# Quantum tomography and compressed sensing via continuous measurement and control

*arXiv:quant-ph/1208.5015*

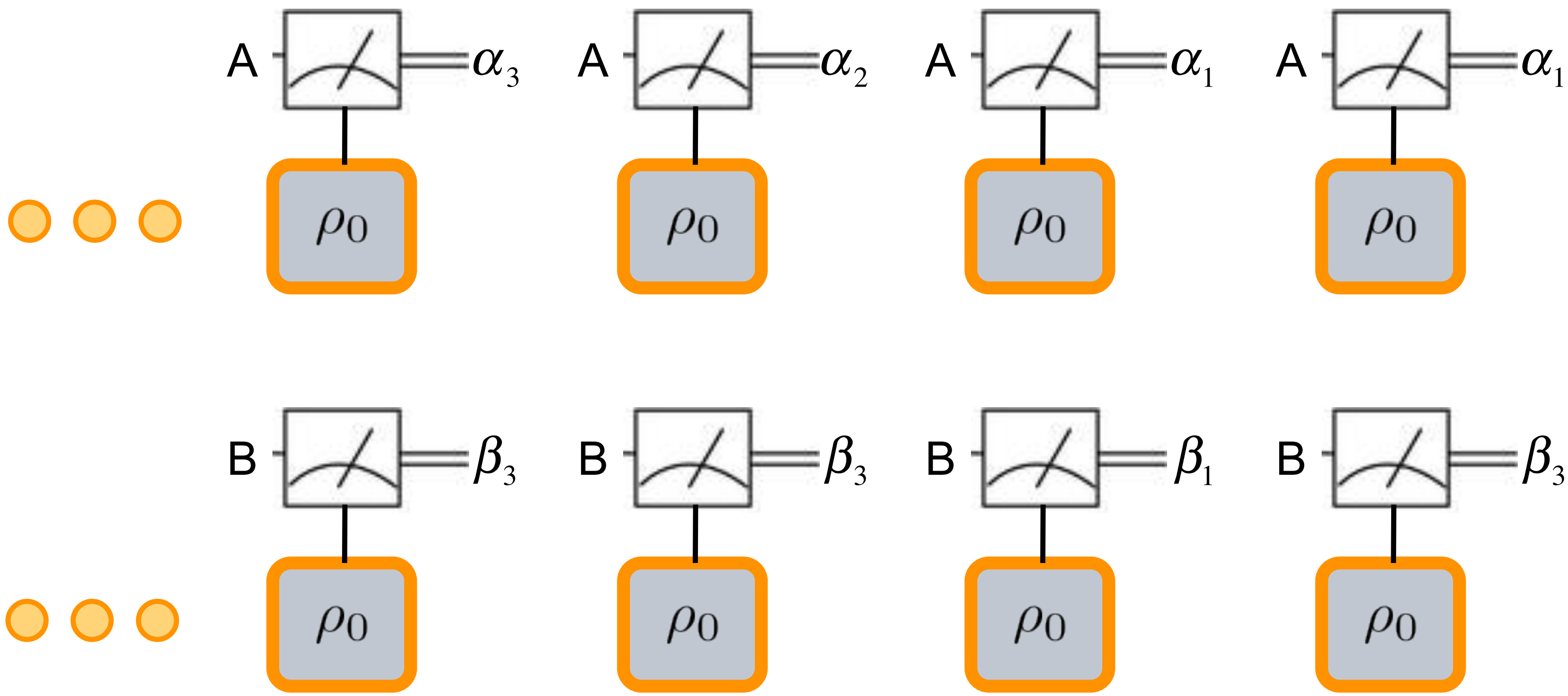
Ivan H. Deutsch, Carlos Riofrío

*University of New Mexico*

Poul S. Jessen, Aaron Smith, Brian Anderson

*University of Arizona*

# Standard Quantum Tomography




Informationally complete set of observables

{A,B, C,…}

Measurement statistics

$f(\alpha_i), f(\beta_i), \ldots$

$\rho_0$

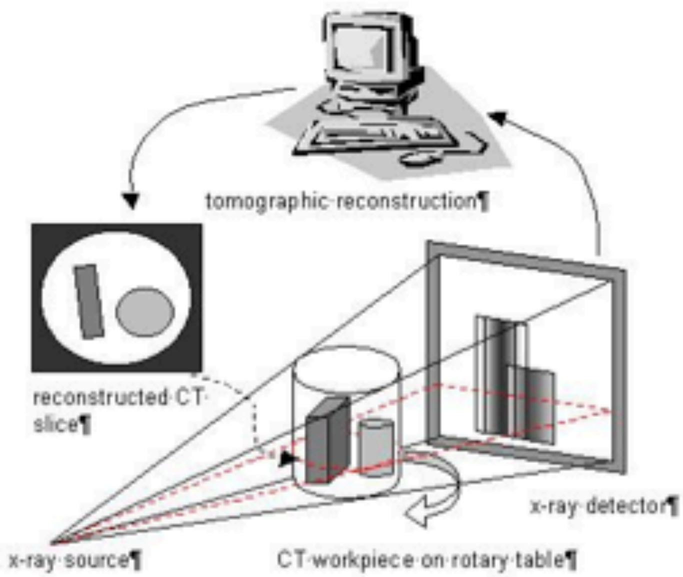
# Challenges to Standard Tomography

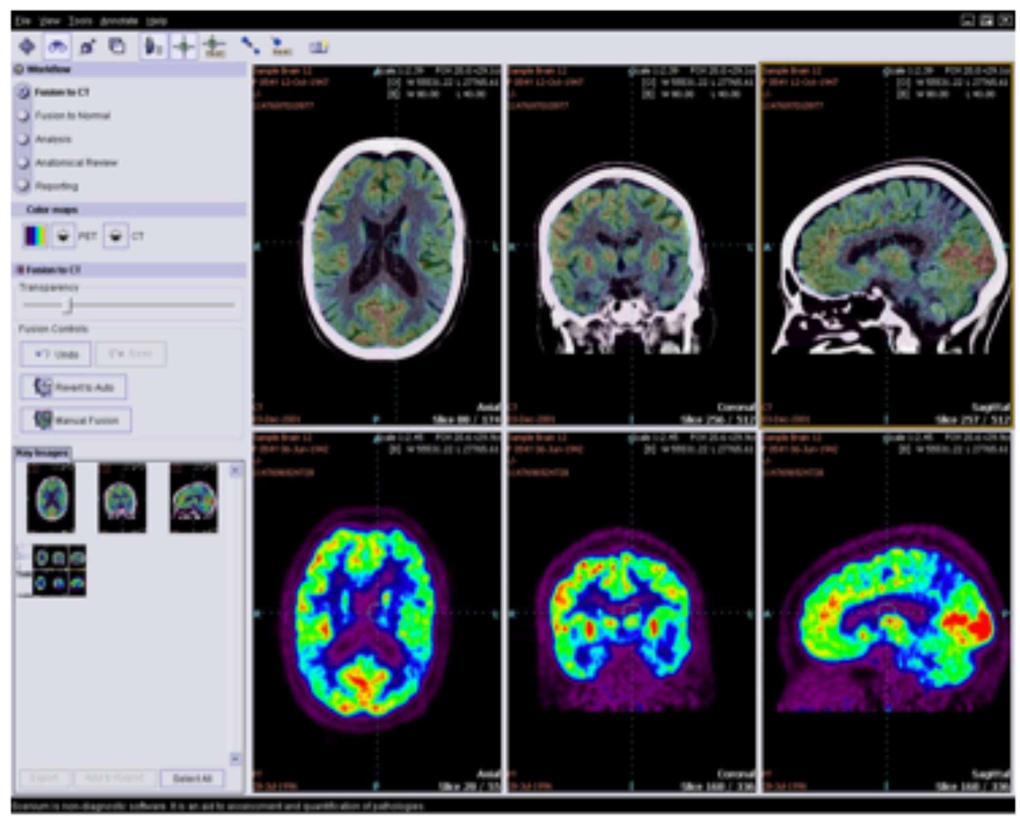
- Extremely time consuming to collect statistics. Informationally complete set: $D^2 - 1 = d^{2N} - 1$ (*n*-qudits)

- Requires repeated preparation of exactly the same state.

- Not robust to errors in measurement settings/repeated preparation. *Systematic errors*

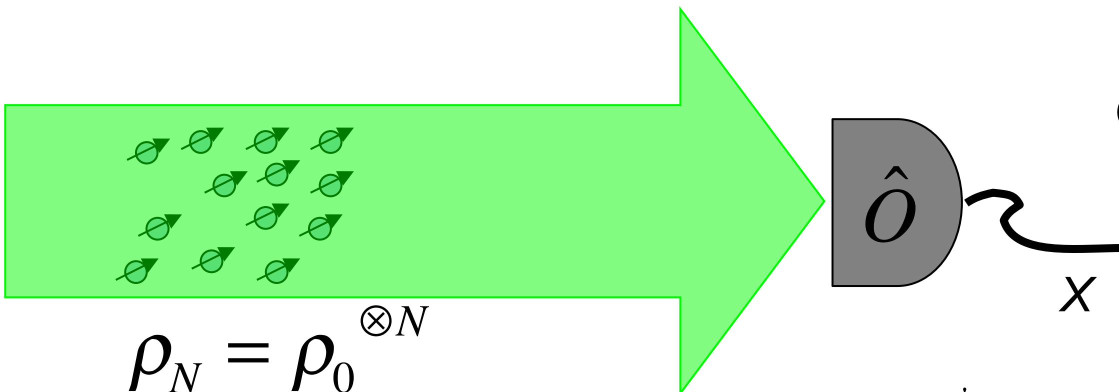
# Classical Tomography



# Quantum Tomography via Weak Continuous Measurement and Control

- Ensemble identically coupled to weak probe (ancilla)




$$\rho_N = \rho_0^{\otimes N}$$

Generalized measurement

$$P_X = Tr\left(E_X \rho_0^{\otimes N}\right)$$

$$E_X = A_X^\dagger A_X = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(X-\hat{O})^2}{2\sigma^2}\right]$$

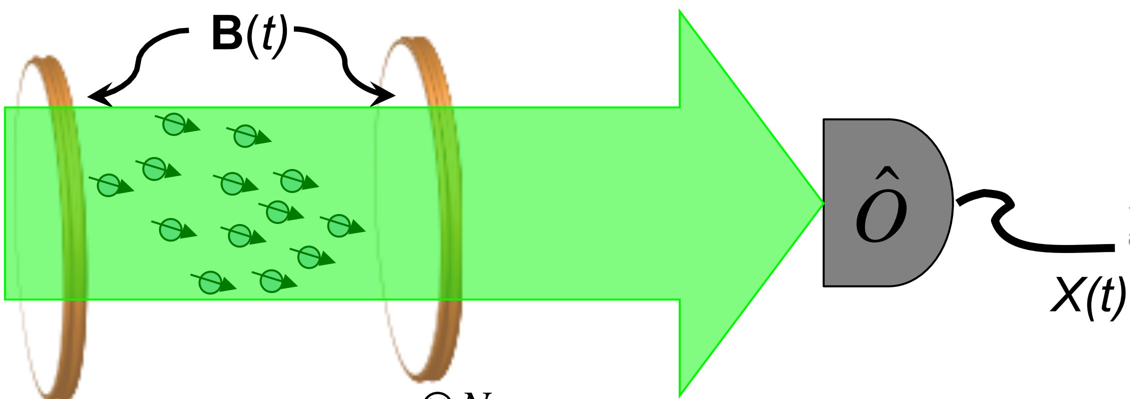
- Multiple copies give excellent statistics.

- Weak backaction: Noise dominated by probe rather than projection noise.

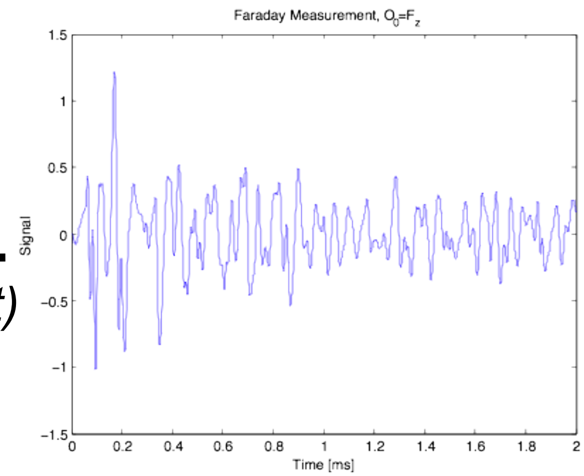
$$\rho_N^{out} = \frac{A_X \rho_N A_X^\dagger}{P_X} \approx \rho_N$$

# Quantum Tomography via Weak Continuous Measurement and Control

• Time-dependent control generates new information

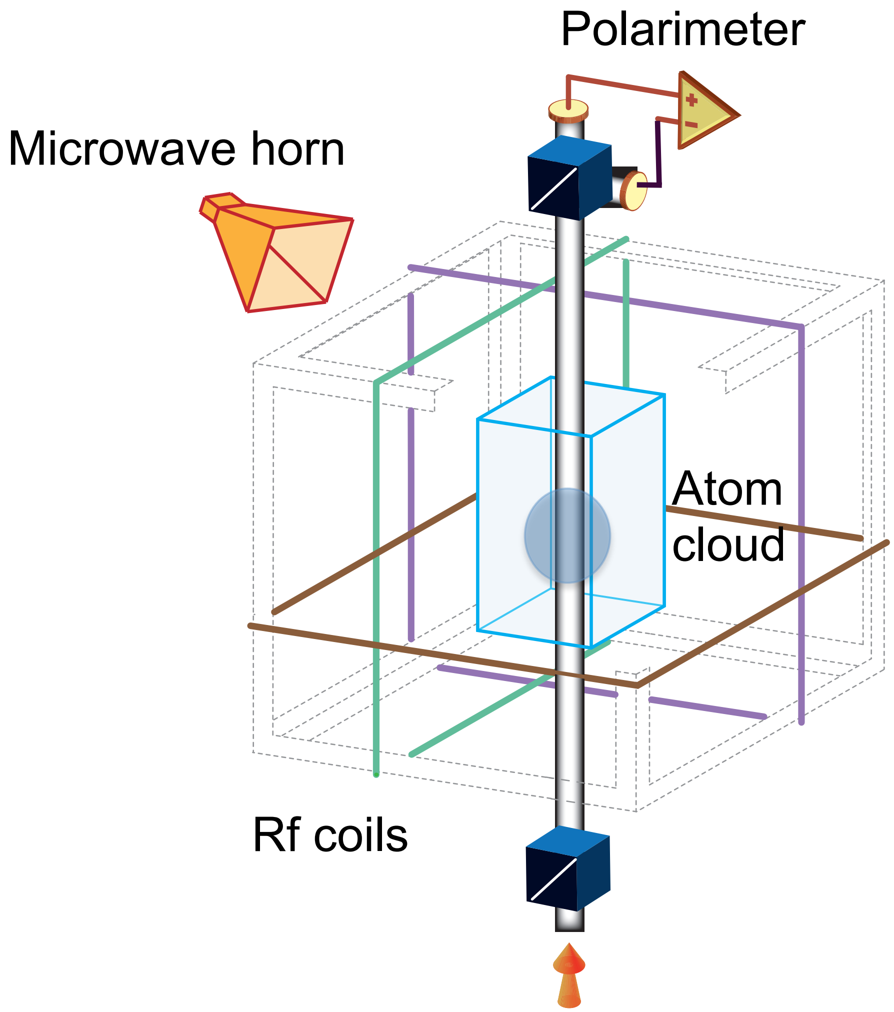


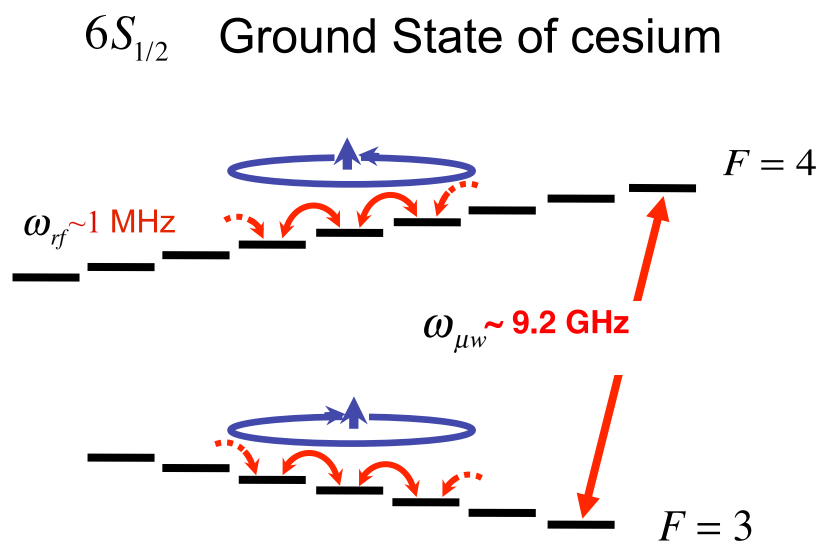

$$\rho_N(t) = \rho(t)^{\otimes N}$$




• Quantum-state estimation problem
  - Choose **B**(*t*) so that *X*(*t*) is "informationally complete"
  - Invert measurement record to find $\rho(t=0) = \rho_0$

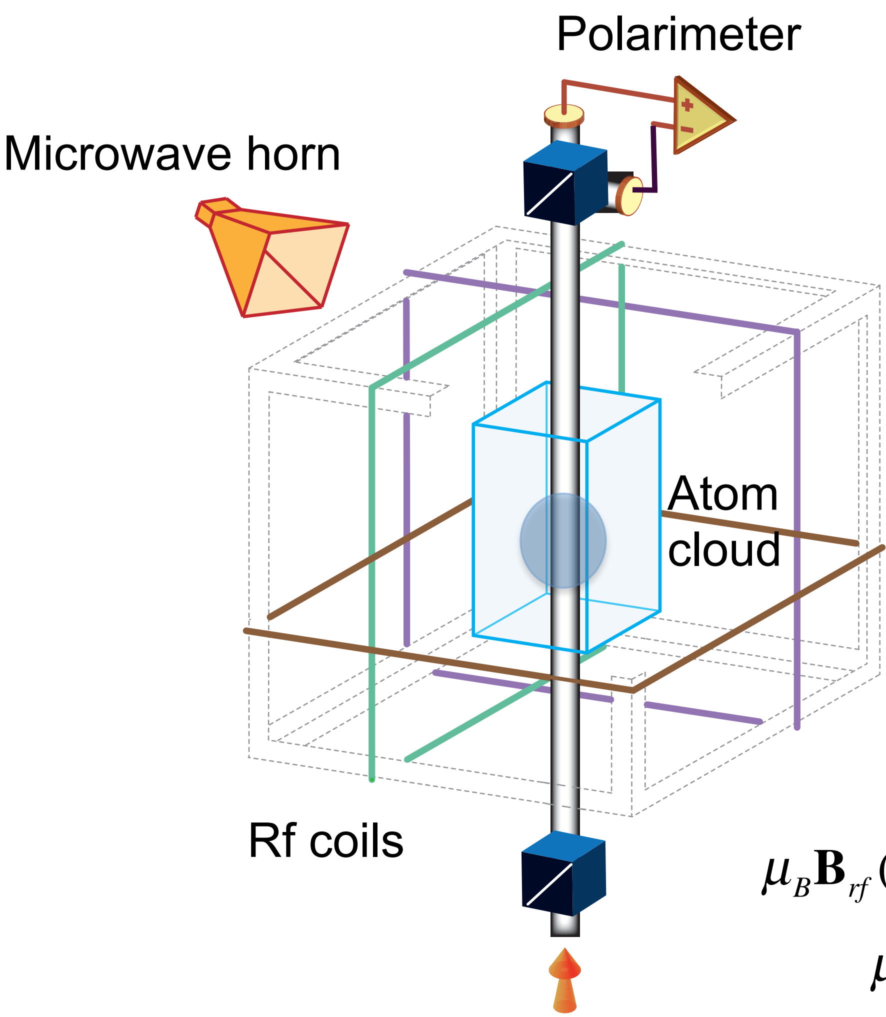
# Our Platform: Atomic Hyperfine Spins

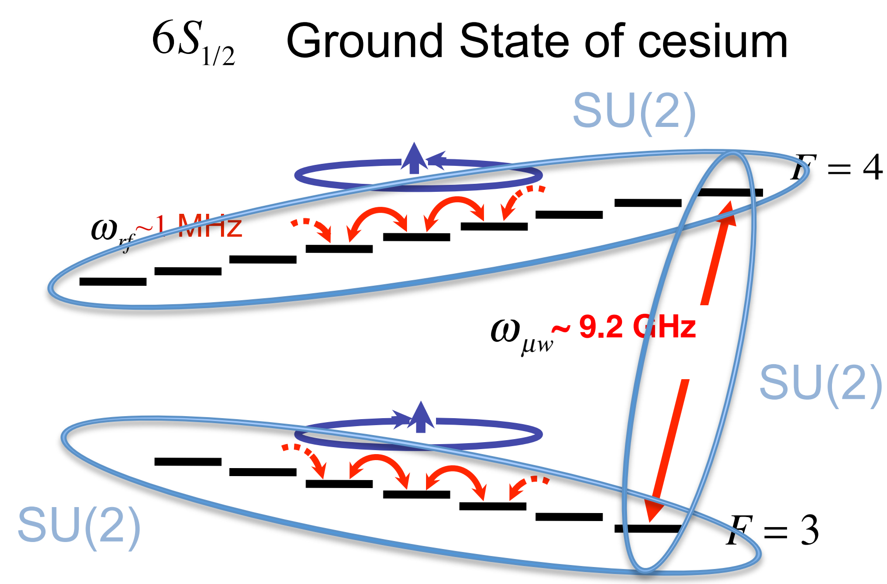



16 dimensional Hilbert space of magnetic sublevels.

# Our Platform: Atomic Hyperfine Spins




Control Hamiltonian

$$H_c(t) = -\vec{\mu}\cdot\left(\mathbf{B}_0 + \mathbf{B}_{rf}(t) + \mathbf{B}_{\mu w}(t)\right)$$

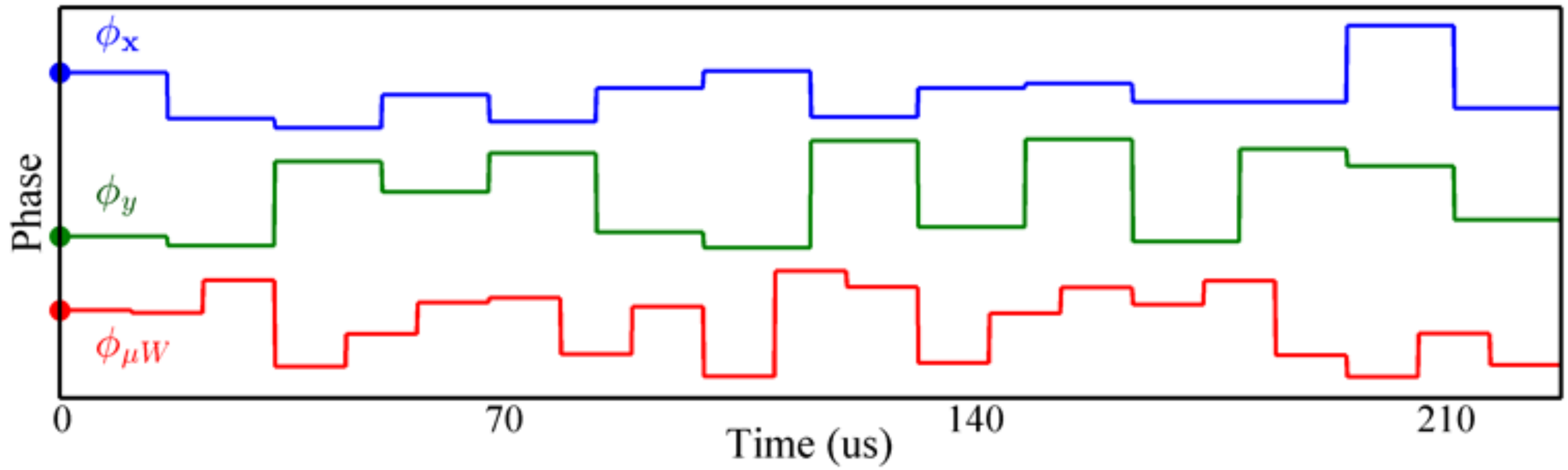
$$\mu_B \mathbf{B}_{rf}(t) \sim \Omega_{rf}\left(\mathbf{e}_x \cos(\omega_{rf}t - \phi_x(t)) + \mathbf{e}_y \cos(\omega_{rf}t - \phi_y(t))\right)$$

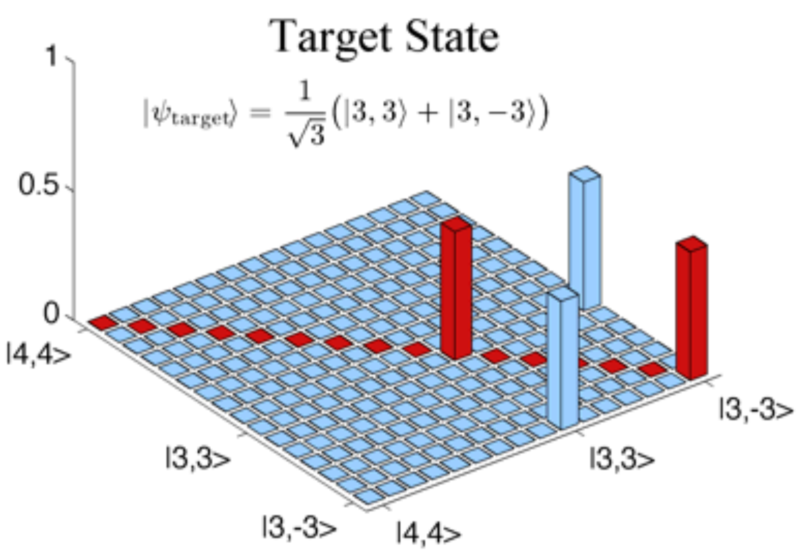
$$\mu_B \mathbf{B}_{\mu w}(t) \sim \Omega_{\mu W}\,\mathrm{Re}\left(\vec{\varepsilon}\exp\left\{-i(\omega_{\mu w}t - \phi_{\mu w}(t))\right\}\right)$$

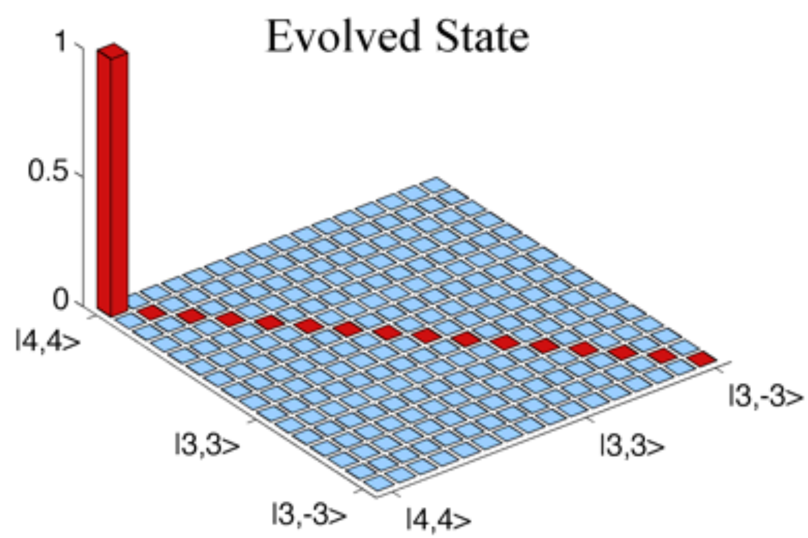
***Controllable → Open-loop control to find*** $\left\{\phi_x(t), \phi_y(t), \phi_{\mu w}(t)\right\}$ SU(16)

# Example: State Mapping

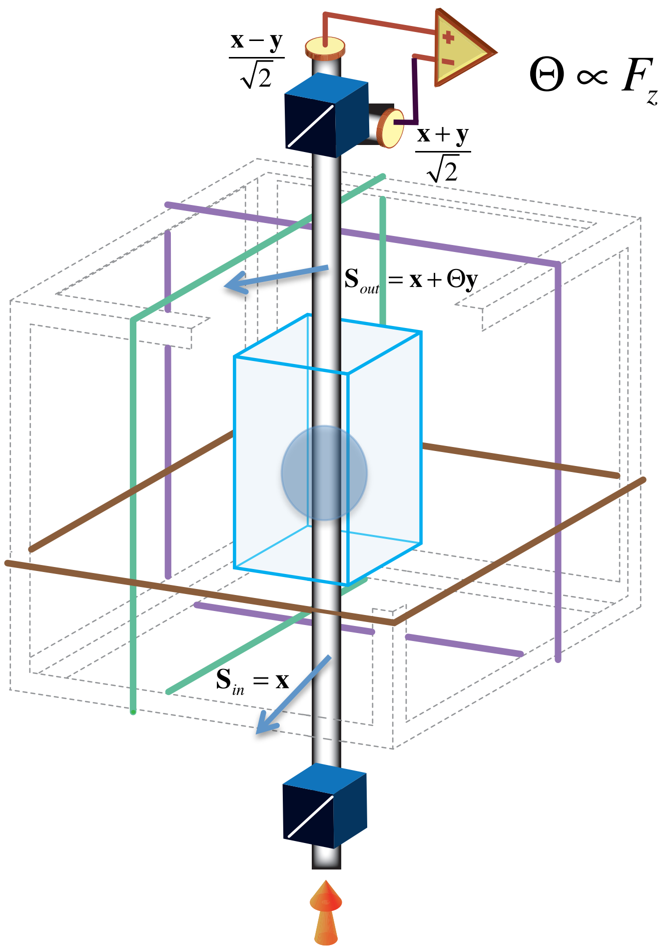
$\Omega_x = \Omega_y = 2\pi \cdot 15\text{kHz}$ $\qquad$ $\Omega_{\mu w} = 2\pi \cdot 33\text{kHz}$



# Measuring Atomic Spins: Faraday Spectroscopy




Faraday Interaction: Laser polarization (ancilla) rotates proportional atomic magnetization:

$$U_{AL} = e^{-i\chi F_z S_3}$$

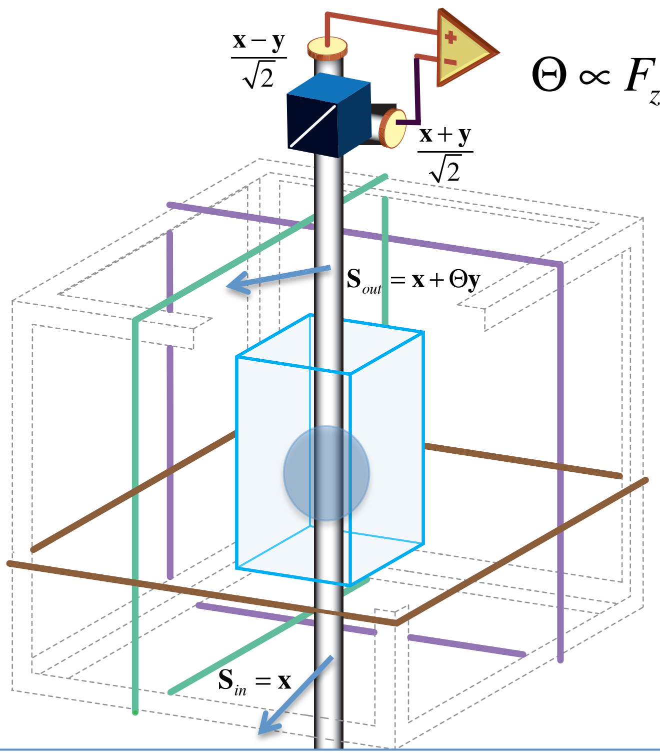
Note: **Ensemble**

Collective Spin

$$F_z = \sum_{i=1}^{N_A} f_z^{(i)}$$

Rotation about 3-axis of Poincaré sphere proportional to $F_z$ component of atomic spin

# Measuring Atomic Spins: Faraday Spectroscopy

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Kraus Operator: $A_M = \langle S_2 = M_S | e^{-i\chi F_z S_3} | S_2 = 0 \rangle \approx e^{-\frac{\chi^2 N_L}{4}(M_S - F_z)^2}$

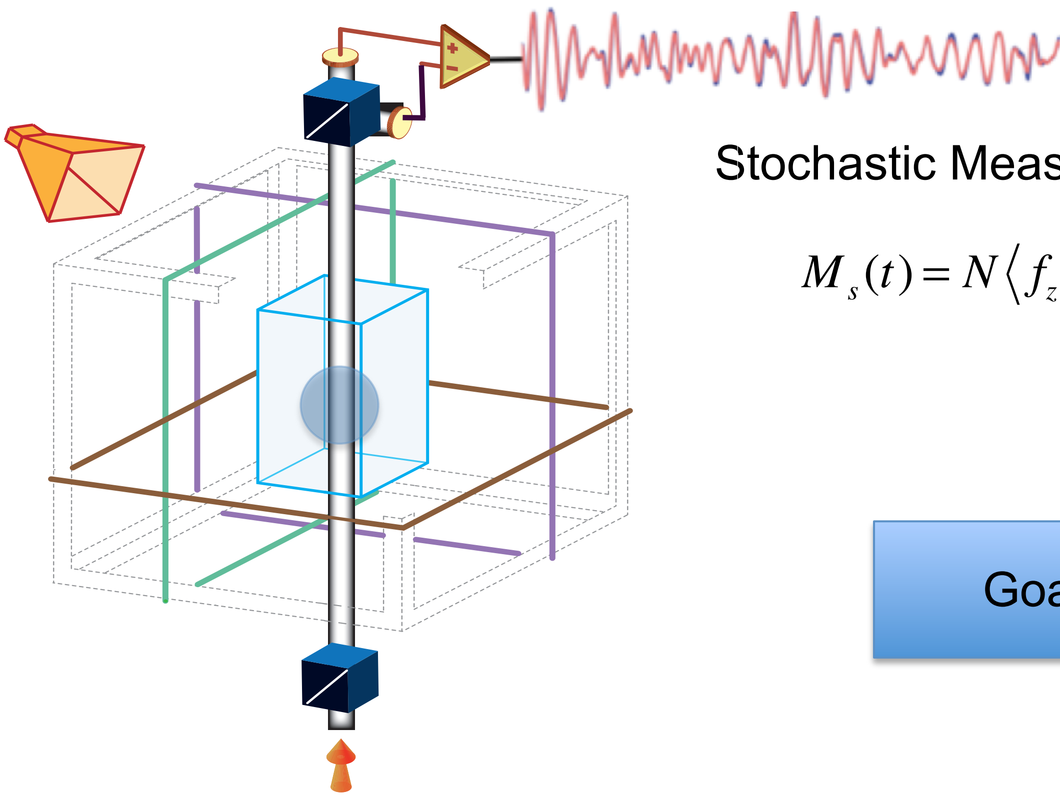
Shot noise resolution of detector: $\left(\Delta F_z^2\right)_{SN} = \left(\chi^2 N_L\right)^{-1}$

Regime of negligible backaction: $\left(\chi^2 N_L\right)^{-1} >> N\,Tr\left(\rho_0\, \Delta f_z^2\right) = \text{"Projection noise"}$

# Measurement-Record Likelihood Function

Negligible Backaction → Mean-field Approximation: $\rho_N(t) = \rho(t)^{\otimes N}$

$$P\left(M_s \middle| \rho_N(t)\right) = Tr\left( e^{-\frac{\chi^2 N_L}{2}\left(M_S - F_z\right)^2} \rho_N(t) \right) \approx e^{-\frac{\chi^2 N_L}{2}\left(M_S - \langle F_z \rangle_t\right)^2} = e^{-\frac{\chi^2 N_L}{2}\left(M_S - N\langle f_z \rangle_t\right)^2}$$



Stochastic Measurement Record:

$$M_s(t) = N\left\langle f_z \right\rangle_t + W_{SN}(t)$$

← Laser shot noise

**Goal: Given $M_s(t)$, estimate $\rho_0$**

# Reconstruction Algorithm

Goal: Given $M_s(t)$, estimate $\rho_0$

- Work in Heisenberg Picture: $M_s(t) = NTr\left(f_z(t)\rho_0\right) + W_{SN}(t)$
- Informational completeness and control: $f_z(t) = U^{\dagger}_{control}(t) f_z U_{control}(t)$

- Discretize measurement record: $M_i = Tr\left(O_i\rho_0\right) + W_i \qquad O_i = N U^{\dagger}_{control}(t_i) f_z U_{control}(t_i)$

- Generalized Bloch vector representation: $\rho_0 = \sum_{a=0}^{d^2-1} r_\alpha E_\alpha = r_0 I + \sum_{a=1}^{d^2-1} r_\alpha E_\alpha \qquad \rho_0 \Leftrightarrow \{r_\alpha\}$

$$M_i = \sum_\alpha r_\alpha O_{\alpha i} + W_i$$

$$O_{\alpha i} = Tr\left(E_\alpha O_i\right)$$

Stochastic linear estimation:
Given $\{M_i\}$ find $\{r_\alpha\} = \rho_0$

- *Key ingredient: All dynamics, i.e.* $\{O_i\}$, *must be exactly known so that all new information in the signal is about the initial state* $\rho_0$.

# Reconstruction Algorithm

Goal: Given $M_s(t)$, estimate $\rho_0$

- Likelihood function: $P\left(\{M_i\}|\rho_0\right) = \exp\left\{-\frac{1}{2\sigma^2}\sum_i \left|M_i - Tr\left(\rho_0 O_i\right)\right|^2\right\}$

**Maximum likelihood (least squares)**

$$\rho_0 = \arg\min_{\rho} \sum_i \left|M_i - Tr\left(\rho O_i\right)\right|^2 \quad s.t. \quad \rho = \rho^\dagger,\ Tr\left(\rho\right) = 1,\ \rho \geq 0$$

**Compressed sensing: Matrix completion for low rank (highly pure states)**

$$\rho_0 = \arg\min_{\rho} Tr\left(\rho\right) \quad s.t. \quad \rho = \rho^\dagger,\ \rho \geq 0,\ \sum_i \left|M_i - Tr\left(\rho O_i\right)\right|^2 \leq \varepsilon$$

Gross *et al.*, Phys. Rev. Lett. 105, 150401 (2010)

Solve by convex optimization (Matlab canned routine)

# Control parameters

- Leave the amplitudes of the control fields constant

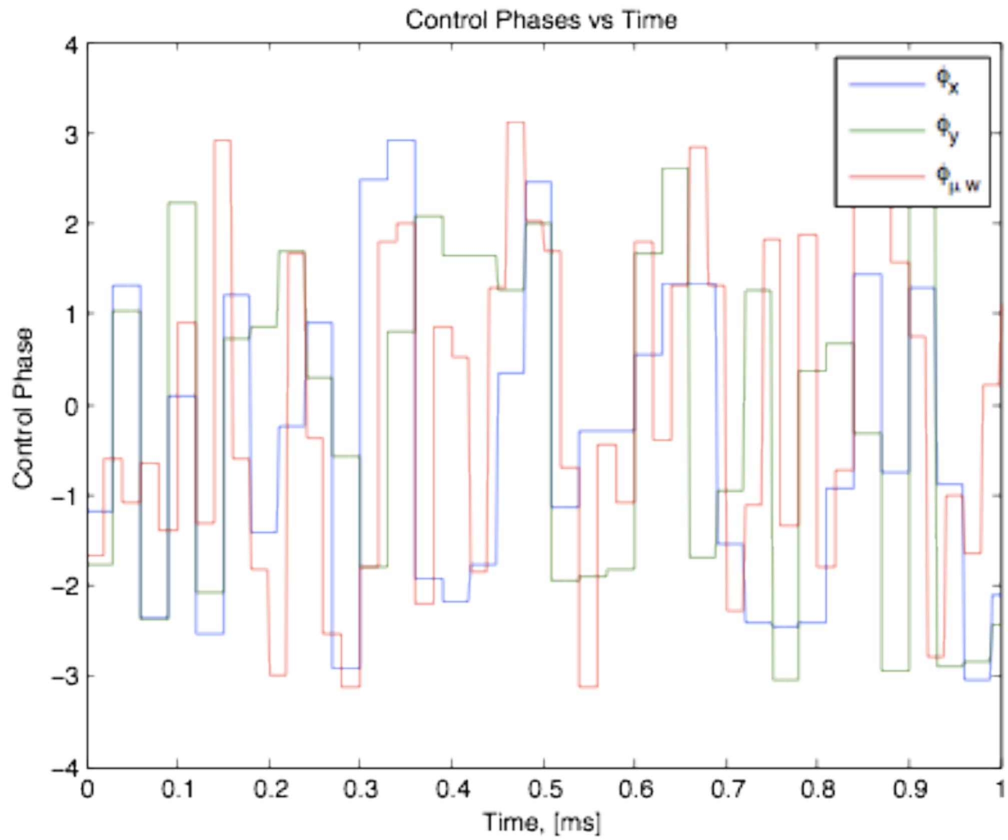
$$\Omega_x = \Omega_y = 2\pi \cdot 15\text{kHz}$$

$$\Omega_{\mu w} = 2\pi \cdot 33\text{kHz}$$

- Choose the phases of the control fields as piece-wise constant random functions

$$\phi_x(t),\ \phi_y(t),\ \phi_{\mu w}(t)$$

- Ensure informationally complete measurement record



# Dynamics of Observables

Unitary Dynamics (microwave, rf, light shifts):

$$O_n = U^\dagger(nt) O_0 U^\dagger(nt)$$

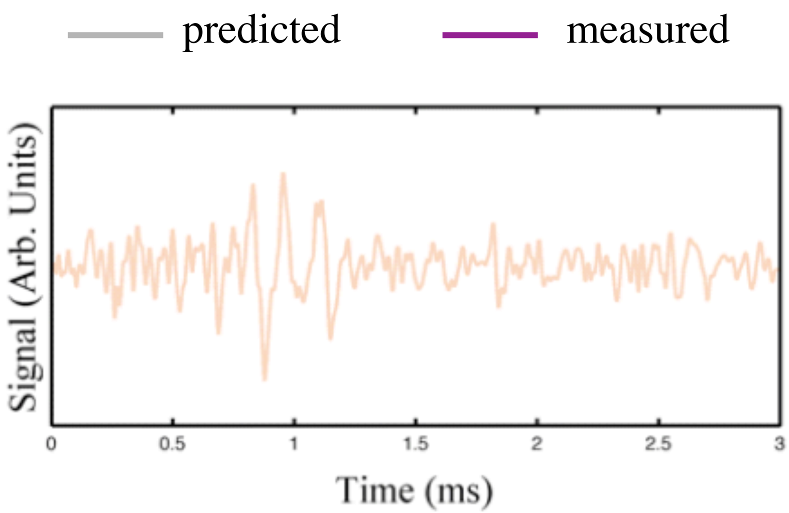
$$\frac{\partial U(t)}{\partial t} = -iH(t)U(t) \qquad H\left[\phi_x(t), \phi_y(t), \phi_{\mu w}(t), \Omega_0, \Omega_{rf}, \Omega_{\mu w}, I, \Delta\right]$$

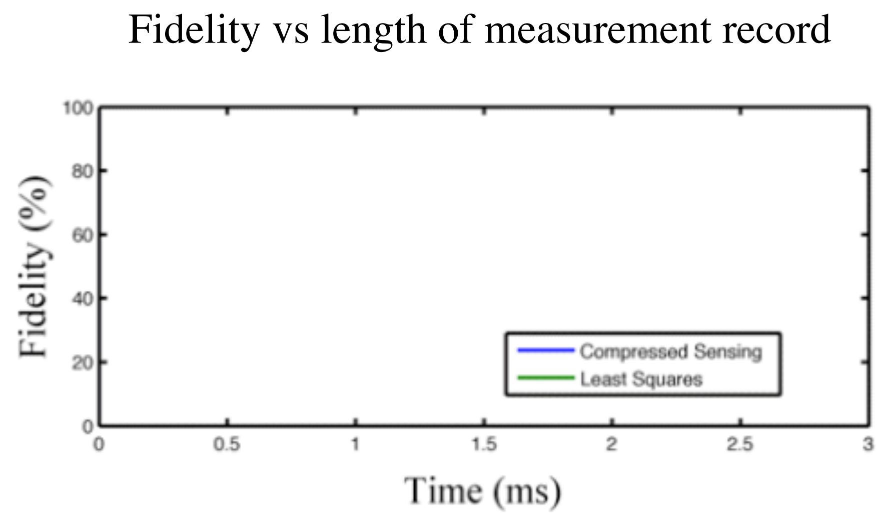
Including Decoherence (photon scattering):

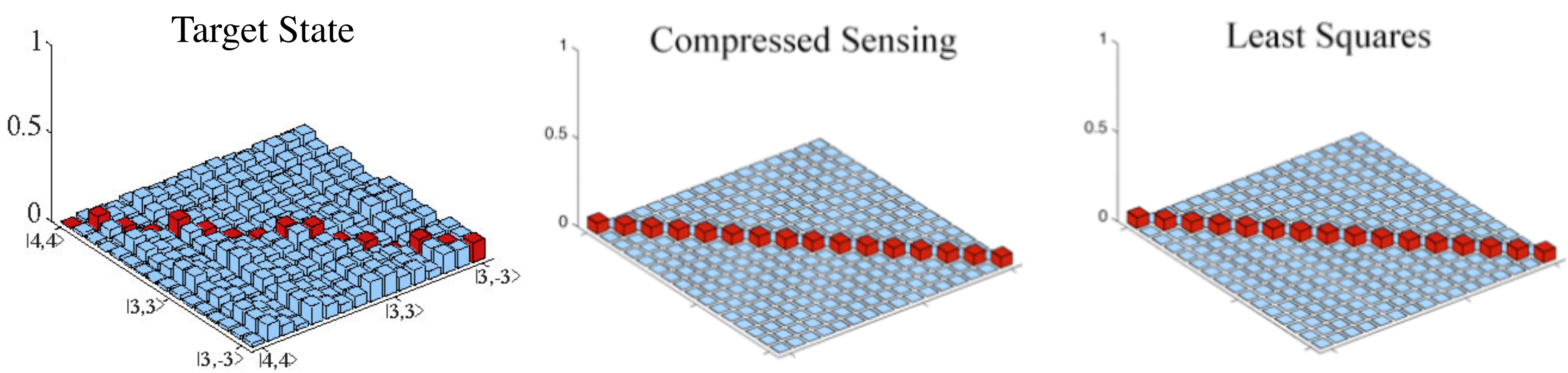
$$O_n = \Upsilon^\dagger_{nT}\left[O_0\right]$$

$$\frac{\partial \Upsilon^\dagger(t)}{\partial t} = \Upsilon^\dagger(t)\mathcal{L}^\dagger(t) \qquad \mathcal{L}(t) = \mathcal{L}\left[H(t), \gamma_s\right]$$

# Example of Tomographic Reconstruction



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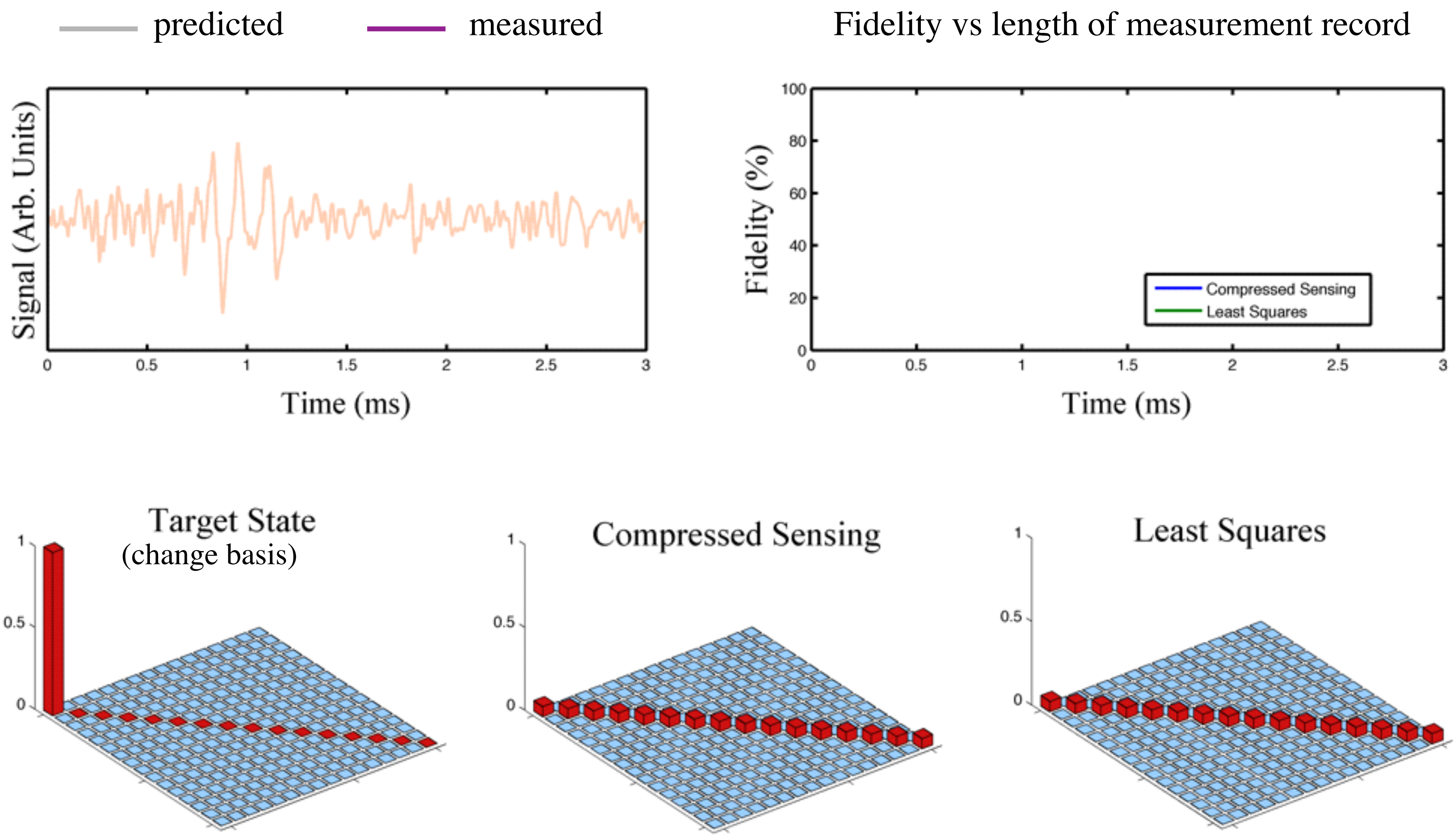
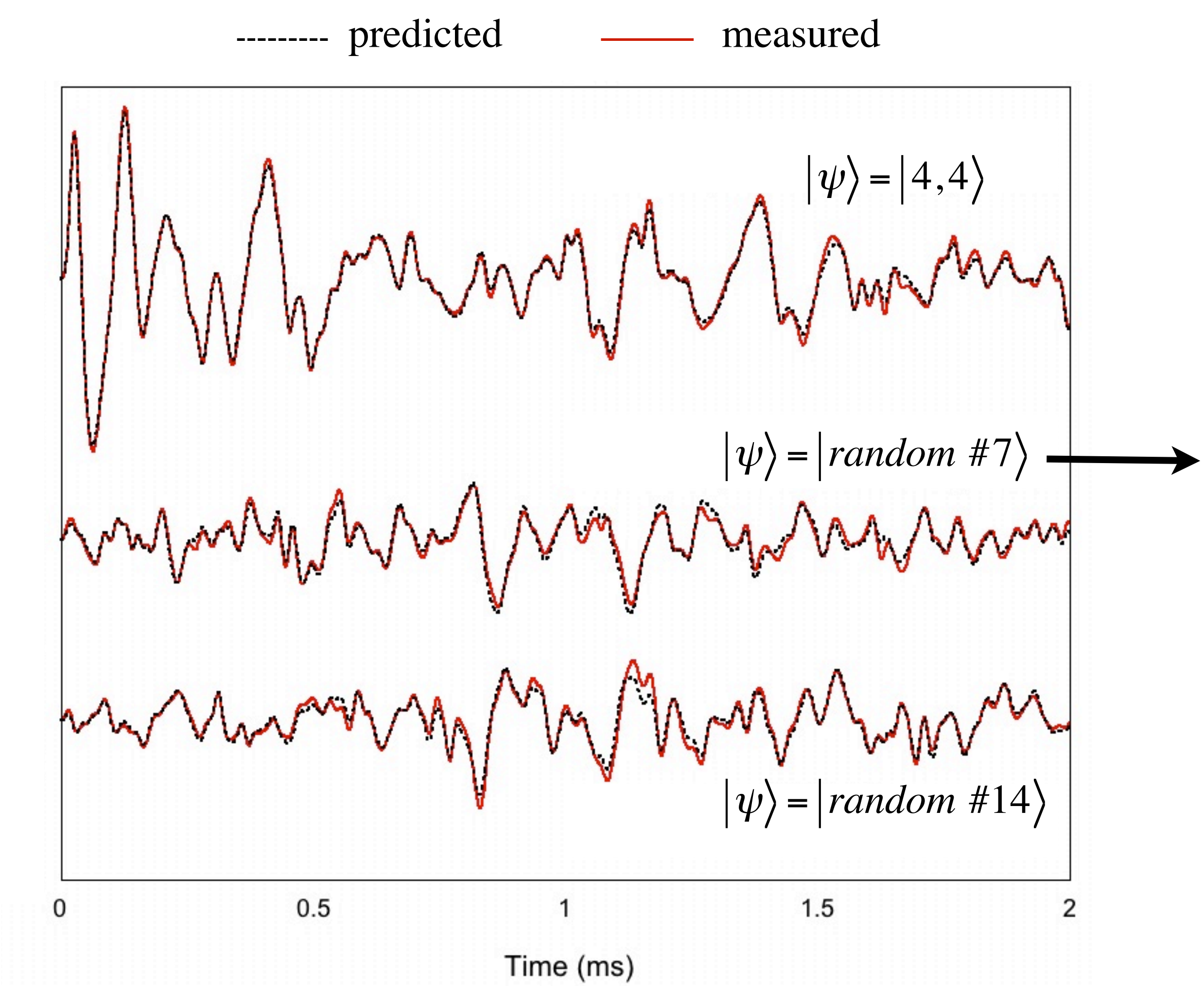


# Polarimetry signal provides unique “fingerprint” for different states

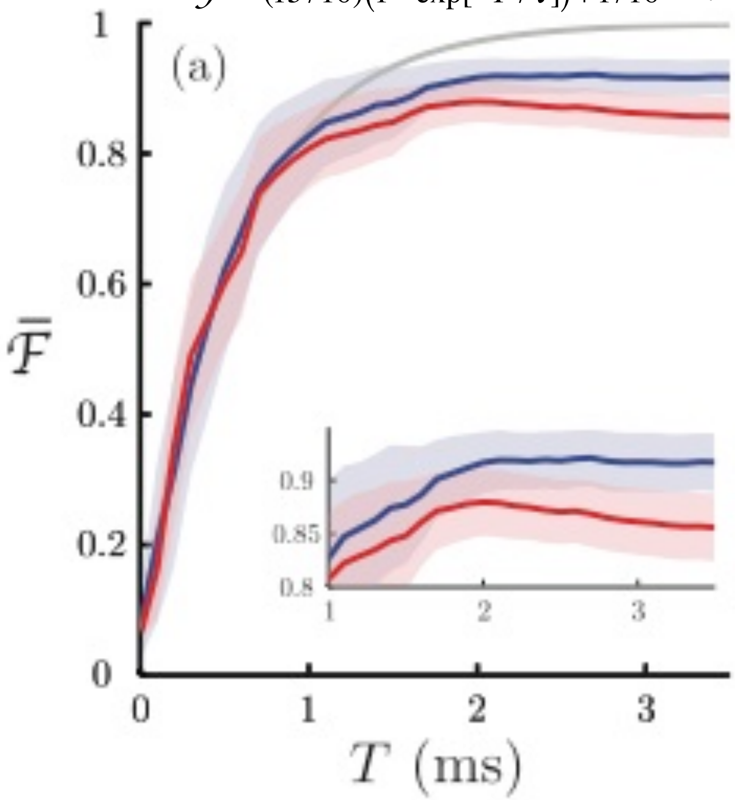



$$
\begin{pmatrix} -0.01-0.26i \\ -0.10+0.27i \\ -0.04+0.02i \\ 0.39-0.04i \\ -0.24+0.14i \\ 0.01+0.14i \\ -0.06-0.04i \\ 0.65+0.18i \\ 0.31+0.06i \\ 0.16+0.19i \\ 0.11+0.05i \\ -0.25-0.03i \\ 0.15+0.30i \\ -0.17-0.13i \\ -0.12+0.11i \\ 0.08-0.35i \end{pmatrix}
\quad
\begin{matrix} |4,4\rangle \\ |4,3\rangle \\ |4,2\rangle \\ |4,1\rangle \\ |4,0\rangle \\ |4,-1\rangle \\ |4,-2\rangle \\ |4,-3\rangle \\ |4,-4\rangle \\ |3,3\rangle \\ |3,2\rangle \\ |3,1\rangle \\ |3,0\rangle \\ |3,-1\rangle \\ |3,-2\rangle \\ |3,-3\rangle \end{matrix}
$$

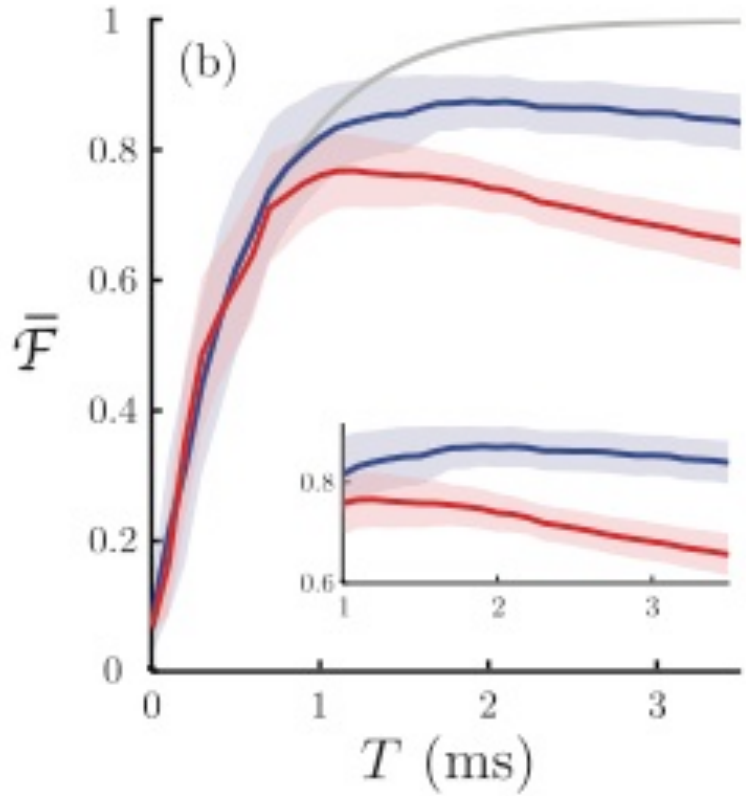
# Fidelity of Tomographic Reconstruction

**Average fidelity for 2 data sets, each of 48 Haar Random pure states**

$\bar{\mathcal{F}} = (15/16)(1-\exp[-T/\tau]) + 1/16 \qquad \tau = 0.57\,ms$




Including inhomogeneity

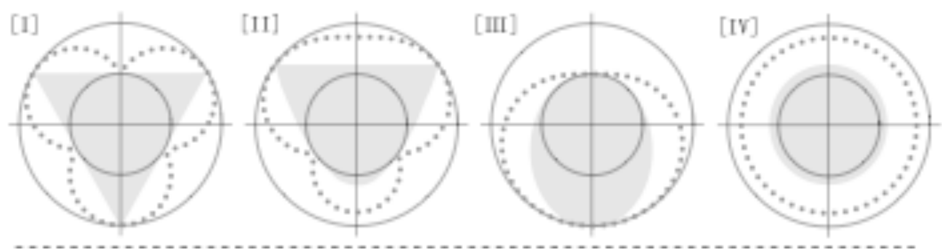



Ignoring inhomogeneity

- LS & CS Estimation comparable at short times (well before time needed for mixed state)
- Fidelity limited by uncertainty in dynamics,  CS estimation is more robust

# The Power of Positivity

- Bloch vector: $$\rho_0 = \sum_{a=0}^{d^2-1} r_\alpha E_\alpha = \frac{1}{d} I + \sum_{a=1}^{d^2-1} r_\alpha E_\alpha$$
- Space of positive states a convex set, not a vector space.




The Bloch-Vector Space for $N$-Level Systems: the Spherical-Coordinate Point of View

Gen Kimura Open Sys. & Information Dyn. (2005) 12: 207–229
DOI: 10.1007/s11080-005-0919-y

- Most highly constrained near space of *pure states:*

→*Accomplishes high fidelity reconstruction just as compressed sensing.*

# Conclusions

• Quantum tomography by weak continuous measurement on a large identical ensemble can be fast and robust.

• Compressed sensing more robust to uncertainty in measurement settings.

• CS vs. LS, conditions for improved performance?
  - Role Positivity
  - Role of the nature of the noise
  - Minimally complete vs. overcomplete observable set

• Outlook → Process Tomography

Shabani *et al.*, Phys. Rev. Lett. 106, 100401 (2011)

• Outlook → Including measurement backaction