QUANTUM CONTROL OF LIGHT AND MATTER

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Fundamentals of Dynamical Decoupling (DD)

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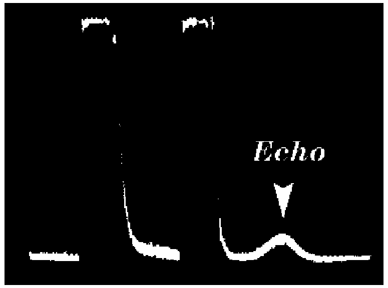
I. Motivation & Overview

What is DD?...

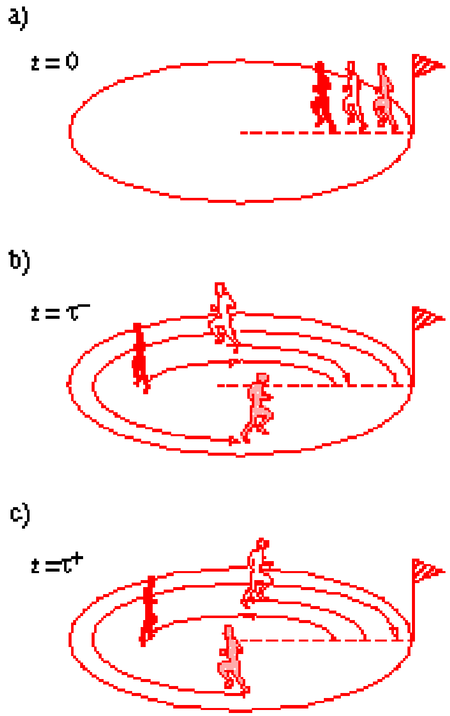
From: Coherent averaging techniques

Coherent control of nuclear spin Hamiltonians in high-resolution NMR spectroscopy.

E.L. Hahn, PR **80**, 580 (1950);
U. Haeberlen & J.S. Waugh, PR **175**, 453 (1968).



Paradigmatic example: Spin echo



The "race-track" echo: Effective time reversal

- Refocusing – Use of tailored pulse sequences to selectively 'turn off' undesired spin couplings over a time interval.

$$H = \omega_1 \sigma_z^1 + \omega_2 \sigma_z^2 \quad \rightarrow \quad H_{eff} = 0$$ **Non-selective**

$$H = \omega_1 \sigma_z^1 + \omega_2 \sigma_z^2 \quad \rightarrow \quad H_{eff} = \omega_1 \sigma_z^1$$ **Selective**

- Decoupling – Refocusing of couplings to a specific subset of degrees of freedom, which are effectively 'traced out'.

$$H = \omega_1 \sigma_z^1 + \omega_2 \sigma_z^2 + J \sigma_z^1 \sigma_z^2 \quad \rightarrow \quad H_{eff} = \omega_1 \sigma_z^1$$

Motivation:

- Elimination of couplings during signal acquisition;
- Enhancement of spectral resolution.

...Why do we need it in QIP?

To: Dynamical decoupling techniques ↔ Open-loop Hamiltonian engineering

Open-loop dynamical control schemes relying on the application of unitary control operations drawn from a basic (finite) repertoire.

Motivation: Need for effectively removing unwanted coherent and decoherent evolution ubiquitous in physical realizations of QIP!

- Dynamical coherent control of unitary (closed-system) evolution:
  - Halt natural evolution: 'no-op'/quantum storage
  - Switch off selected qubit couplings: Universal Hamiltonian simulation
  - Remove couplings to non-computational degrees of freedom: Leakage suppression
  - … …
- Dynamical coherent control of non-unitary (open-system) evolution:
  - Remove coupling to environment: Decoherence suppression (no redundancy, no measurement)
  - Symmetrize coupling to environment: DFS/NS synthesis
  - … …

Remark: Control tasks meaningful for both physical and logical (encoded) qubits...

A paradigmatic example: DD of qubit dephasing-I

LV & S. Lloyd, PRA **58**, 2733 (1998).

Dephasing spin-boson model:

$$H_0 = H_S \otimes I_E + I_S \otimes H_E + H_{SE} = \omega_0 \sigma_z \otimes I_E + I_S \otimes \sum_k \omega_k b_k^\dagger b_k + \sigma_z \otimes \sum_k g_k \left(b_k + b_k^\dagger\right)$$

➞ Preferred (z) basis: $[H_S, H_{SE}] = 0$ – Hovever, genuinely quantum bath: $[H_E, H_{SE}] \neq 0$

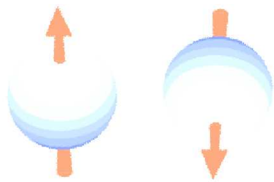
Free [exact] coherence dynamics:

$$\rho_{01}(t) \equiv \langle 0 | \rho(t) | 1 \rangle = \rho_{01}(0) \exp(2i\omega_0 t) \exp(-\Gamma_0(t))$$

$$\Gamma_0(t) = \int_0^\infty d\omega\, I(\omega) [2\bar{n}(\omega, T) + 1] \frac{1 - \cos(\omega t)}{\omega^2}$$

Decoherence function

Spectral density



Control action:

A train of identical, resonant π pulses, with separation Δt – arbitrarily strong and fast (BB).

Elementary spin-flip cycle of duration $T_c = 2\Delta t$:

$$U(T_c) = P U_0(\Delta t) P U_0(\Delta t) = e^{-i H'_0 \Delta t} e^{-i H_0 \Delta t} \equiv e^{-i H_{eff} T_c}, \quad H'_0 = P^\dagger H_0 P = -H_S + H_E - H_{SE}$$

$$H_{eff} = H'_0 + H_0 + O(\Delta t) \approx I_S \otimes H_E$$

Approximate time reversal as long as bath is 'frozen'!

A paradigmatic example: DD of qubit dephasing-2

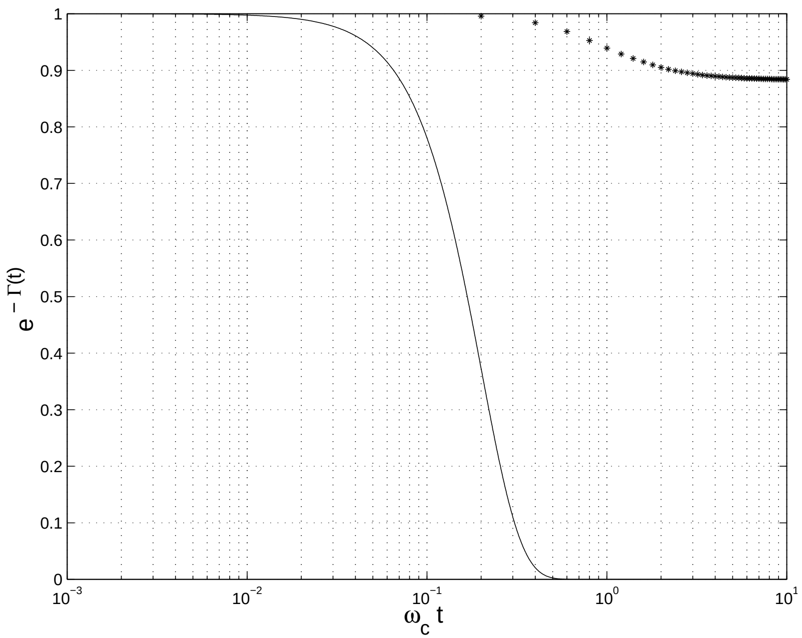
Controlled [exact] stroboscopic coherence dynamics, $t_N = NT_c$:

$$\Gamma_0(t_N) \rightarrow \Gamma_c(t_N) = \int_0^\infty d\omega \; I_c(\omega, T_c)[2\bar{n}(\omega, T)+1] \frac{1-\cos(\omega t_N)}{\omega^2}, \qquad I_c(\omega, T_c) = I(\omega) \tan^2\left(\frac{\omega T_c}{4}\right)$$

DD-renormalized spectral density

$$\lim_{T_c \rightarrow 0, N \rightarrow \infty} \rho_{10}(t_N = N T_c) = \rho_{10}(0)$$

Decoherence suppression if control period T_c shorter than memory correlation time.



$$\frac{\omega_c}{T} = 0.01 \qquad \frac{\Delta t}{\tau_c} = 0.1 \qquad \omega_c t_N = 2N\left(\frac{\Delta t}{\tau_c}\right),$$

$$N = 1, \ldots, N_{max} = 50$$

➡ Formal analogy with quantum Zeno (and anti-Zeno) physics

➡ Requirement for 'coherent averaging':

$$T_{DD} \ll \tau_c \sim 1/\omega_c = \min\{\tau^{corr}\}$$

Non-Markovian error regime

10+ yrs of DD: An (incomplete) overview...

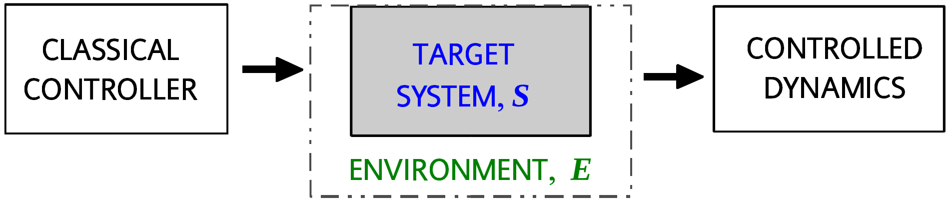
1998	BB control for single qubit.	Viola & Lloyd, PRA **58**, 2733
1999	Error suppression/symmetrization.	Viola, Knill, Lloyd, PRL **82**, 2417; Zanardi, PLA **258**, 77
	Universal decoupled control.	Viola, Lloyd, Knill, PRL **83**, 4888; Duan & Guo, PLA **261**, 139
	Parity kicks for quantum oscillator.	Vitali & Tombesi, PRA **59**, 4178
2000	Dynamical generation of NSs/DFSs.	Viola, Knill, Lloyd, PRL **85**, 3520
	Algebraic framework.	Knill, Laflamme, Viola, PRL **84**, 2525; Zanardi, PRA **63**, 012301
	Collisional decoherence suppression.	Search & Berman, PRL **85**, 2272; PRA **62**, 053405
	Off-resonant effect suppression.	Tian & Lloyd, PRA **62**, 050301
2001	Exp. BB suppression of single-photon dephasing.	Berglund, quant-ph/0010001
	Inhibition of decay to continuum.	Agarwal, Scully, Walther, PRL **86**, 4271
	Encoded dynamical decoupling.	Lidar & Wu, PRL **88**, 017905; Viola, PRA **66**, 012307
	Decoupling based on orthogonal arrays.	Stollsteimer & Mahler, PRA **64**, 052301
2002	Exp. realization of encoded dynamical decoupling.	Fortunato, Viola, NJP **4**, 5.1
	Universal quantum simulation.	Wocjan et al, QIC **2**, 133; Lloyd & Viola, PRA **65**, 010101
	Heating/finite temperature reservoir.	Vitali & Tombesi, PRA **65**, 012305
	DFS dynamical generation.	Wu & Lidar, PRL **88**, 207902
	Solid-state QC design and decoherence.	Byrd & Lidar, PRL **89**, 047901
	Universal leakage suppression.	Wu, Byrd, Lidar, PRL **89**, 127901
	Geometric interpretation of BB control.	Byrd & Lidar, QIP **1**, 19
2003	Robust bounded-strength design.	Viola & Knill, PRL **90**, 037901

An (incomplete) overview – Continued...

2004	Connection with quantum Zeno physics.	Facchi, Lidar, Pascazio, PRA **69**, 032314
	Application to 1/*f* spectral densities/power spectra.	Shiokawa & Lidar, PRA **69**, 030302
		Faoro & Viola, PRL **92**, 1179051
	Connection with universal dynamical control.	Kofman & Kuritzki, PRL **93**, 130406
	Decoupling based on Hamilton cycles.	Roetteler, quant-ph/0408078
	Equivalence with orthogonal arrays.	Roetteler & Wocjan, quant-ph/0409135
2005	BB control of nuclear quadrupolar qubit.	Fraval et al, PRL **95**, 030506
	Concatenated dynamical decoupling.	Khodjasteh & Lidar, PRL **95**, 180501
	Random dynamical decoupling.	Viola & Knill, PRL**94**, 060502
		Santos & Viola, PRL **97**, 150501
	PAREC/Embedded dynamical decoupling.	Kern & Alber, PRL **95**, 250501
2006	Decoupling based on Eulerian orthogonal arrays.	Wocjan, PRA **73**, 062317
	Exp. BB control of fullerene qubits.	Morton et al, Nature Phys. **2**, 40
2007	Optimal decoupling for dephasing spin-boson model.	Uhrig, PRL **98**, 100504
	DD for QDs/Long-time decoherence freezing.	Zhang *et al*, PRB **75**, 201302
2008	Universality of Uhrig DD for dephasing.	Yang & Liu, PRL **101**, 180403
	Exp. DD-enhanced nuclear spin memory in P:Si.	Morton et al, Nature **455**, 1085
	Exp. BB control of polarization qubits.	Damodarakurup et al, arXiv:0811.2654
2009	Exp. Uhrig DD in trapped ions.	Biercuk *et al*, Nature **458**, 996
	Dynamically corrected gates.	Khodjasteh & Viola, PRL **102**, 080501

II. Control-theoretic Framework

System assumptions



$$H_{tot}(t) = [H_S + H_{ctrl}(t)] \otimes I_E + I_S \otimes H_E + \Sigma_a E_a \otimes B_a$$

- Target system S is in general an open quantum system: Total 'drift' Hamiltonian

$$H_0 = H_S \otimes \boldsymbol{I}_E + \boldsymbol{I}_S \otimes H_E + H_{SE}\ , \qquad H_S \text{ traceless}$$

$$\mathcal{H} \simeq \mathcal{H}_S \otimes \mathcal{H}_E\, ,\ \mathcal{H}_S \simeq \mathbb{C}^d \text{ for some } d,\ d = 2^n \text{ for } n \text{ qubits}$$

Reduced system dynamics:

$$\rho_S(t) = Trace\left\{U(t)\rho_S(0) \otimes \rho_E(0) U^\dagger(t)\right\}$$

Closed-system limit (unitary dynamics) recovered for $H_{SE} = 0$.

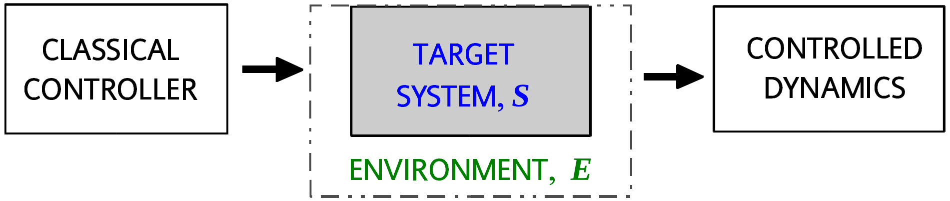
- H_{SE} responsible in general for unwanted non-unitary/decoherence effects:

$$H_{SE} = \sum_a S_a \otimes B_a\ , \qquad S_a \text{ traceless}$$

← **Error generators**

→ Bath operators H_S, B_a are assumed to be bounded but otherwise (potentially) unknown.

Control assumptions



$$H_{tot}(t) = [H_S + H_{ctrl}(t)] \otimes I_E + I_S \otimes H_E + \Sigma_a\, E_a \otimes B_a$$

- Environment E is uncontrollable: Adjoin controller acting on S only,

$$H_c(t) \equiv H_c(t) \otimes I_E, \quad H_c(t) = \sum_m \left(H_m \otimes \boldsymbol{I}_E\right) u_m(t)$$

Design object: $$U_c(t) = T\exp\{-i \int_0^t dx\; H_c(x)\}$$ ← **Control propagator**

- Controlled evolutions are most easily described in a frame that follows applied control,

$$\tilde{\rho}_{SE}(t) = U_c^{\dagger}(t)\, \rho_{SE}(t)\, U_c(t)$$

$$\tilde{U}(t) = U_c^{\dagger}(t)\, U(t) = T\exp\{-i \int_0^t dx\; U_c^{\dagger}(x) H_0 U_c(x)\}$$ ← **Logical (or toggling-frame) propagator**

→ Logical-frame evolution is ruled by time-dependent Hamiltonian $\tilde{H}(t) = U_c^{\dagger}(t) H_0 U_c(t)$.

Control objective and performance

DD problem = Open-loop steering problem for the (joint) propagator in an appropriate frame, subject to relevant control constraints.

- Assume that control inputs $u_m(t)$ can realize a set of instantaneous BB pulses.

DD benchmark: Suppress evolution due to H_{SE} and/or H_S over desired evolution time T

$$\tilde{U}(T) \approx \boldsymbol{I}_S \otimes U_E(T), \quad T > 0$$

NOOP gate/'time suspension'

Arbitrary state preservation:

$$\tilde{\rho}_{SE}(T) = \rho_S(0) \otimes [U_E(T)\rho_E(0)U_E^\dagger(T)] \quad \Rightarrow \quad \rho_S(T) = U_c(T)\rho_S(0)U_c^\dagger(T) = \rho_S(0) = |\psi\rangle\langle\psi|$$

- Characterize DD quality by appropriate performance indicators, *e.g.*:

  - Worst-case pure-state error (probability):

$$\epsilon_T = Max_{|\psi\rangle}\left\{Trace_S\left(P_S^\perp \tilde{\rho}_S(t)\right)\right\}, \quad P_S = |\psi\rangle\langle\psi|, \; P_S^\perp = \boldsymbol{I}_S - P_S$$

$1 - \epsilon_T =$ Minimum (input-output) fidelity

  - Average input-output fidelity or gate entanglement fidelity...
  - Fidelity error for a fixed [generic] initial state...

Classification of DD schemes

Design of DD schemes largely influenced by:

- Assumptions on control resources:

(1) Type of control operations –

Pulsed vs Continuous
Unbounded vs Bounded strength
Unbounded vs Bounded rate
...

(2) Mode of applying control operations –

Deterministic vs Randomized
(Cyclic vs Acyclic)
...

(3) Accuracy of control operations –

Perfect vs Faulty
(Systematic vs Random errors)
...

- Assumptions on target Hamiltonian:

Known vs Unknown (Model uncertainty)
Time-independent vs Time-varying
...

Generic vs Local structure (Efficiency)
...

No claim (hope) of completeness – focus on basic DD design...

III. Bang-Bang DD Protocols

Periodic DD: Average Hamiltonian description

- Assume that controller operates cyclically: $U_c(t + T_c) = U_c(t)$ for $T_c > 0$. **Cycle time**

  → Stroboscopic controlled evolution: $U_c(T_c) = \boldsymbol{I}_S, \quad U(t_M = M T_c) = \tilde{U}(t_M = M T_c)$

- Assume that drift Hamiltonian is time-independent, with $\|H_0\| \leq K = Max|eig(H_0)|$

  → A time-independent average Hamiltonian exists such that

$$U(t_M = M T_c) = \exp\{-i \overline{H} t_M\} = \exp\{-i(\overline{H}^{(0)} + \overline{H}^{(1)} + \ldots) t_M\}$$

Magnus series

$$\overline{H}^{(0)} \equiv \overline{H}_0 = \frac{1}{T_c} \int_0^{T_c} dt_1 \, \tilde{H}(t_1), \qquad \overline{H}^{(1)} = -\frac{i}{2T_c} \int_0^{T_c} dt_2 \int_0^{t_2} dt_1 \, [\tilde{H}(t_2), \tilde{H}(t_1)]$$

Convergent for $KT_c \ll 1$. Higher-order terms $\overline{H}^{(m)} T_c = O(KT_c)^m, \; m \geq 1$.

- First-order decoupling: Generate $\overline{H}$ to lowest order in T_c, by noticing that

  $\overline{H}_0$ approaches $\overline{H}$ in the fast control limit, $T_c = T/M, \; M \to \infty$.

- Physical requirement for manipulation:

  Coupling must remains coherent over manipulation time scale, $T_c \ll \tau_c = min_i \{\tau_i^{corr}\}$

Focus on $\overline{H}_0$ design

Group-based DD design

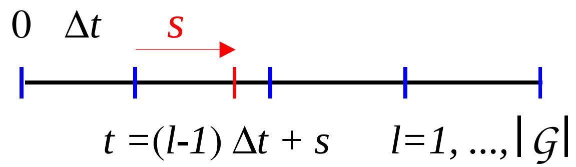
Keyword: Map time-average into group-theoretical average.

- Decoupling group: $\mathcal{G} = \{g_j\},\ j = 0,\ldots,|\mathcal{G}|-1$. $\mathcal{G}$ acts on state space $\mathcal{H}_S$ via a faithful, unitary, projective representation,

$$\mu(g_j) = \hat{g}_j \in \mathrm{Mat}_d(\mathbb{C}), \qquad \widehat{g_j g_k} = \hat{g}_j \hat{g}_k \text{ up to phase}, \quad \hat{g}_0 = \boldsymbol{I}_S .$$

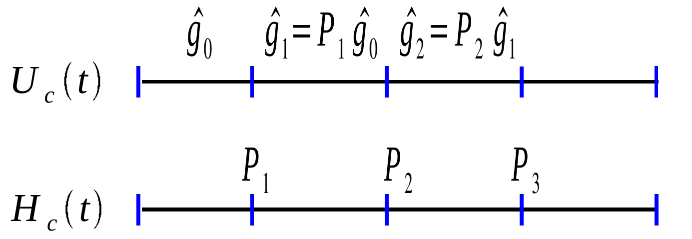
- PDD protocol: Let $T_c = |\mathcal{G}|\Delta t$ and assign $U_c(t)$ over T_c as

$$U_c\left[(l-1)\Delta t + s\right] = \hat{g}_{l-1}$$



$$U_c(t) = \begin{cases} \hat{g}_0 & t \in \Delta t_1 \\ \hat{g}_1 & t \in \Delta t_2 \\ \cdots & \cdots \\ \hat{g}_{|\mathcal{G}|-1} & t \in \Delta t_{|\mathcal{G}|} \end{cases}$$

Instantaneously change control propagator at the end of each control subinterval.



Sequence of BB pulses

$$P_k = \hat{g}_k \hat{g}_{k-1}^\dagger$$

Decoupling by symmetrization

- Lowest-order BB effective Hamiltonian:

$$\overline{H}_0 = \frac{1}{|\mathcal{G}|}\sum_j \hat{g}_j^\dagger H_0 \hat{g}_j = \left(\Pi_{\hat{\mathcal{G}}} \otimes I_E\right)(H_0)$$

$\mathcal{G}$-symmetrization

$\Pi_{\hat{\mathcal{G}}}$ is the projector onto the commutant $\widehat{\mathbb{C}\mathcal{G}}' = \{$operators commuting with all $\hat{g}_j\}$.

Filter out unwanted evolution using symmetry

(1) Closed-system setting:

→ $\Pi_{\hat{\mathcal{G}}}(H_0)=0$: Non-selective (maximal) refocusing (aka: 'annihilation')
→ $\Pi_{\hat{\mathcal{G}}}(H_0)\neq 0$: Selective refocusing

(2) Open-system setting:

Refocusing   Error suppression

→ $\Pi_{\hat{\mathcal{G}}}(S_a)\equiv 0$: $\overline{H}_0 = \left(\frac{1}{|\mathcal{G}|}\sum_j g_j^\dagger H_S g_j\right)\otimes \boldsymbol{I}_E + \boldsymbol{I}_S \otimes H_E + \sum_a \left(\frac{1}{|\mathcal{G}|}\sum_j g_j^\dagger S_a g_j\right)\otimes B_a$

→ $\Pi_{\hat{\mathcal{G}}}(S_a)=\overline{S}_a$: Error symmetrization

Remark: If $\mathcal{G}$ acts irreducibly, averaging is always maximal by Schur's Lemma, $\widehat{\mathbb{C}\mathcal{G}}' = \mathbb{C}\boldsymbol{I}$.

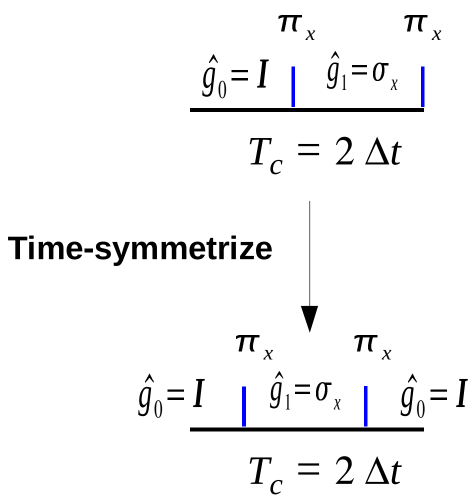
BB DD by example-I

1. Single qubit with pure dephasing:

$$H_0 = \omega_0 \sigma_z \otimes I_E + I_S \otimes H_E + \sigma_z \otimes B_z$$

$\mathcal{G} = \mathbb{Z}_2 = \{0, 1\}$, represented as $\widehat{\mathcal{G}} = \{ I, \sigma_x \}$

$$\bar{\sigma}_z = \frac{1}{2}\left[I \sigma_z I + \sigma_x \sigma_z \sigma_x \right] = 0$$



➡ Second-order PDD by rearranging control path,

$$U_c(T_c - t) = U_c(t) \Rightarrow$$

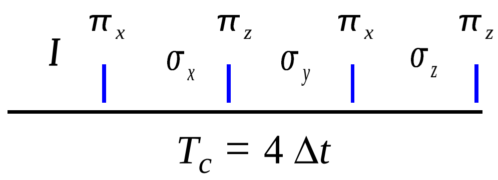
Leading corrections of order $O[(K T_c)^3]$.

2. Single qubit with arbitrary decoherence:

$$H_0 = \omega_0 \sigma_z \otimes I_E + I_S \otimes H_E + \sum_{a=x,y,z} \sigma_a \otimes B_a$$

$\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,1), (1,0)\}$, represented as $\widehat{\mathcal{G}} = \{ I, \sigma_x, \sigma_y, \sigma_z \}$

$$\bar{\sigma}_x = \frac{1}{4}\left[I \sigma_x I + \sigma_x \sigma_x \sigma_x + \sigma_y \sigma_x \sigma_y + \sigma_z \sigma_x \sigma_z \right] = 0, \quad etc.$$



➡ Different group paths yield same $\overline{H}^{(0)}$ but different $\overline{H}^{(1)}$.

BB DD by example-2

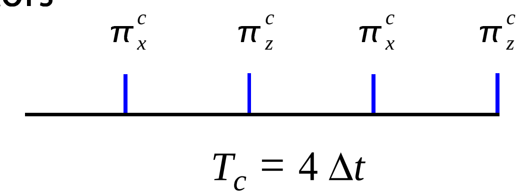
3. Quantum register with linear decoherence:

$$H_0 = \sum_{i=1}^{n} \omega_i \sigma_z^{(i)} \otimes \boldsymbol{I}_E + \boldsymbol{I}_S \otimes H_E + \sum_{i=1}^{n} \sum_{a=x,y,z} \sigma_a^{(i)} \otimes B_a^{(i)}$$

➞ Independent vs collective decoherence: $3n$ vs 3 error generators

$\mathcal{G} = \mathcal{Z}_2 \times \mathcal{Z}_2 = \{(0,1), (1,0)\}$, represented as $\hat{\mathcal{G}} = \{\boldsymbol{I}, X, Y, Z\}$

$X = \otimes_{i=1}^{n} \sigma_x^{i} \equiv \pi_x^c$, *etc.* (collective pulses) $\Rightarrow \quad \overline{\sigma}_a^i = 0 \quad \forall\, a, i.$



4. Quantum register with arbitrary evolution and/or decoherence, $d=2^n$:

➞ Fully generic [unknown] H_S and/or H_{SE} contain arbitrary n-body qubit operators.

$\mathcal{G} = \mathcal{Z}_d \times \mathcal{Z}_d$, represented as $\hat{\mathcal{G}}_P = \{\boldsymbol{I}, \sigma_x, \sigma_y, \sigma_z\}^1 \otimes \ldots \otimes \{\boldsymbol{I}, \sigma_x, \sigma_y, \sigma_z\}^n, \ |\hat{\mathcal{G}}_P| = 4^n$

➞ Recursively cycle each qubit through Pauli basis, *e.g.*, for arbitrary 2-body couplings.

$$H_0 = \sum_{i=1}^{n} \omega_i \sigma_z^{(i)} + \sum_{i<j=1}^{n} \sum_{a,b=x,y,z} J_{ab}^{ij} \sigma_a^{(i)} \otimes \sigma_b^{(j)}$$

$$n=3 \quad \begin{pmatrix} I & I & I & I & I & I & I & I & I & I & I & I & I & I & I & I \\ I & I & I & I & X & X & X & X & Z & Z & Z & Z & Y & Y & Y & Y \\ I & X & Z & Y & I & X & Z & Y & I & X & Z & Y & I & X & Z & Y \end{pmatrix}$$

Exp inefficient...

Complexity grows as 4^{n-1}

The need for improved DD design...

BB PDD is very simple and attractive in principle, unfortunately way too idealized...

- Poor efficiency: Averaging becomes unpractical with growing group size $|\mathcal{G}|$...

- Low-level averaging: Unwanted interactions are removed only to the first order...

$$F_{min}(T) \geq 1 - O\left(T^2 \Delta t^2 \|H_{error}\|^4\right)$$

(1) Fidelity loss directly dominated by second-order corrections...
(2) Residual error amplitudes add up coherently over multiple cycles...

⇒ High-level DD design

- Extremely unrealistic control resources...

(1) Instantaneous control pulses imply unbounded control strengths even for finite T_c...

➡ Poor spectral selectivity.

➡ Inappropriate for including drift during pulses.

⇒ DD beyond BB assumption

(2) No build-up tolerance/reduced sensitivity against control imperfections...
(3) Estimated control rates may be/seem prohibitively high for realistic open systems....

⇒ Improved convergence analysis

Improving DD efficiency: Combinatorial design

M. Stollsteimer & G. Mahler, PRA **64**, 052301 (2001); M. Roetteler & P. Wocjan, quant-ph/0409135.

Keyword: Make explicit reference to multipartite structure of the target system.

- Exploit combinatorial concept of Orthogonal Array (OA): *e.g.*, OA(16,5,4,2)

$$H_0 = \sum_{i=1} \sum_{i<j=1}^{n} \sum_{a,b=x,y,z} J_{ab}^{ij} \sigma_a^{(i)} \otimes \sigma_b^{(j)}$$

M: # control time-slots (→ 16) *n:* # qubits (→ 5)

$$\begin{pmatrix} I & I & I & I & X & X & X & X & Y & Y & Y & Y & Z & Z & Z & Z \\ I & X & Y & Z & I & X & Y & Z & I & X & Y & Z & I & X & Y & Z \\ I & X & Y & Z & X & I & Z & Y & Y & Z & I & X & Z & Y & X & I \\ I & X & Y & Z & Y & Z & I & X & Z & Y & X & I & X & I & Z & Y \\ I & X & Y & Z & Z & Y & X & I & X & I & Z & Y & Y & Z & I & X \end{pmatrix}$$

Every pair of rows contains all 16 possible pairs of symbols precisely once.

Big saving over 4^5!

- Any OA(M,n,4,2) can be used to decouple $n+1$ qubits governed by an arbitrary 2-local (bilinear) Hamiltonian using M control time-slots. OA(M,k,4,2) may be constructed from QEC codes with parameters

$$k = \frac{4^m - 1}{3}, \quad M = 4^m, \quad m \in \mathbf{N}$$

Efficient scaling *M* vs *k*

<u>Remark</u>: OA approach may be extended to qudits with t-local interactions.

Boosting DD performance: High-level DD design

Design of high-level DD protocols in the BB limit has explored different venues...

(1) Concatenated DD: Recursively apply a lower-order periodic sequence.
Optimize short-time performance by effective renormalization of H_{error}:

$$F_T = 1 - O\left(T^2 \left\| H_{error}^{eff} \right\|^2\right)$$

Khodjasteh & Lidar, PRL **95** (2005).

➞ Number of required pulses grows exponentially with concatenation level...
➞ Very successful in single-qubit decoherence settings...

Zhang *et al*, PRB **75** (2007); **77** (2008).

(2) Optimal DD: Achieve exact cancellation of H_{error} to desired order:

$$\Delta t_k = T \sin^2 \frac{k\pi}{2(N+1)}, \quad k = 1,2,\ldots,N$$

Uhrig, PRL **98** (2007).

➞ Linear complexity, however only for purely dephasing interactions...

(3) Randomized DD: Pick control operations and/or path at random.
Optimize long-time performance by enforcing probabilistic cancellation of H_{error}:

$$F_T = 1 - O\left(T \Delta t^5 \left\| H_{error} \right\|^6\right)$$

LV & E. Knill, PRL **94** (2005);
Santos & LV, PRL **97** (2006).

➞ Robust against model uncertainty, however requires tracking of control trajectory...

IV. Bounded-strength DD Protocols

Tools for bounded-strength design

Keyword: Eulerian cycles on Cayley graphs.

- Given $\mathcal{G}$, choose a set of generators, $\Gamma = \{ \gamma_\lambda \}$, $\lambda = 1, \ldots, |\Gamma|$.

  Rules for constructing the Cayley graph $G(\mathcal{G}, \Gamma)$:

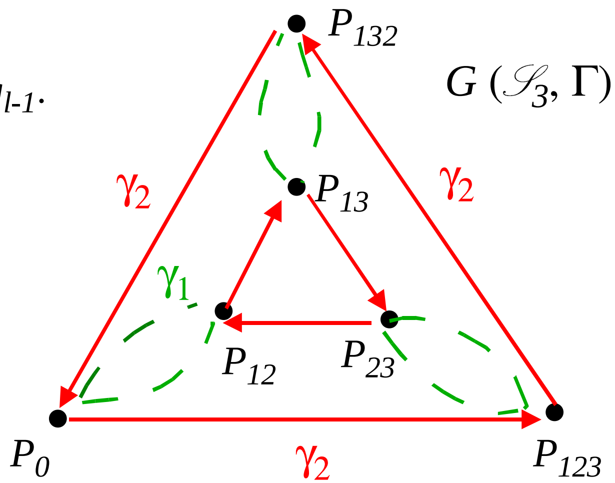
  → Assign a vertex to each group element.

  → Assign a color to each generator.

  → Join vertex g_{l-1} to g_l by an edge of color λ iff $g_l = \gamma_\lambda g_{l-1}$.

$$\mathcal{G} = \mathcal{S}_3 = \{ P_0, P_{12}, P_{23}, P_{13}, P_{123}, P_{132} \}$$

$$\Gamma = \{ \gamma_1, \gamma_2 \} = \{ P_{12}, P_{123} \} \qquad \gamma_2 \gamma_1 = \gamma_1 \gamma_2^2$$



- Def: An Eulerian cycle on $\mathcal{G}$ is a cycle that uses each edge exactly once. [Euler, 1766!]

- A Cayley graph supports Eulerian cycles of length

$$L = |\mathcal{G}|\,|\Gamma| \qquad \mathcal{P}_E(G(\mathcal{S}_3, \Gamma)) = (\gamma_2, \gamma_2, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_1, \gamma_2, \gamma_1, \gamma_1, \gamma_2, \gamma_1)$$

Eulerian DD (EDD)

LV & E. Knill, PRL **90**, 037901 (2003).

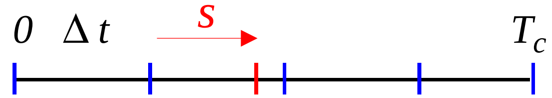
Keyword: Design continuous $U_c(t)$ according to Eulerian cycle.

- Control resources: Assume ability to implement group generators,

$$\hat{\gamma}_\lambda = T\exp\{-i\int_0^{\Delta t} ds\ h_\lambda(s)\} \equiv u_\lambda(\Delta t)\,, \quad \lambda = 1,\ldots,|\Gamma|.$$

- Eulerian protocol: Choose Eulerian cycle $\mathcal{P}_E = (\gamma_{\lambda 1}, \gamma_{\lambda 2}, \ldots, \gamma_{\lambda L})$ on $G\,(\mathcal{G}, \Gamma)$.
  Let $T_c = L\,\Delta t$ and assign $U_c(t)$ over T_c as

$$U_c(t_{l-1}+s) = u_l(s)\,U_c(t_{l-1})$$



$t_{l-1} = (l-1)\Delta t$ $\quad l=1,\ldots,L$

$$U_c(t) = \begin{array}{ll} u_1(s) & t=s \\ u_2(s)\,\hat{\gamma}_{\lambda_1} & t=\Delta t+s \\ u_3(s)\,\hat{\gamma}_{\lambda_2}\hat{\gamma}_{\lambda_1} & t=2\Delta t+s \\ \ldots & \ldots \\ u_L(s)\,\hat{\gamma}_{\lambda_{L-1}} & t=(L-1)\Delta t+s \end{array}$$

During the l-th interval, use as a control Hamiltonian the one that implements the generator $\hat{\gamma}_{\lambda_l}$, with γ_{λ_l} colouring the l-th edge in $\mathcal{P}_E$.

Eulerian symmetrization and robustness

- Lowest-order Eulerian effective Hamiltonian:

$$\overline{H}_0 = Q_{\hat{\mathcal{G}}}(H_0) = \Pi_{\hat{\mathcal{G}}}\left(F_{\hat{\Gamma}}(H_0)\right), \qquad F_{\hat{\Gamma}}(X) = \frac{1}{|\Gamma|}\sum_{\lambda=1}^{|\Gamma|}\frac{1}{\Delta t}\int_0^{\Delta t} ds\; u_\lambda^\dagger(s)\, X\, u_\lambda(s).$$

Still $\mathcal{G}$-invariant, but average is over both the group generators and the control interval.

- Provided that the errors generated during each interval are correctable by the DD group, EDD achieves the same $\mathcal{G}$-symmetrization of the BB limit with finite control strengths:

$$\overline{H}_0 = Q_{\hat{\mathcal{G}}}(H_0) = \frac{1}{|\mathcal{G}|}\sum_j \hat{g}_j^\dagger H_0\, \hat{g}_j$$

→ BB limit formally recovered for $F_{\hat{\Gamma}}(X) = X$ $i.e.\ h_l = 0$ during Δt_l.

→ A 2nd-order protocol may be obtained by a time-symmetric 'Euler supercycle' (SEDD).

- EDD automatically incorporates robustness properties against systematic control errors:

$$H'_c(t) = H_c(t) + \Delta H_c(t) = \text{ideal control} + \text{error component}$$

→ Control errors are also symmetrized: $Q_{\hat{\mathcal{G}}}[\Delta H_c(s)] \in \mathbb{C}\widehat{\mathcal{G}}\,'$

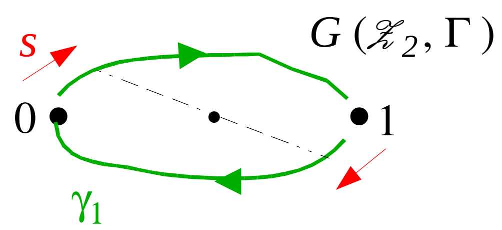
→ Full 'fault-tolerance' if DD group acts irreducibly.

EDD by example-I

1. Single qubit with pure dephasing revisited:

$\mathcal{G} = \mathbb{Z}_2 = \{0, 1\}$, represented as $\widehat{\mathcal{G}} = \{ \boldsymbol{I}, \sigma_x \}$

Generating set $\Gamma = \{\gamma_1\} = \{1\}$, $\mathcal{P}_E = (\gamma_1, \gamma_1)$



$$u_x(t) = \exp\{-i \int_0^t ds\; h_x(s) \sigma_x\}$$

$$u_x(\Delta t) = \sigma_x \equiv X$$

Euler cycle: $\gamma_1, \gamma_1 = \boldsymbol{X}\,\boldsymbol{X}$

SEDD: $(\boldsymbol{X}\,\boldsymbol{X})(\boldsymbol{X}^\dagger \boldsymbol{X}^\dagger)$

$$\overline{\sigma}_z = \frac{1}{2\Delta t}\left[\int_0^{\Delta t} ds\; u_x^\dagger(s) \sigma_z u_x(s) + \int_0^{\Delta t} ds\; u_x^\dagger(s) [\sigma_x \sigma_z \sigma_x] u_x(s) \right] =$$

$$= \frac{1}{2\Delta t} \int_0^{\Delta t} ds\, [u_x^\dagger(s) \sigma_z u_x(s) - u_x^\dagger(s) \sigma_z] = 0$$

'Continuous-time spin-echo'

➞ No time-overhead wrto BB case.

➞ Second-order DD implemented by pairs of alternating-phase pulses.

➞ Robustness (1^{st} order) against any systematic error along z, y:

$$Q_{\widehat{\mathcal{G}}}[\sigma_z(t)] = 0 \;,\; Q_{\widehat{\mathcal{G}}}[\sigma_y(t)] = 0 .$$

LV, JMO **51**, 2357 (2004).

EDD by example-2

2. Single qubit with arbitrary decoherence revisited:

$\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,1), (1,0)\}$

Generating set $\Gamma = \{ \gamma_1, \gamma_2 \} = \{(0,1), (1,0)\}$,

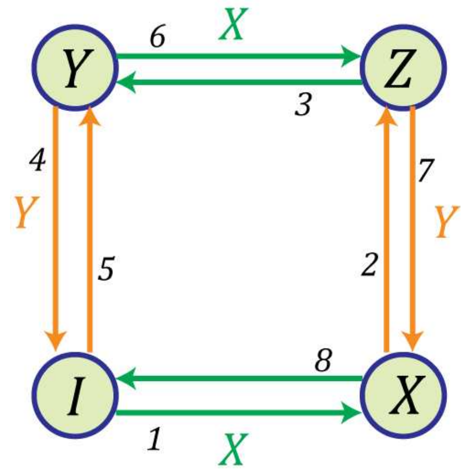
$$\mathcal{P}_E = (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_2, \gamma_1, \gamma_2, \gamma_1)$$

$$u_x(t) = \exp\{-i \int_0^t ds\ h_x(s) \sigma_x\}$$

$$u_y(t) = \exp\{-i \int_0^t ds\ h_y(s) \sigma_y\}$$

➡ Time-overhead wrto BB case: Factor of 2.

➡ Fully robust (to 1^{st} order) against systematic faults.



Generators: X , Y

Euler cycle: $X\ Y\ X\ Y\ Y\ X\ Y\ X$

3. Quantum register with linear decoherence revisited:

$\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,1), (1,0)\}$

Now work under the collective (tensor power) representation,

$\mathcal{P}_E = (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_2, \gamma_1, \gamma_2, \gamma_1)$, with collective generators, $\gamma_1 = X^{(all)} = X_1 \otimes \ldots \otimes X_n$

$\gamma_2 = Y^{(all)} = Y_1 \otimes \ldots \otimes Y_n$

➡ Control may be realized through collective Hamiltonians:

$$u_x^{(all)} = \exp\left[-i \int_0^t ds\ h_x(s) \left(\sum_j^n \sigma_x^{(j)} \right)\right]$$

Khodjasteh & LV, arXiv:0906.0525.

V. (Some) Experimental Demonstrations

Optical polarization qubits: Single-axis BB DD

Berglund [with P. Kwiat], quant-ph/00100001.

Qubit computational basis: $\{|H\rangle, |V\rangle\}$

Input photon state:

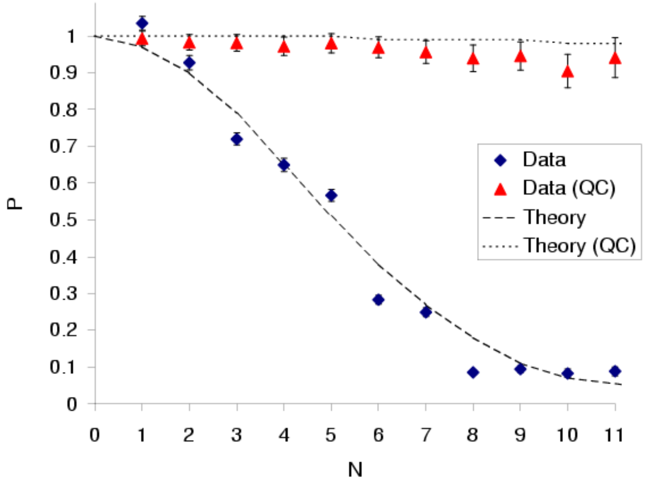
$$|\Psi\rangle_{input} = \int d\omega\ A(\omega)|\omega\rangle \otimes |\psi\rangle,\ \ |\psi\rangle = c_H|H\rangle + c_V|V\rangle$$

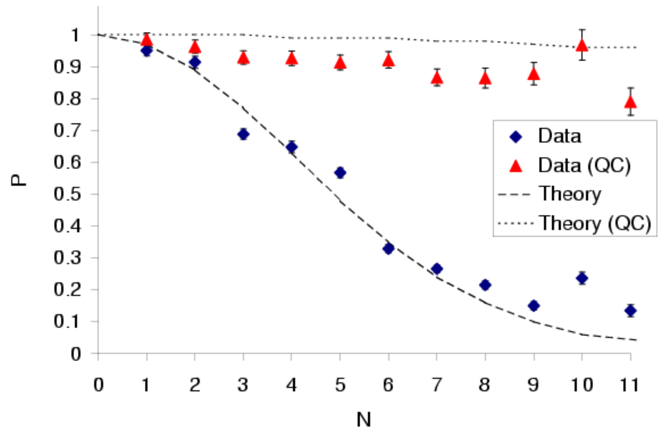
➞ Decoherence introduced by unwanted coupling between polarization and frequency degrees of freedom (frequency-insensitive detection).

➞ BB control according to $\mathcal{G} = \mathscr{Z}_2 = (\mathrm{I}, \mathrm{X})$ implemented by periodically interchanging polarization eigenstates, faster than optical correlation time $\tau_c \approx L_c / c$,

$$\boldsymbol{R}|H\rangle = |V\rangle,\ \boldsymbol{R}|V\rangle = \pm|H\rangle$$

λ/2 or λ/4 waveplate, or
optically active quartz rotator





Optical polarization qubits: Two-axis BB DD

Damokarakurup *et al*, arXiv:0811.2654 (Nov 2008).

Qubit computational basis: $\{|H\rangle, |V\rangle\}$

➞ Decoherence in an arbitrary basis engineered through suitable cavity design:
Insert a Soleil-Babinet element in front of each of two 45-deg plane mirrors

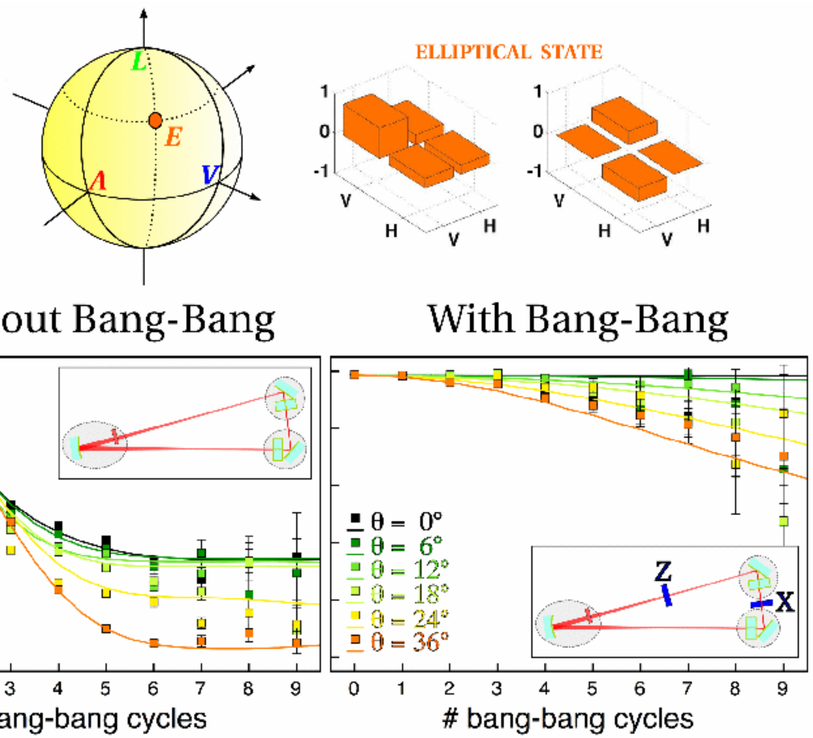
➞ BB polarization control achieved through full Pauli-group DD (two active operation/cycle)

$\mathcal{G} = \mathcal{Z}_2 \times \mathcal{Z}_2 = (I, X, Y, Z) \dashrightarrow$ XZXZ

$Z = |H\rangle\langle H| - |V\rangle\langle V|,$

$X = |H\rangle\langle V| + |V\rangle\langle H|$

Active DD-correction of arbitrary single-qubit errors



DD in solid-state nuclear-quadrupole qubits

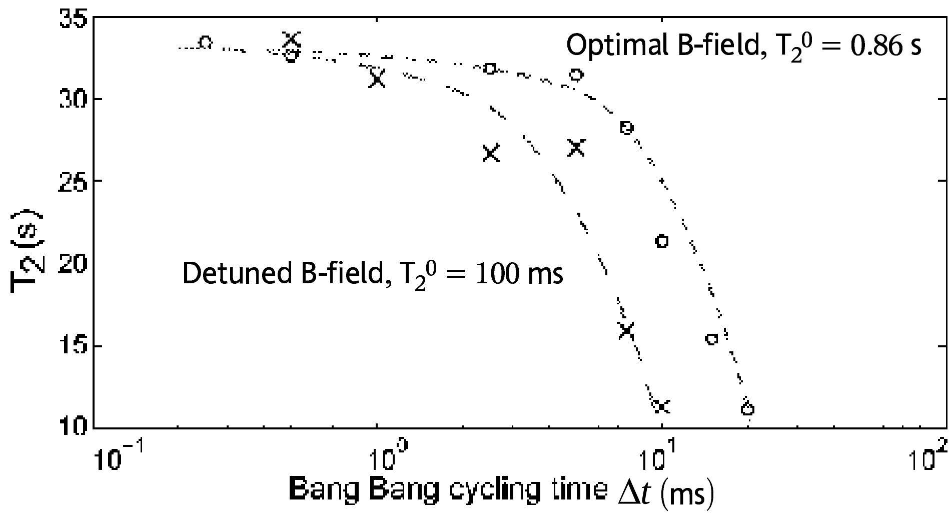
Fraval, Sellars & Longdell, PRL **95**, 030506 (2005).

System: Crystal of Y_2SiO_5 doped with Pr^{3+} ions.

Electronic ground state splits into 3 doubly degenerate levels with $m_I = \pm 5/2, \pm 3/2, \pm 1/2$

➡ Quantum coherence stored in hyperfine transition $m_I = -1/2 \rightarrow +3/2$

➡ Decoherence primarily due to slowly fluctuating magnetic fields as a result of cross-relaxation



DD according to

$\mathcal{G} = \mathbb{Z}_2 = (I, X)$

Hyperfine coherence time extended to > 30s.

DD in electron-nuclear spin qubits

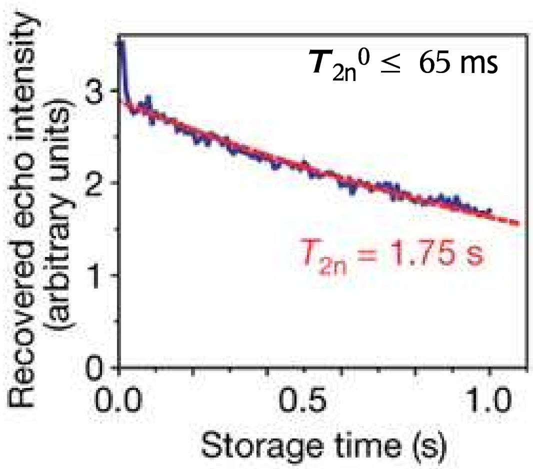
Morton *et al*, Nature Phys. **2**, 40 (2006).

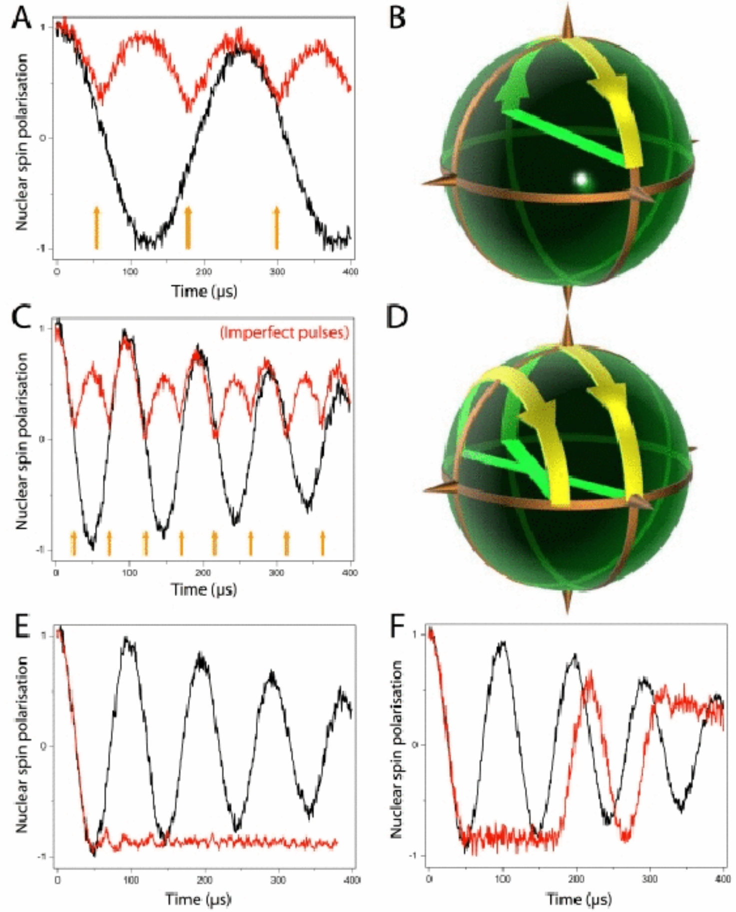
2-qubit system supported by 4 levels in 12-level spin manifold of fullerene N@C60 molecule:

$$m_S^{electron} = +3/2, -3/2, \quad m_I^{nucl} = 0, 1$$

➞ Strong engineered coupling (RF driving) is introduced to drive Rabi nuclear oscillations, and removed by DD on the electron.

$$\mathcal{G} = \mathscr{Z}_2 = (\mathrm{I}, \mathrm{Z})$$





➞ DD-enhanced quantum storage of electron-spin state in the nuclear-spin state of ^{38}P in ^{28}Si single crystal.

Morton *et al*, Nature **455**, 1085 (2008).

DD in trapped-ion qubits

Biercuk *et al*, Nature **458**, 996 (2009); arXiv:0905.0286 (May 2009).

System: Array of $^9Be^+$ ions in a Penning trap.
Qubit states: Ground-state electron-spin-flip transition.

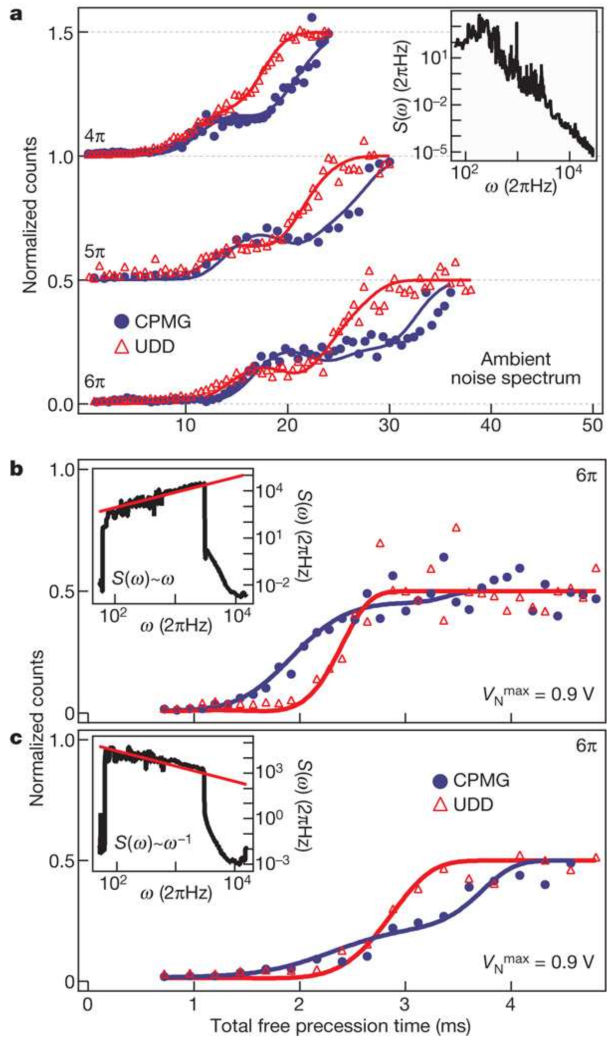
$$(m_I, m_J) = +3/2, \pm 1/2$$

➞ Phase decoherence due to ambient [magnetic field] noise as well as engineered phase noise

$$S(\omega)_{ambient} \approx \frac{1}{\omega^4}, \quad S(\omega)_{eng} \approx \omega\, \Theta(\omega - \omega_c)$$

➞ Quantitative comparison between BB control based on uniformly spaced vs optimally spaced π_x pulses in different purely-dephasing environments

UDD significantly outperforms uniform DD in the presence of noise with a 'hard' spectral cut-off.



Essential reading on DD theory

- *Dynamical suppression of decoherence in two-state quantum systems* –
  LV & S. Lloyd, *Phys. Rev. A* **58**, 2733 (1998).
- *DD of open quantum systems* –
  LV, E. Knill & S. Lloyd, *Phys. Rev. Lett.* **82**, 2417 (1999).
- *Robust DD of quantum systems with bounded controls* –
  LV & E. Knill, *Phys. Rev. Lett.* **90**, 037901 (2003).
- *Enhanced convergence and robust performance of randomized DD* –
  L.F. Santos & LV, *Phys. Rev. Lett.* **97**, 0150501 (2006).
- *Performance of deterministic DD schemes: Concatenated and periodic sequences* –
  K. Khodjasteh & D. A. Lidar, *Phys. Rev. A* **75**, 062310 (2007).
- *Keeping a quantum bit alive by optimized* π*-pulse sequences* –
  G.S. Uhrig, *Phys. Rev. Lett.* **98**, 100504 (2007).
- *Introduction to quantum dynamical decoupling* –
  LV, Book chapter, forthcoming [check my research group webpage@Dartmouth...]

...Thank you for your attention...