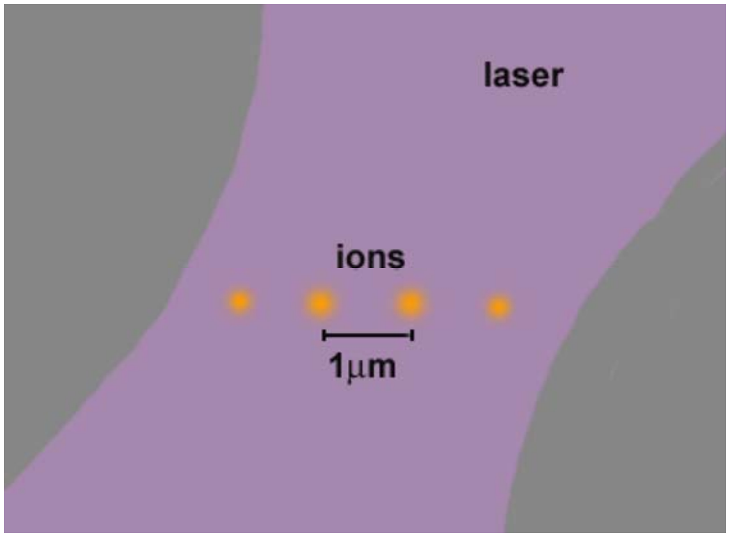
Resonant control

Control via truncation

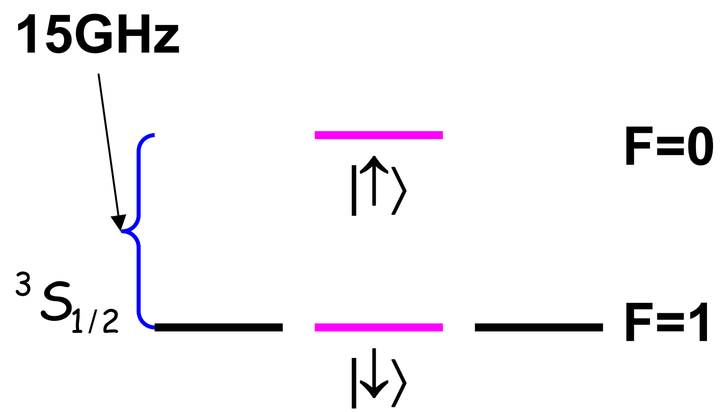
Optimal control?

Ultrafast / Adiabatic control (if time permits)

Trapped ions (E.g. Cadmium)



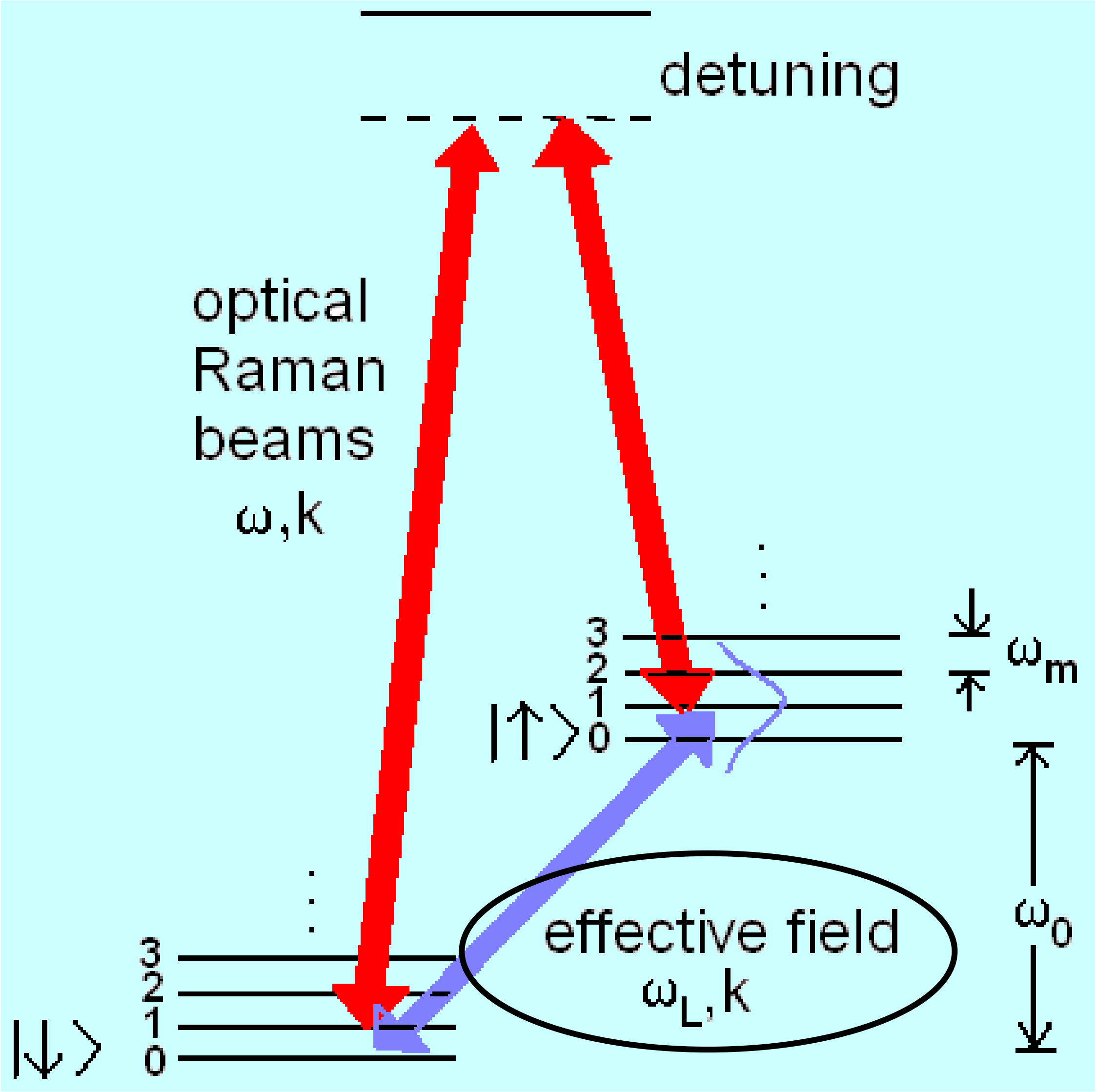
Qubit



Qubits coupled by harmonic oscillators

Comprehensive review: "Quantum dynamics of single trapped ions" by D. Leibfried, R. Blatt, C. Monroe, D. Wineland. **Review of Modern Physics, vol. 75, p. 281 (2003).**

Single ion energy levels



Mathematical formulation

Field-free Hamiltonian:

$$H_0 = (^1/_2)\omega_0\sigma_z + \omega_m a^t a$$

Field: $\mathbf{E}(\xi,t) = \mathbf{x}\, E(t)\cos(k\xi - \omega_L t);\ \omega_L \approx \omega_0$

Interaction Hamiltonian:

$$H_I = -\mu_S \cdot \mathbf{E}(\xi,t)$$

$$= (^1/_2)\,\mu\sigma_x\, E(t)\cos(k\xi_0(a+a^t) - \omega_L t)$$

Lamb-Dicke parameter $\eta = k\xi_0$

Resonant transitions

Carrier: $\omega_L = \omega_0$

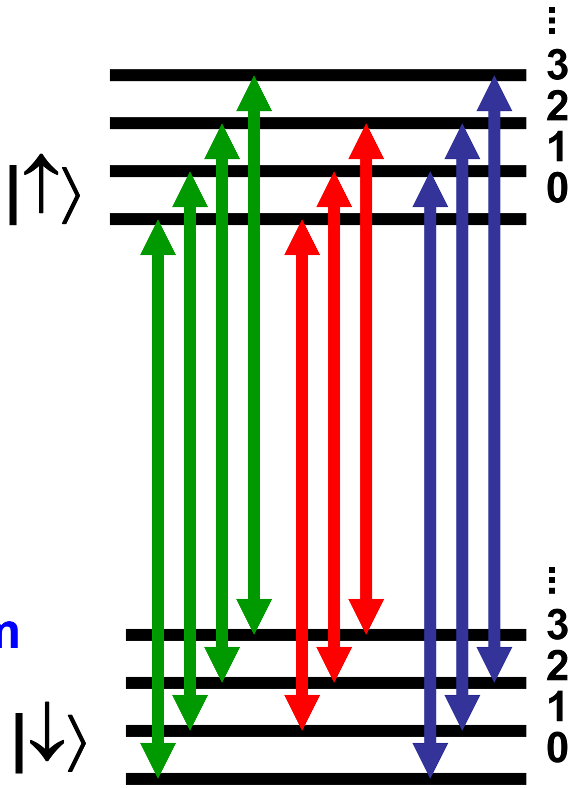
$|\downarrow, n\rangle$ to $|\uparrow, n\rangle$

First red sideband: $\omega_L = \omega_0 - \omega_m$

$|\downarrow, n\rangle$ to $|\uparrow, n-1\rangle$

First blue sideband: $\omega_L = \omega_0 + \omega_m$

$|\downarrow, n\rangle$ to $|\uparrow, n+1\rangle$

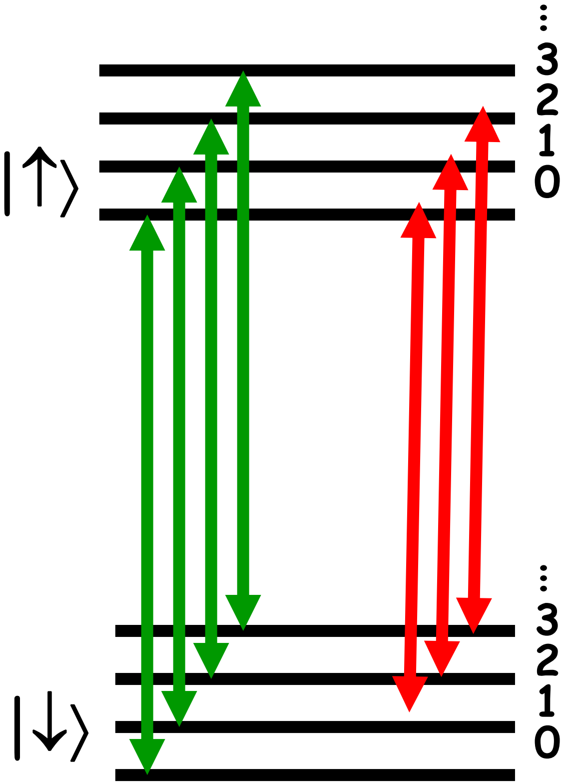


Transition couplings

Trapped-ion quantum states

Spin ½ system coupled to H.O.:

Eigenstates are transitively connected by only two resonant fields



Lamb-Dicke limit

Interaction Hamiltonian:

$$H_I = \sigma_x \Omega(t)\cos(\eta(a+a^t)-\omega_L t)$$

Transition matrix elements: $\langle A|\, H_I\, |B\rangle$

Carrier:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\rangle \sim L_n(\eta^2)$

First red sideband:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim i\, L^{(1)}{}_n(\eta^2)$

Lamb-Dicke limit (LDL): (motional cooling)

$\xi_0 \ll \lambda$, $\eta \ll 1$, keep terms to $\vartheta(a+a^t)$

$H_I = \Omega(t)[\sigma_+ + \sigma_-]$

$H_I = \Omega(t)i[\sigma_+ a - \sigma_- a^+]$

Carrier:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\rangle \sim 1$

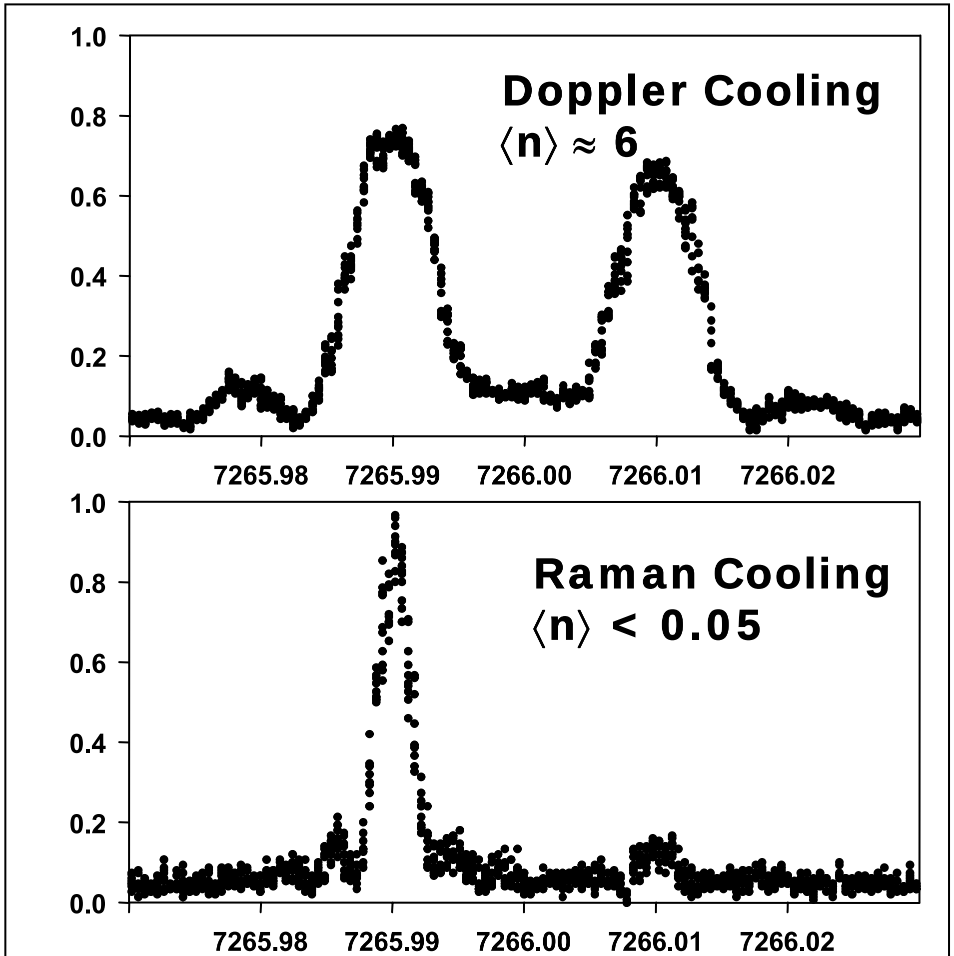
First red sideband:

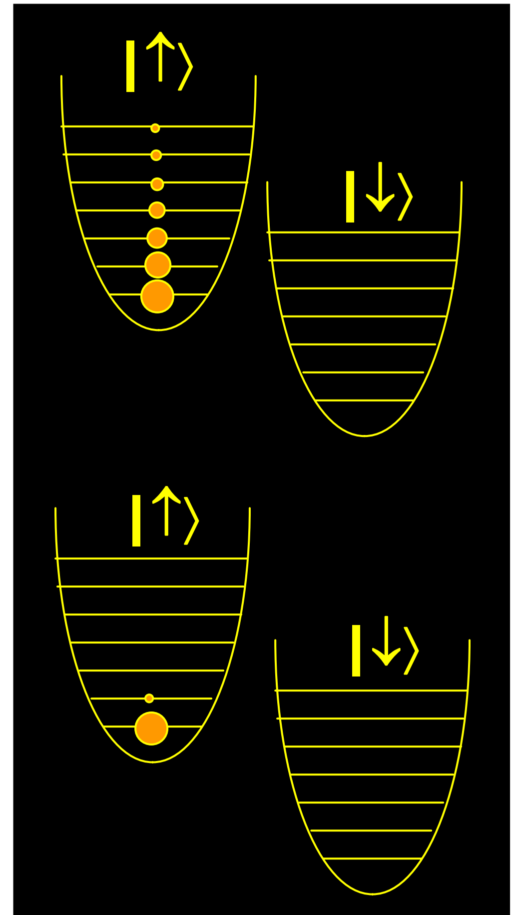
$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim \sqrt{n}$

Laser-cooling Cd^+ to n=0
(Slide from Chris Monroe)

Thermometry:

$$\frac{I_{AS}}{I_S} = \frac{\langle n \rangle}{1+\langle n \rangle}$$





Stokes
$|\uparrow,n\rangle \Rightarrow |\downarrow,n+1\rangle$

Anti-Stokes
$|\uparrow,n\rangle \Rightarrow |\downarrow,n-1\rangle$

L. Deslauriers et al. quant-ph/0404142

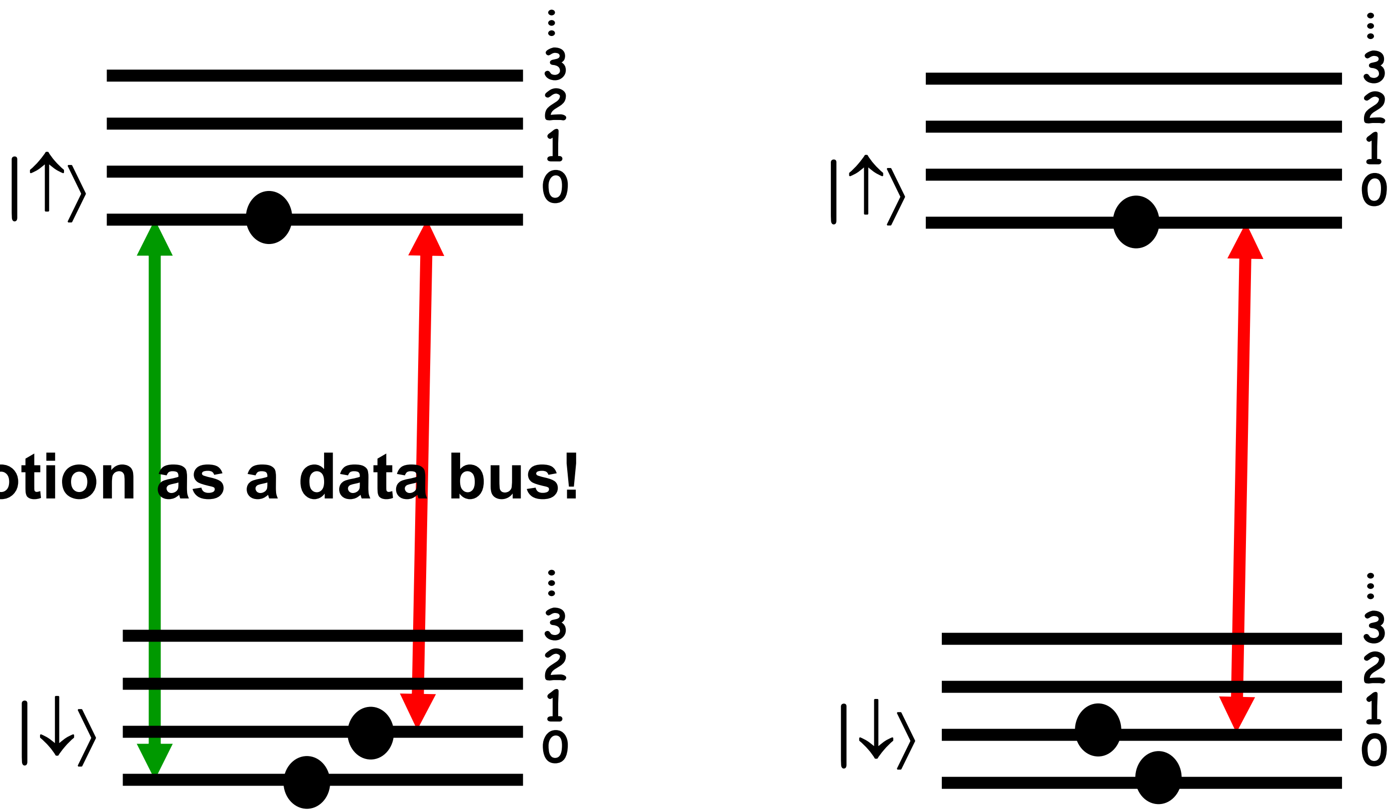
Δx_{rms} = 3 nm
"Lamb-Dicke" regime

Cirac-Zoller QC scheme

VOLUME 74, NUMBER 20 PHYSICAL REVIEW LETTERS 15 MAY 1995

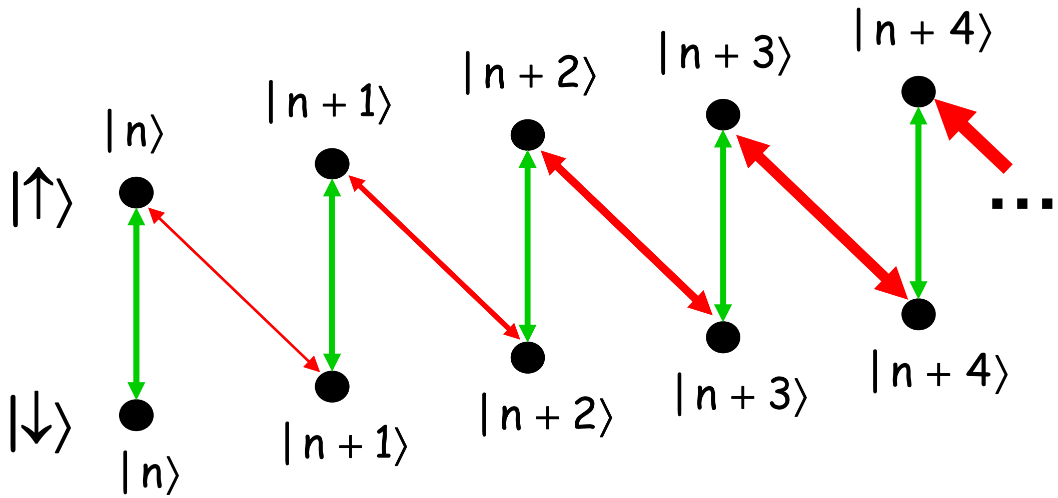
Quantum Computations with Cold Trapped Ions

J. I. Cirac and P. Zoller*
Institut für Theoretische Physik, Universität Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria
(Received 30 November 1994)



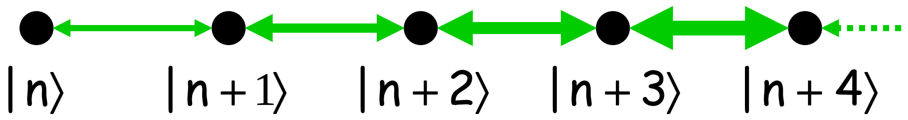
Use motion as a data bus!

Controllability?



Expect: system is uncontrollable

Harmonic oscillator:



Challenges in infinite-D

- How to define controllability?

$|n\rangle \longrightarrow |m\rangle \neq$

$\Sigma_{\text{(finite number)}} c_n|n\rangle \longrightarrow \Sigma_{\text{(finite number)}} d_m|m\rangle \neq$

$\Sigma_{\text{(infinite number)}} c_n|n\rangle \longrightarrow \Sigma_{\text{(infinite number)}} d_m|m\rangle \neq$

- Lie algebra might be ∞-dim, but does it span the space? Don't know.
- Using piecewise-constant controls, global controllability cannot be achieved with a finite number of operations (Huang, Tarn & Clark, J. Math. Phys., 1983)

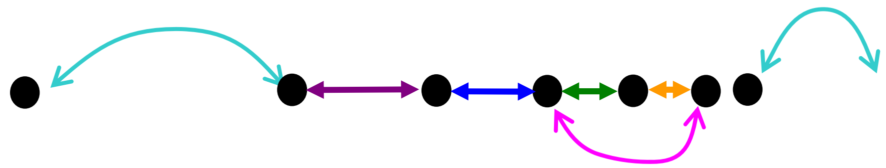
Methods from Classical Control

I. Graphical methods (transfer graphs)

Classical: Turinici & Rabitz, Chem. Phys. 2001

Quantum: Rangan & Bloch, J. Math. Phys. 2005

Eigenstates of the field-free Hamiltonian: nodes
Transition matrix elements of interaction Hamiltonian: edges



Methods from Classical Control

II. Lie algebraic methods

Brockett, IEEE Trans. Auto. Control, 1969

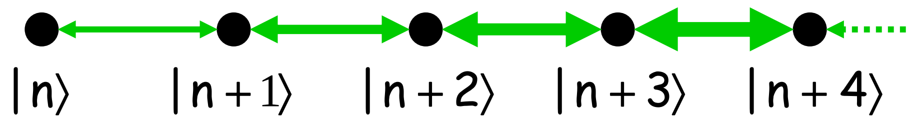
Ramakrishna et al., Phys. Rev. A, 1995

If $\iota\dot{\Psi}=(H_0+H_i)\Psi$ is controllable, the Lie algebra formed by H_0 , H_i , and all possible linearly independent commutators spans U(N).

This is the Gold Standard for Finite-D systems

For ∞-D systems, these methods have restricted use

Ex. Driven Harmonic Oscillator



# of elements in the control algebra = 4 (does not span Hilbert space)

$|0\rangle \longrightarrow |\alpha\rangle$:R.J. Glauber, *Phys. Rev.* (1963)

$|\alpha\rangle \longrightarrow |\beta\rangle$

Schemes to control ∞-D systems

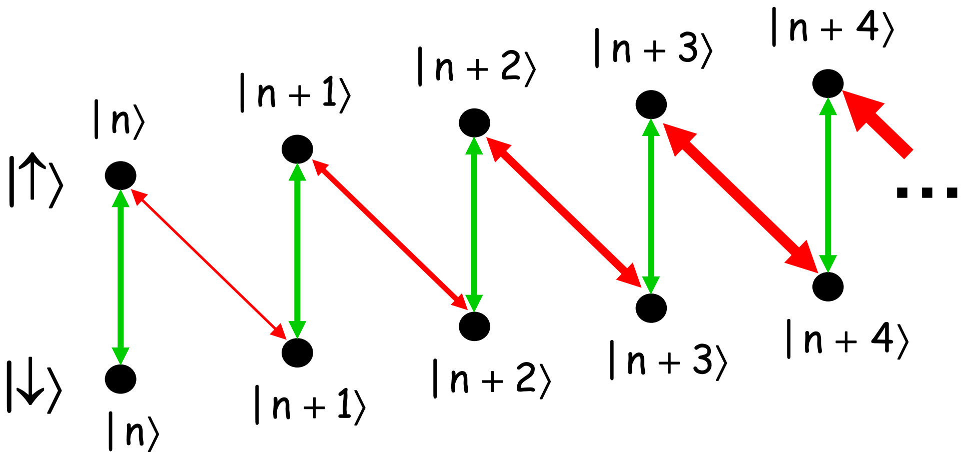
a. **Truncate infinite-dimensional space** (Rangan, Monroe, Bucksbaum, Bloch, Phys. Rev. Lett., 2004, Yuan & Lloyd, Phys. Rev. A, 2007)

b. **Coarse-grained controllability** (E. Shapiro, Ivanov & Billig, J. Chem. Phys., 2004)

c. **Analytic domain controllability** (Lan, Tarn, Chi & Clark, J. Math. Phys., 2005)

d. **Finite controllability** (Bloch, Brockett, Rangan, 2009)

Infinite Lie algebra



Lie algebra is ∞-D (Bloch, Brockett, Rangan, quant-ph/0608075)

$$\exp(-iH\Delta t)$$
$$= \exp(-i(H_c+H_r)\Delta t)$$
$$= \exp(-iH_c\Delta t) . \exp(-iH_r\Delta t)$$
$$.\exp(-1/2[H_c,H_r](\Delta t)^2)$$
$$.\exp(1/12[H_c,[H_c,H_r]](\Delta t)^3)$$
$$.\exp(1/12[[H_c,H_r],H_r](\Delta t)^3)\ldots$$

The *alternate* application of control fields removes a chirp instability in unitary flows. (Brockett, Rangan, & Bloch, CDC 2003)

I. Finite controllability

Definition

Given

- **-a system, and**
- **-a nested set of finite dimensional subspaces**

it will be said to be <u>finitely controllable</u> if

- **- it can be transferred from any point in one of the subspaces to any other point in that subspace**
- **- with a trajectory lying entirely within the subspace.**

Finite controllability theorem

Consider a complex Hilbert space X together with a nested set of finite-dimensional subsets

$$\mathrm{H} = \{\mathrm{H}_1 \subset \mathrm{H}_2 \subset \mathrm{H}_3 \ldots \}$$

Consider $$i\dot{\Psi} = \left(\sum_{i=1}^{m} u_i B_i \right) \Psi$$

where the B_i are Hermitian control operators.

Assume

- **H_1 is an invariant subspace for B_1**
- **the system is unit vector controllable on H_1 using only B_1**

Finite controllability theorem (cont'd)

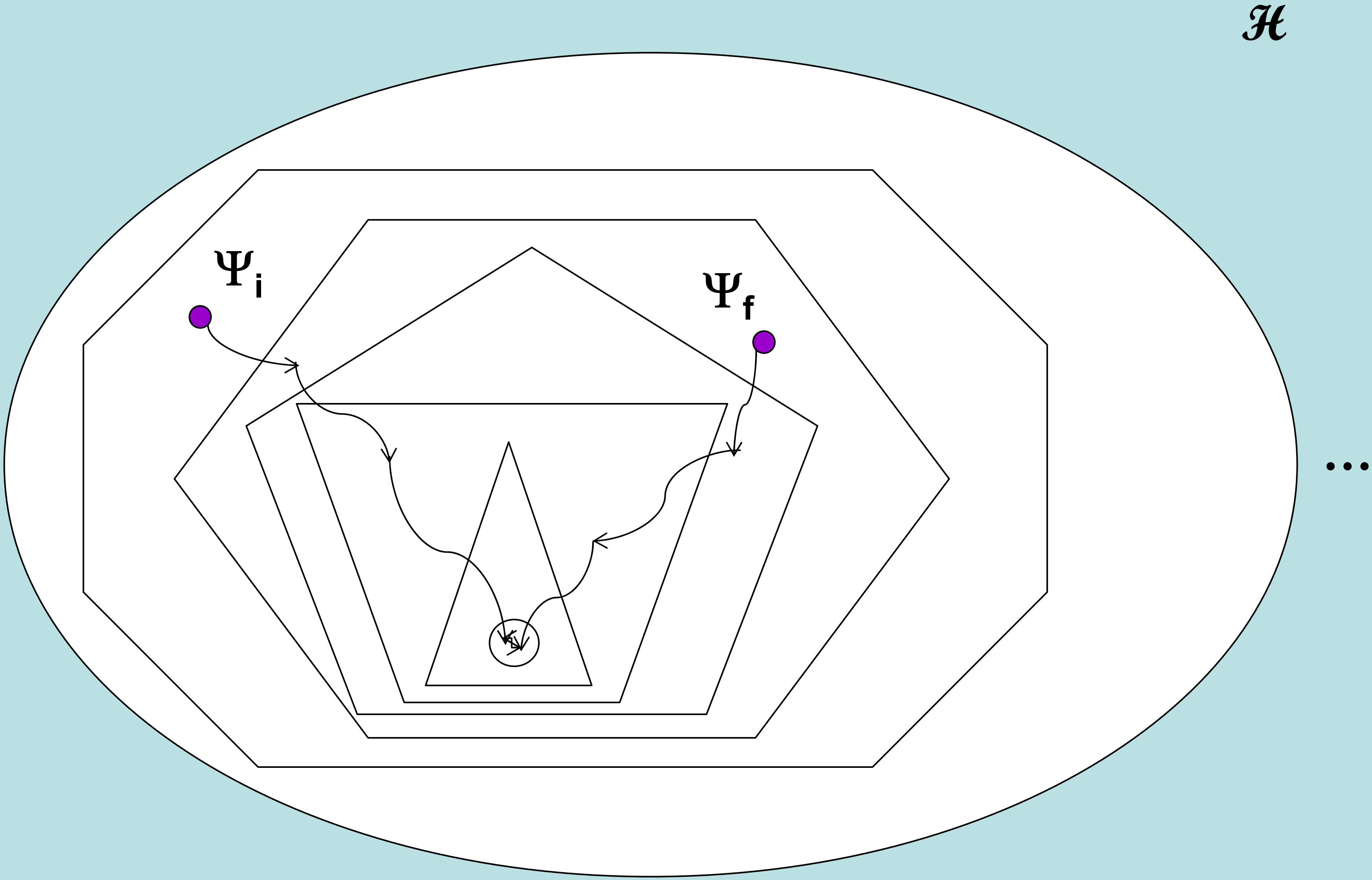
If

- **for each $H_\alpha ; \alpha \neq 1$ there is a B_α that leaves H_α invariant, and**
- **for any unit vector in H_α the orbit generated by $\exp(iB_\alpha)$ contains a point in one of the lower dimensional subspaces H_β**

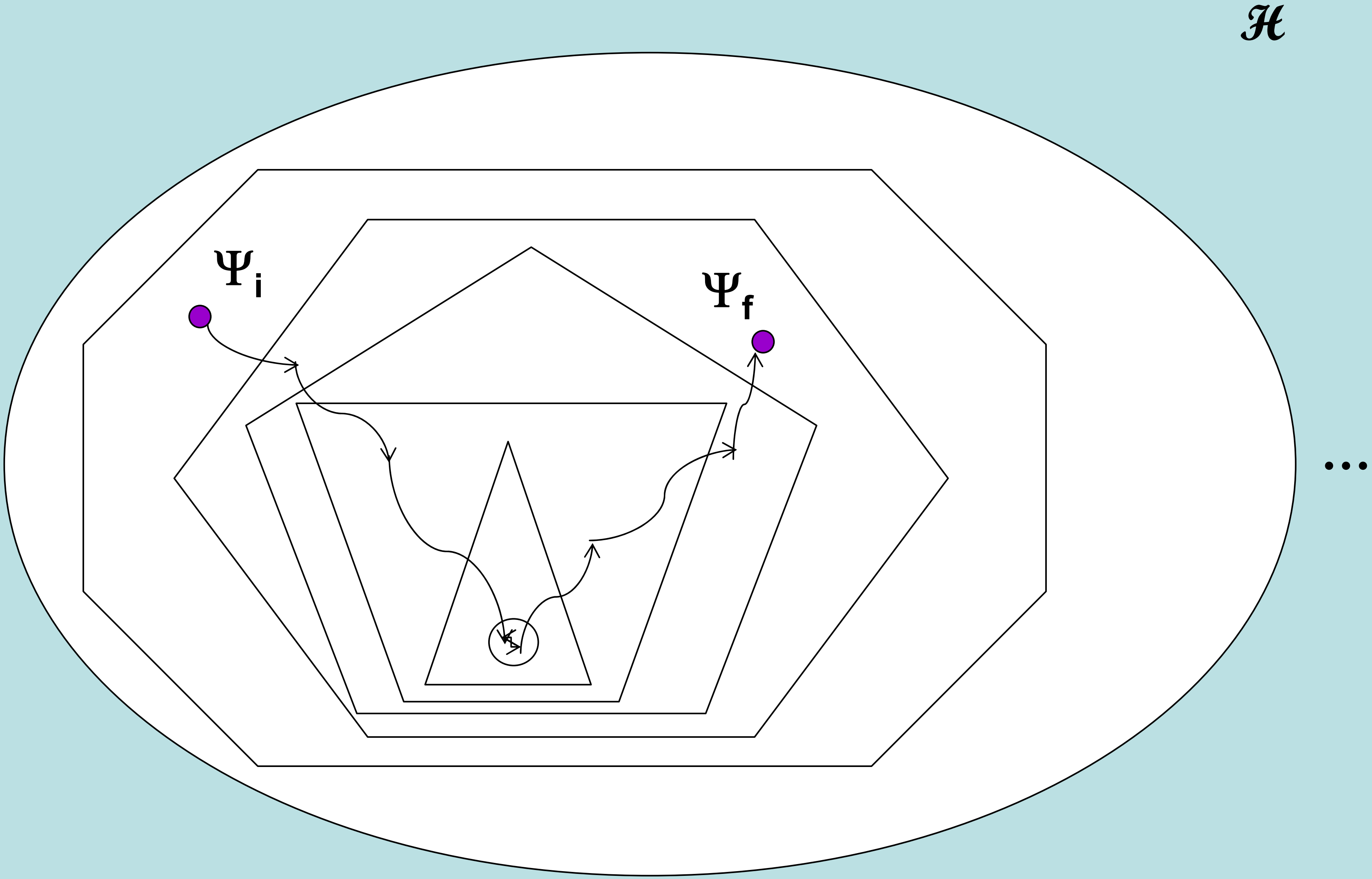
then any unit vector in any of the H_i can be steered to any other unit vector in any other H_j using a finite number of piecewise constant controls.

(Bloch, Brockett, Rangan, IEEE TAC, ~~2005~~, ~~2006~~, ~~2007~~, ~~2008~~, 2009)

Finite controllability



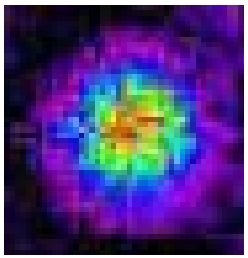
Explicit scheme



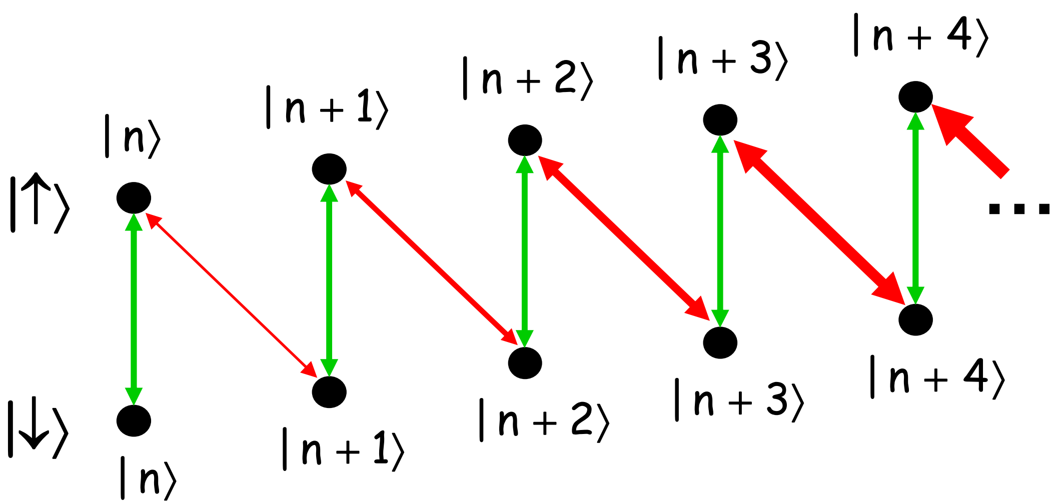
Example: trapped-ion qubit

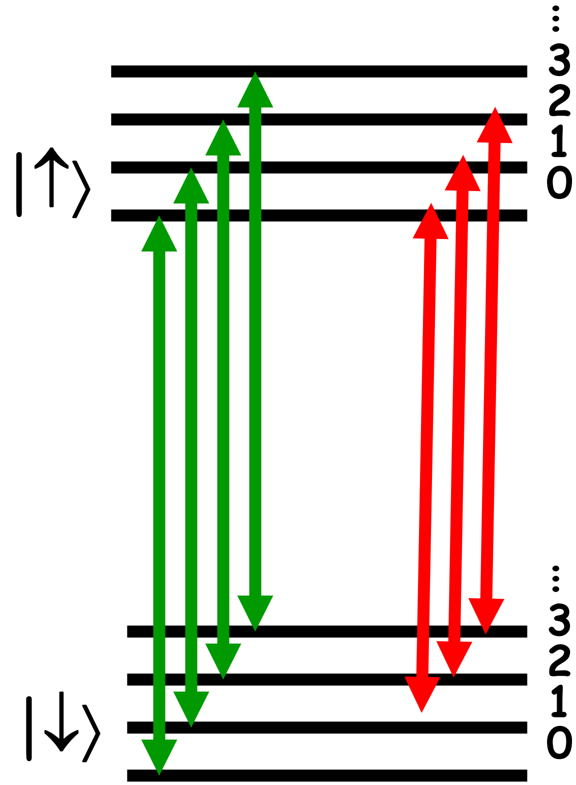
Trapped-ion quantum states

Spin ½ system coupled to H.O.:



Transitively connected by two resonant fields



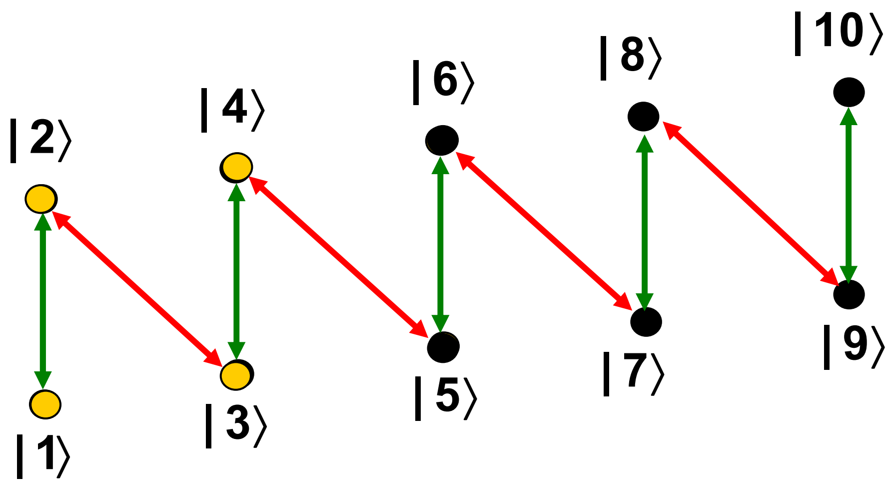


Finite Controllability Example

Kneer-Law-Eberly scheme, PRA <u>57</u>, 2096 (1998)

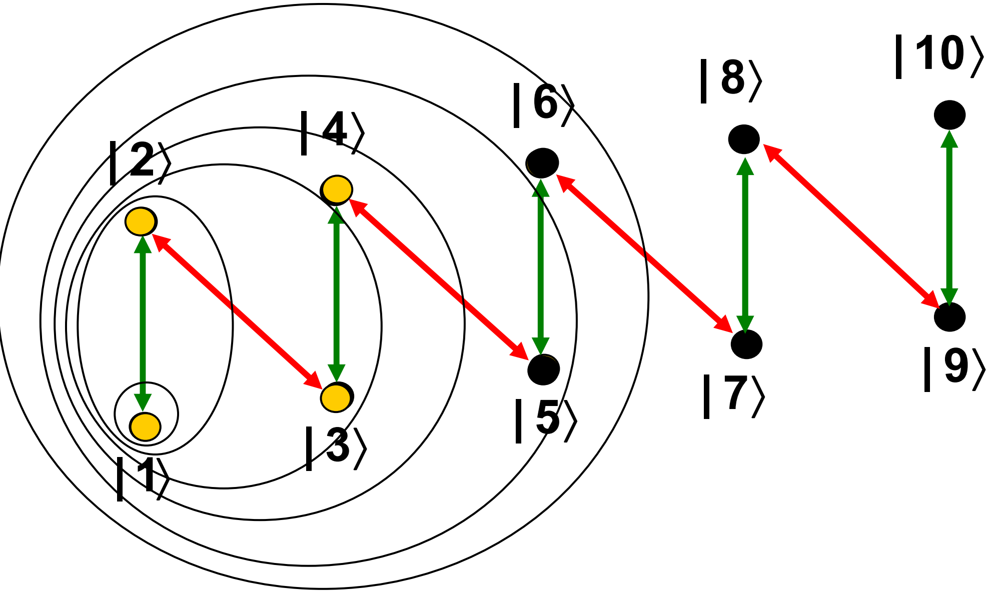
<u>Aim</u>: Start from ground state and create a finite superposition of trapped-ion energy eigenstates

<u>Method</u>: reverse engineer



- The key to controllability is that each operator has different invariant subspaces within the set of finite superpositions, and one never in fact turns on both operators simultaneously.

Finite controllability of trapped-ion



$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B' & \cdots \\ 0 & 0 & 0 & B' & 0 & \cdots \\ \cdots & & & & & \end{pmatrix}$$

Reachable set includes superpositions of finite numbers of eigenstates. (BBR, quant-ph/0608075)

II. Eigenstate controllability

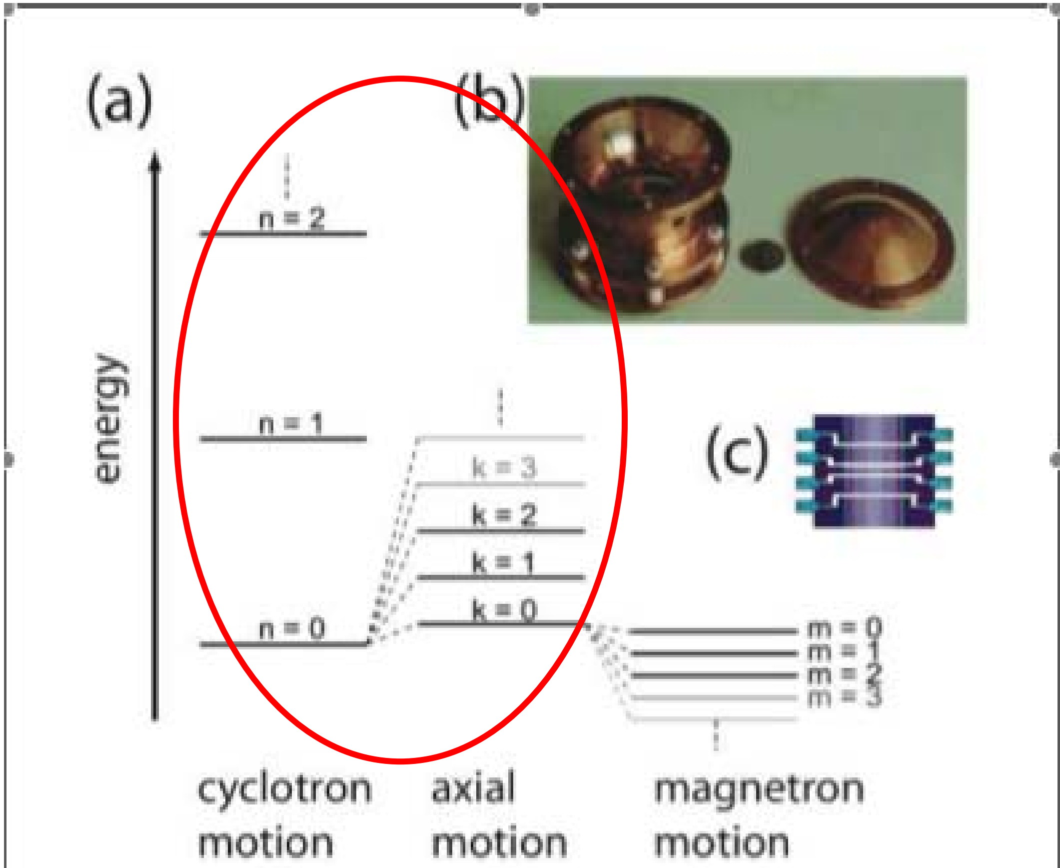
A system is eigenstate controllable if the population can be coherently transferred from any eigenstate to any other eigenstate.

Example: trapped-electron

Trapped-electron quantum states:

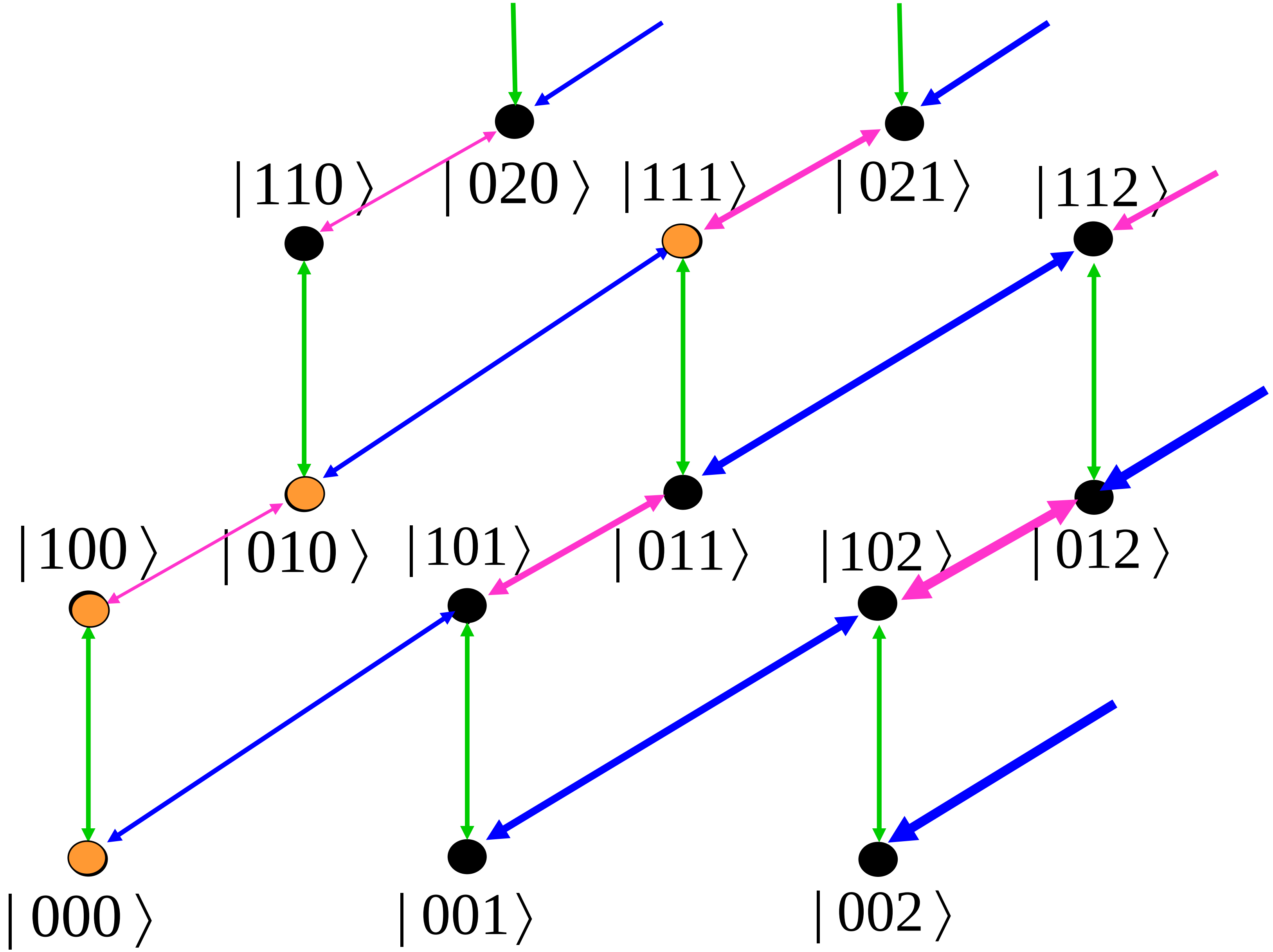
Spin-1/2 system coupled to two S.H.O.'s

(Pedersen & Rangan, Quant. Inf. Proc., 2008.)

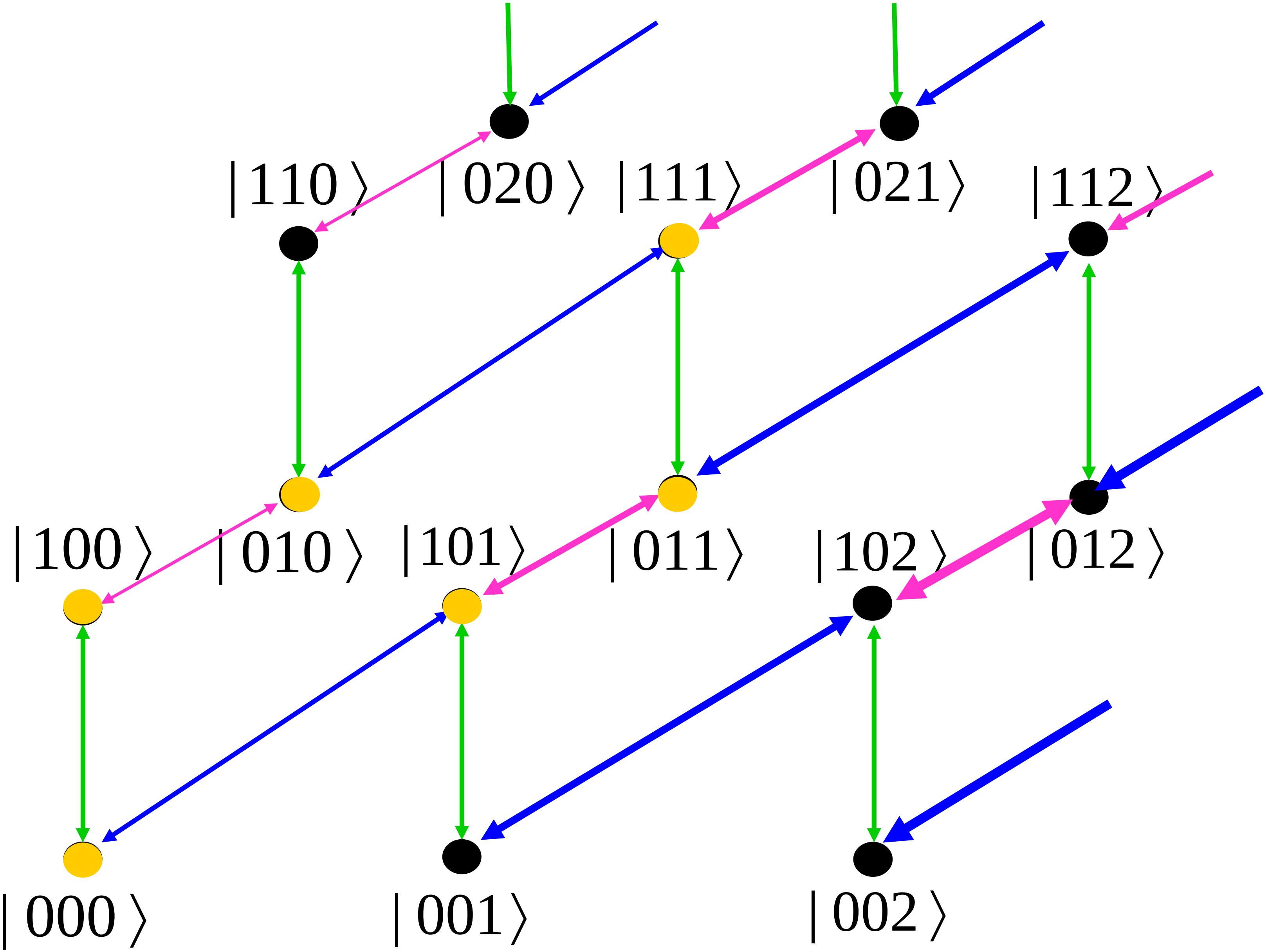


From Marzoli et al., arXiv:0810.4408

Eigenstate controllability



BUT - No finite controllability



Eigenstate controllability

Eigenstate controllability does not imply finite controllability in an infinite-dimensional system.

Control schemes for spin-1/2 HO

μs fields:

- Alternating pulse schemes (Cirac-Zoller)
- Off-resonant schemes (Molmer-Sorensen)
- Spin-dependent forces (Milburn-Schneider-James)
- Bichromatic scheme (Rangan, Monroe, Bloch, Bucksbaum)

ns fields:

- Fast pulse scheme (Garcia-Ripoll,Cirac, Zoller)

Adiabatic schemes:

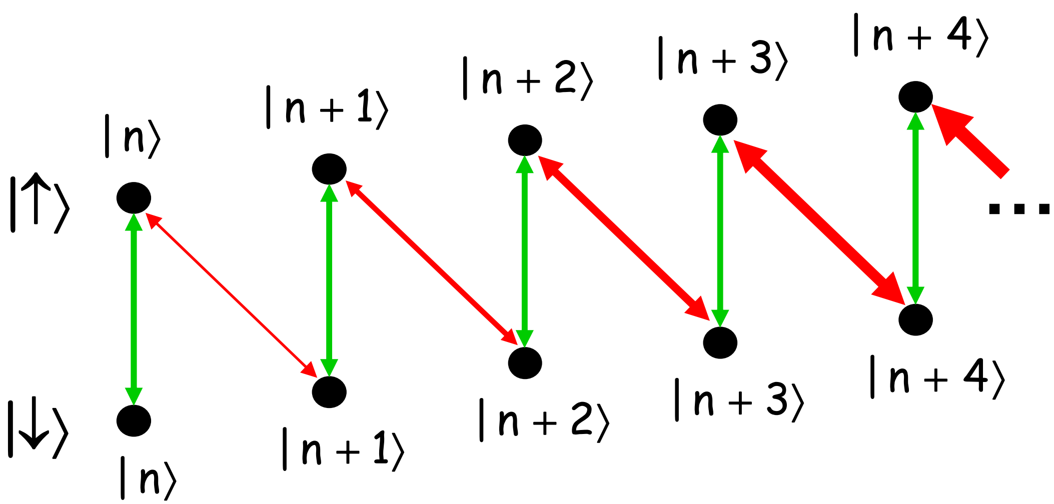
Control by truncating Hilbert space

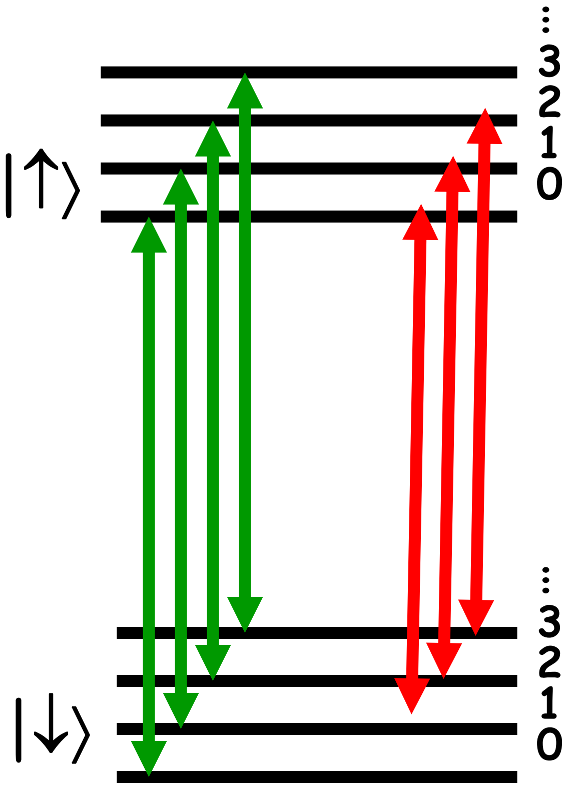
Trapped-ion quantum states

Spin ½ system coupled to H.O.:

Transitively connected by a

Bichromatic resonant field



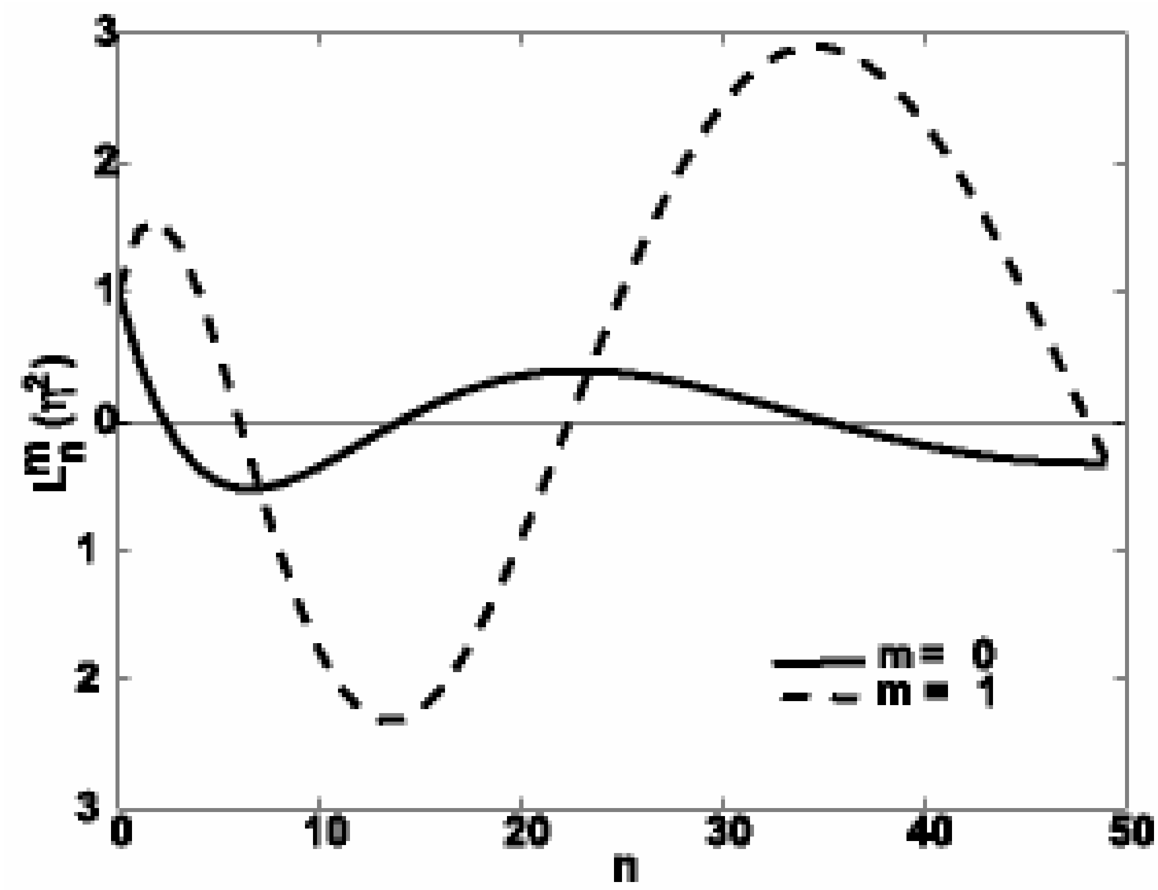


Transition matrix elements

Carrier: $|\downarrow n\rangle \leftrightarrow |\uparrow n\rangle \sim L_n(\eta^2)$

First red sideband:

$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim i\, L^{(1)}{}_n(\eta^2)$

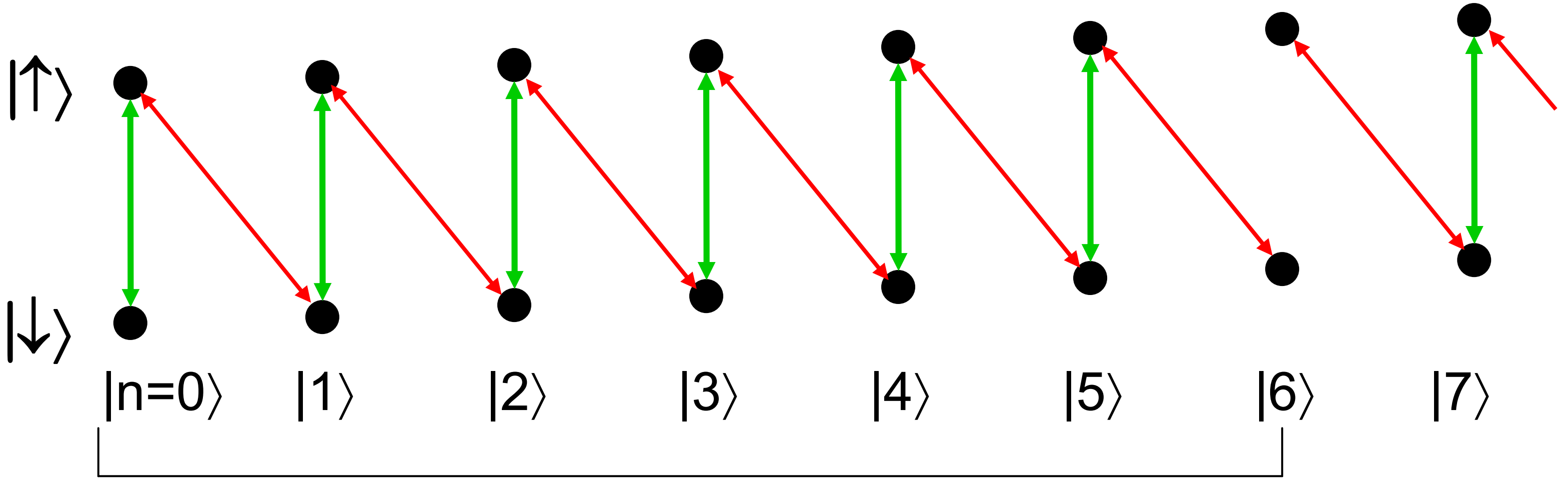


Manipulate coupling transitions

Carrier: **$|\downarrow n\rangle \leftrightarrow |\uparrow n\rangle \sim L_n(\eta^2)$**

First red sideband:
$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim i\, L^{(1)}_n(\eta^2)$

Choose η such that a desired transition is turned off.
E.g., at η **~ 0.53, $|\downarrow 7\rangle \leftrightarrow |\uparrow 6\rangle$** coupling is turned off



Finite sequentially connected system

Rangan, Monroe, Bucksbaum, Bloch, Phys. Rev. Lett., 2004

Lie algebra spans the space

Decompose control Hamiltonian into the roots of the algebra

Using standard notation for a basis of $su(N)$, let $e_{i,j}$ denote the matrix with unit ij entry and zeros elsewhere. Define $x_{i,j}=e_{i,j}-e_{j,i}$ and $y_{i,j}=\imath(e_{i,j}+e_{j,i})$. B is decomposed into the $\imath$-times-symmetric roots

$$S_1 = y_{1,2} = \imath \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3}$$

$$S_2 = y_{2,3}, \tag{4}$$

$$S_3 = y_{3,4}, \tag{5}$$

$$\cdots . \tag{6}$$

The Lie bracket of these roots with each other give the $N-2$ skew-symmetric matrices that represent next-nearest-neighbor coupling as shown below. These matrices form a closed Lie algebra with the matrices from which they were formed, for example, S_1, S_2 and their commutator $K_N=[S_1, S_2]$ form a Lie subalgebra, similarly for S_2, S_3 and their commutator K_{N+1}, and so on. This generation of alternate symmetric and skew-symmetric elements of the algebra has been observed earlier,[3,13]

$$[S_1, S_2] = x_{1,3} \equiv K_N, \tag{7}$$

$$K_{N+1} = x_{2,4}, \tag{8}$$

$$\cdots . \tag{9}$$

Similarly,

$$[x_{1,3}, x_{2,4}] = y_{1,4} \equiv S_{2N-1}. \tag{10}$$

Carrying on in a similar fashion through the matrix that represents the coupling between the first and Nth state (here N is assumed even),

$$S_{N(N-1)/2} = y_{1,N}. \tag{11}$$

It can be shown that the number of linearly independent commutators formed by this set of matrices is $N(N-1)/2$. Thus, the roots of the control Hamiltonian can be used to produce $N(N-1)/2$ independent elements of the algebra.

Lie algebra spans the space

An interesting observation can be made if the control matrices B_i representing the nearest-neighbor couplings are all skew-symmetric. The Lie algebra generated by these matrices consists of the skew-symmetric matrices, i.e., the symmetric matrices S_n are not generated. These matrices also number $N(N-1)/2$. This is the set of generators for the rotation group $O(N)$, each pairwise coupling representing an independent rotation in N-dimensions.[16]

Thus, if the eigenstates are sequentially connected by the transition matrix elements (usually real), then the Lie algebra generated by the roots of the control terms alone span a space of $N(N-1)/2$. If the drift matrix is strongly regular,[12] it can be decomposed into N linearly independent traceless diagonal matrices $h_i = e_{i,i} - e_{i+1,i+1}$. The Lie brackets formed by the drift matrix and the $N(N-1)/2$ matrices computed above yield another $N(N-1)/2$ matrices of the opposite symmetry. For example, $[A, S_1]$ gives K_1, etc. Thus the total number of linearly independent matrices are $2*N(N-1)/2+N=N^2$, which is sufficient to show controllability.

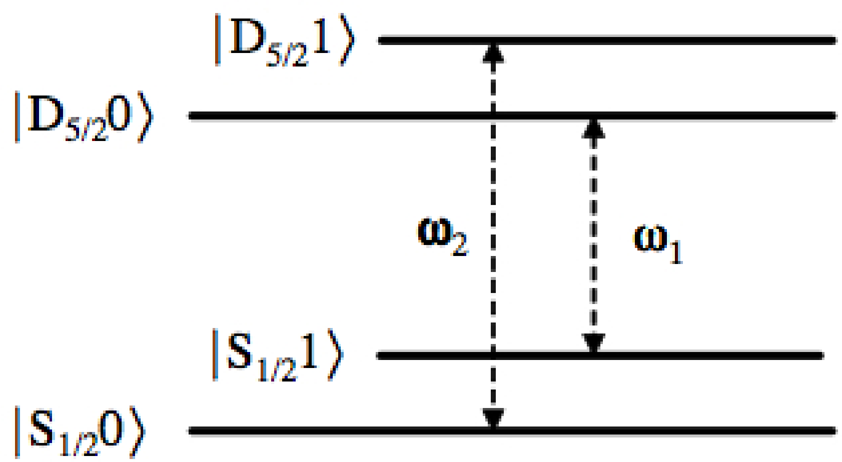
Lie algebra of the spin-1/2 coupled to truncated harmonic oscillator controlled by the carrier and red sideband fields spans the space.

Rangan & Bloch, J. Math. Phys., 2004

Lie algebra of multiple TIQC's

If an n-qubit system has a symmetric distribution of field-free eigenenergies, the system can be controlled by only $2^n(2^n+1)$ elements of the $sp(2^n)$ algebra.

$$H_0 = \begin{pmatrix} \omega_2 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 \\ 0 & 0 & 0 & -\omega_2 \end{pmatrix}$$



(Cabrera, Rangan, Baylis, Phys. Rev. A, 2007)

Manipulate coupling transitions

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First red sideband:
$|\downarrow n\rangle \leftrightarrow |\uparrow n\text{-}1\rangle \sim i\, L^{(1)}{}_n(\eta^2)$

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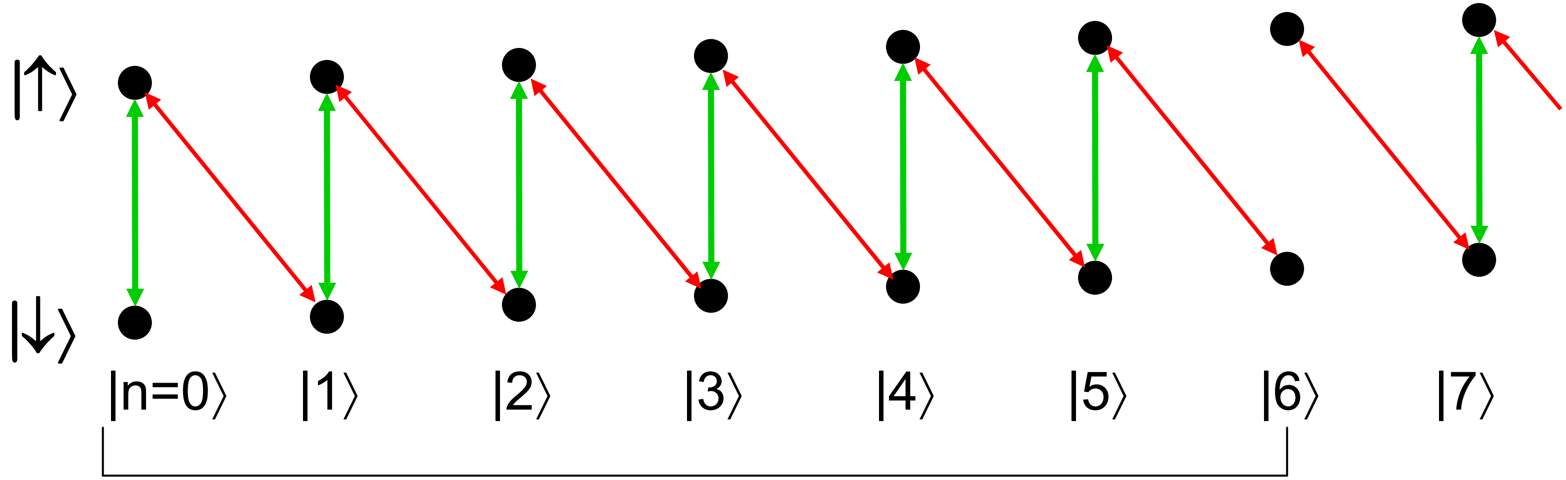
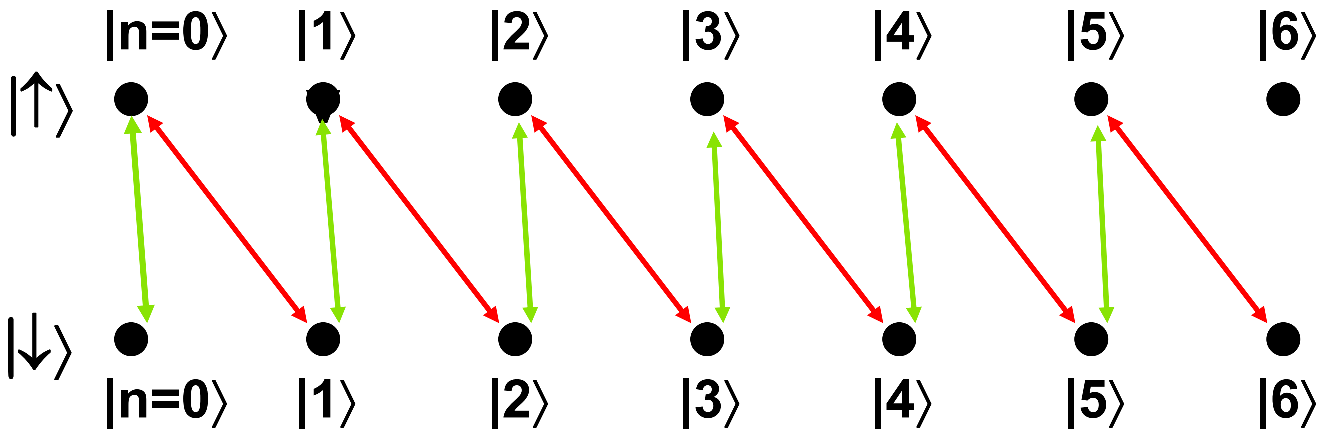
E.g., at η **~ 0.53, $|\downarrow 7\rangle \leftrightarrow |\uparrow 6\rangle$** coupling is turned off



Finite sequentially connected system

Rangan, Monroe, Bucksbaum, Bloch, Phys. Rev. Lett., 2004

Numerical example



$|\Psi(t=0)\rangle = |\downarrow 0\rangle$

$|\Psi(t=0)\rangle = (|\downarrow 4\rangle + |\uparrow 3\rangle)/\sqrt{2}$

3μs pulse produces 30% transfer

10μs pulse produces 99.4% transfer

Good candidate for optimal control problem

Optimal Control Theory

Shi & Rabitz (1988, 1990), Kosloff et al (1989), …

Find the control field E(t), $0 \le t \le T$

Initial state: $|\Psi(t=0)\rangle$

Target functional: $T = \langle \Psi(T) | P_k \rangle \langle P_k | \Psi(T) \rangle$ — **maximize**

Cost functional: $\int_0^T l(t) |E(t)|^2 dt$ — **minimize**; $l(t)$: **penalty parameter**

Constraint: Schrödinger's equation $|\dot{\Psi}(t)\rangle + \imath H(t, E(t)) |\Psi(t)\rangle = 0 + \text{c.c.}$

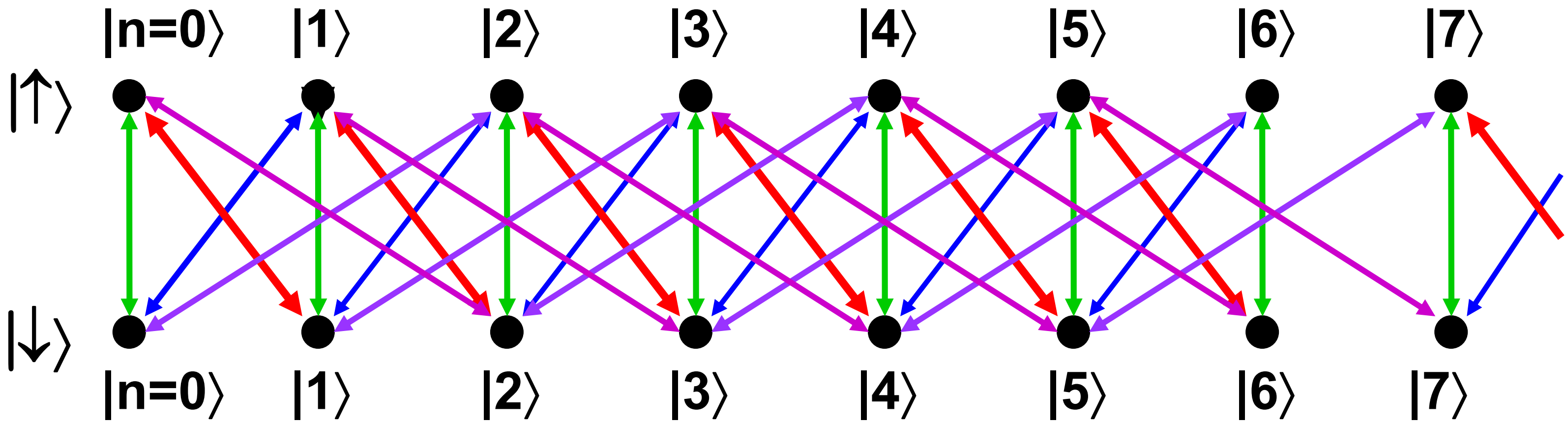
Introduce Lagrange multiplier: $|\lambda(t)\rangle$ Maximize unconstrained functional

$$J = T - \int_0^T l(t) |E(t)|^2 dt - 2\,\mathrm{Re} \int_0^T dt (\langle \lambda(t) | \dot{\Psi}(t) \rangle + \imath H(t, E(t)) | \Psi(t) \rangle)$$

OCT of Quantum Search Algorithm in Rydberg atoms:
Rangan & Bucksbaum, Phys. Rev. A, 64, 33417 (2001)

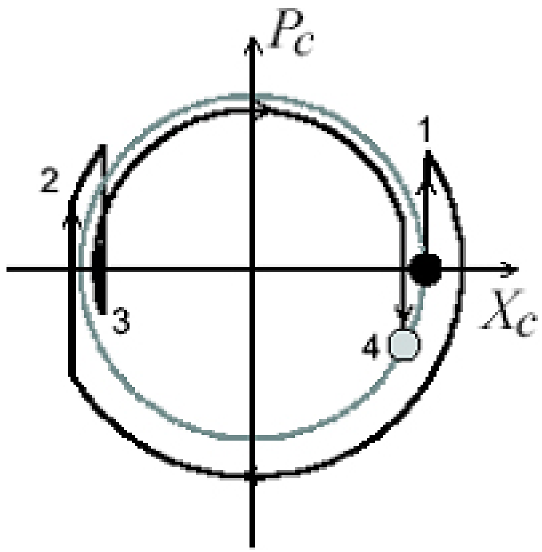
Using shorter pulses?

Faster pulses → larger bandwidth, many colors



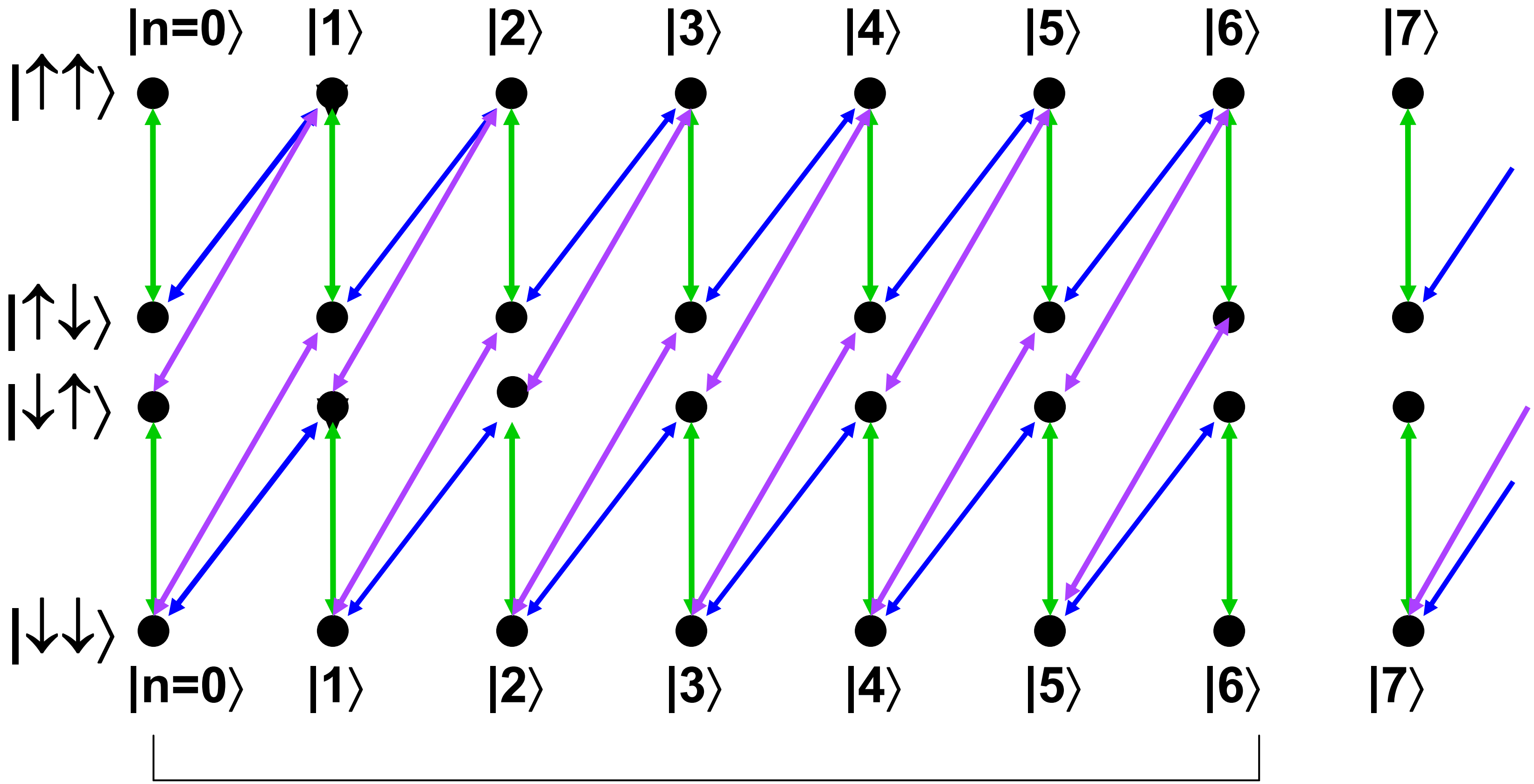
Uncontrollable!

Need faster pulses (ns)



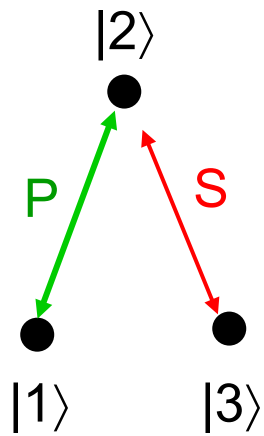
The fast pulse control scheme (Garcia-Ripoll et al, 2003) shows that it is possible to access a finite set of states (2⊗2) by leaving the state space into the HO states (coherent states).

Two-ion entangled states

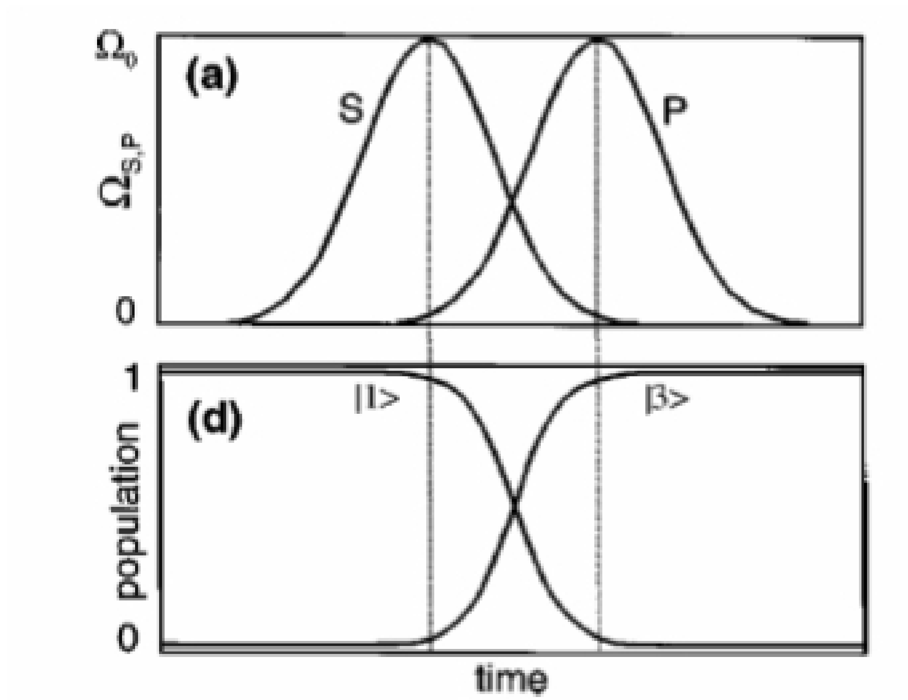


A bichromatic field can be used to produce entangled states of two ions. (Rangan, Monroe, Bucksbaum, Bloch, Phys. Rev. Lett., 2004)

Recap: Coherent control via STIRAP



Aim: adiabatically transfer populatio from $|1\rangle$ to $|3\rangle$



$$H(t) = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega_P(t) & 0 \\ \Omega_P(t) & 2\Delta_P & \Omega_S(t) \\ 0 & \Omega_S(t) & 2(\Delta_P - \Delta_S) \end{bmatrix}$$

From: Bergmann et al., Rev. Mod. Phys., 1998

Also look at David Tannor's book

Adiabatic Hamiltonian for trapped ion

UNPUBLISHED

RWA Hamiltonian in the interaction picture, fields on resonance

$$H_{int} = -\left(\begin{array}{cc|cc|cc|c} 0 & z_{12}E_c & 0 & 0 & 0 & 0 & \cdots \\ z_{21}E_c & 0 & z_{23}E_r & 0 & 0 & 0 & \cdots \\ \hline 0 & z_{32}E_r & 0 & z_{34}E_c & 0 & 0 & \cdots \\ 0 & 0 & z_{43}E_c & 0 & z_{45}E_r & 0 & \cdots \\ \hline 0 & 0 & 0 & z_{54}E_r & 0 & z_{56}E_c & \cdots \\ 0 & 0 & 0 & 0 & z_{65}E_c & 0 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)$$

Dipole matrix elements z_{ij} are complex

Only two colors E_c and E_r

Two-color N-level STIRAP

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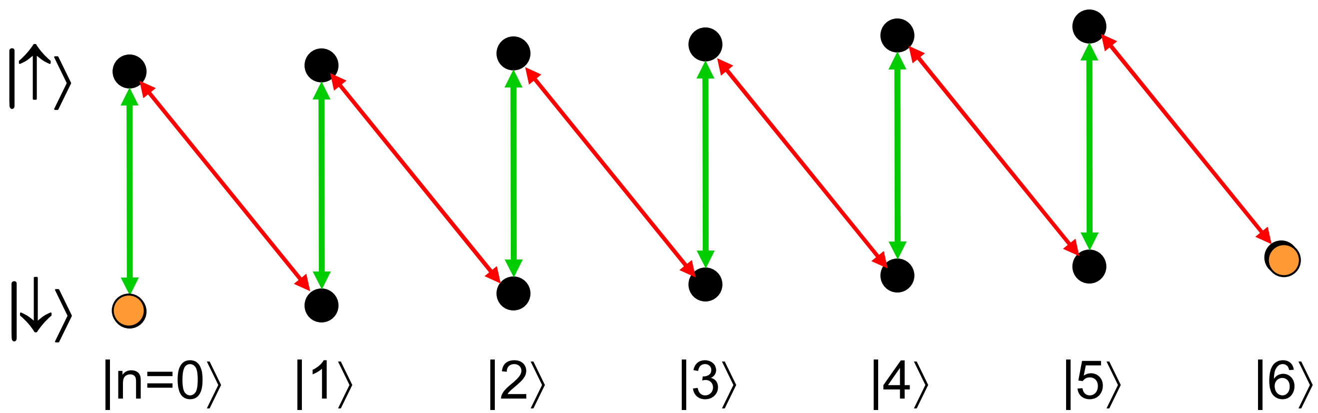
Truncated trapped-ion system:

Adiabatically transfer population from |↓, n=0⟩ to |↓, n=6⟩



Similar to multilevel STIRAP in magnetic sublevel quantum states: Shore, Bergmann et al., Phys. Rev. A, 1995. See also, theory by Vitanov, Phys. Rev. A.

STIRAP with >1 ions?

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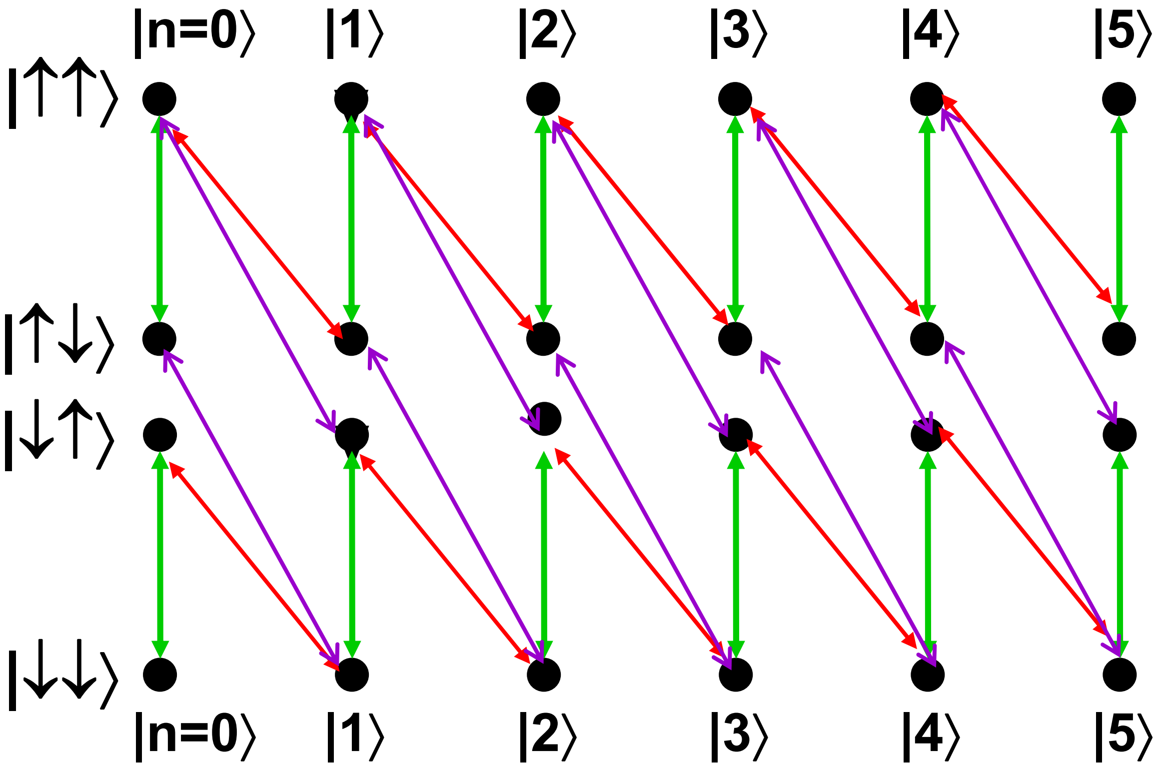


Adiabatic Hamiltonian couples only |↓↓⟩ with |↑↑⟩ ☺

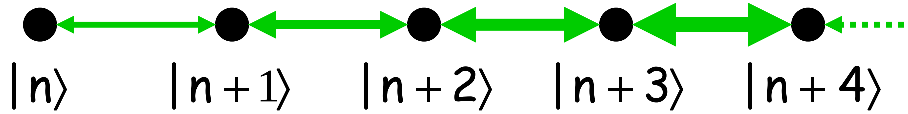
But equations are inconsistent ☹ - WIP

Transfer graphs and Control

- How well do transfer graphs represent quantum control processes?
- Classical transfer graphs: Turinici & Rabitz, Chem. Phys. 2001

  Eigenstates: nodes, transition couplings: edges

  Example:



Trapped-ion transitions

The transition couplings can be complex.

In LDL,

Carrier: $\Delta=0$

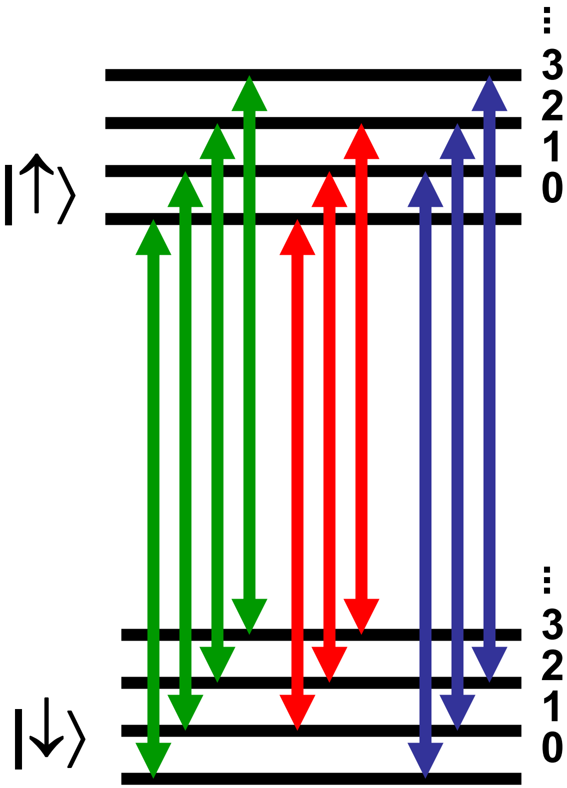
$H_I = \Omega(t)[\sigma_+ + \sigma_-]$

First red sideband: $\Delta=-\omega_m$

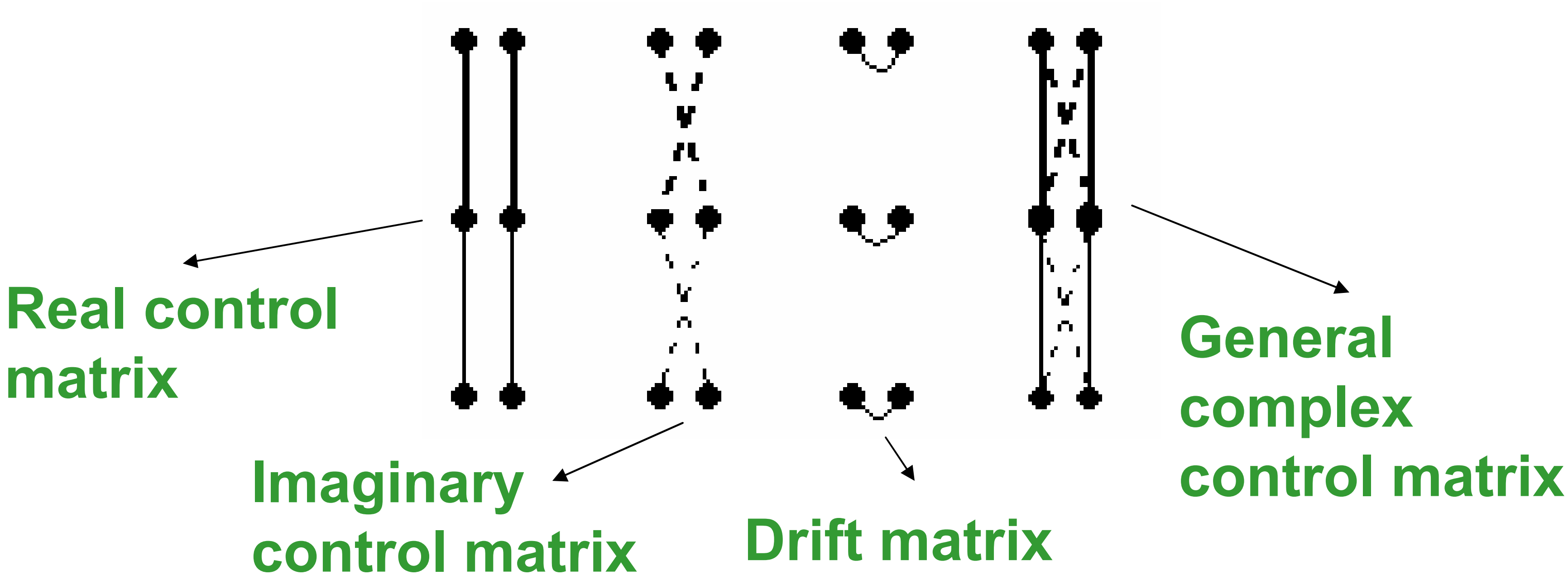
$H_I = \Omega(t)i[\sigma_+ a - \sigma_- a^\dagger]$

First blue sideband: $\Delta=\omega_m$

$H_I = \Omega(t)i[\sigma_+ a^\dagger - \sigma_- a]$

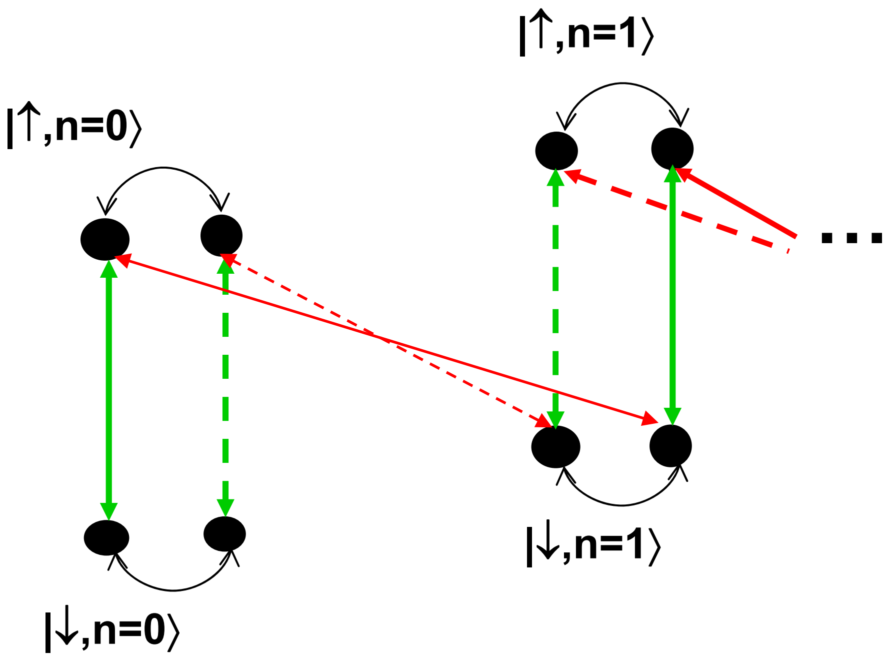


Quantum Transfer Graph



In QTGs, eigenstates represented by a doublet of nodes. (Rangan & Bloch, J. Math. Phys., 2005)

Quantum Transfer Graph



The role of the drift Hamiltonian (field-free evolution) is crucial for controllability of the finite (and ∞) system. This feature is elucidated by the quantum transfer graph. (Rangan & Bloch, J. Math. Phys., 2005)

Summary

Spin-half particle coupled to a quantum harmonic oscillator – model of a trapped-ion

Example of infinite-D control:

-Eigenstate controllability ≠

finite controllability ≠ global controllability

Bichromatic control in the truncated system

-Lie algebra, entanglement, optimal control, STIRAP

Classical transfer graphs have limitations in describing quantum control processes.

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