Renormalization approach to open quantum system dynamics

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"traditional" approach to open system dynamics:
retain minimal info about the environment E and formulate an equation in terms of a superoperator acting on \( \rho_S(t) = \text{Tr}_E \rho(t) \)
⇒ simulate the open system dynamics via a Master Equation
Main limits of the "traditional" approach:

- In general applicable only under Markovian or close-to-Markovian noise approximation (perturbative approach).
- Weak S-E coupling, drastic assumptions on the bath (in general as soon as it has non-trivial structure fail to be fulfilled).
- What to do with a spin bath? (in general no straightforward characterization via a spectral density, no clear a priori characterization of Markovian/non-Markovian, no analytical correlation functions, strong S-E coupling).

Why do we care about non-Markovian dynamics?


Controllability is expected to be better for non-Markovian systems due to information backflow.
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Interesting non-Markovian quantum systems (e.g. strong SE interaction and/or spin bath)

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observations and conjectures

Observation:
in very different contexts numerical simulations of non-Markovian dynamics based on truncating the number of environmental modes often show that a comparatively small number of modes is sufficient to reach converged results (R. Baer, R. Kosloff JCP 106 8862 (1997); C. P. Koch, T. Klüner, H. J. Freund, R. Kosloff PRL 90 117601 (2003); H. J. Hogben, P. Hore, I. Kuprov, JCP 132 174101 (2010), K. H. Hughes, C. D. Christ, I. Burghardt, JCP 131 024109 (2009),...).

Conjecture:
it takes time to establish correlations between system and environment ⇒ the system interacts progressively ⇒ it is a quite general feature of quantum dynamics.

In this talk: prove the conjecture identify the necessary ingredients for an accurate and efficient simulation of open quantum systems

General philosophy: system-bath unitary dynamics + renormalization group VS system non-unitary dynamics + a priori assumptions on the bath
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**In this talk:**

Prove the conjecture.

Identify the necessary ingredients for an accurate and efficient simulation of open quantum systems.
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Giulia Gualdi

Renormalization approach to open quantum system dynamics

- Discrete environments/local interactions ⇒ quasi-locality of quantum dynamics
- Continuous environments/non-local interactions ⇒ quasi-finite resolution of quantum dynamics
- Correspondence between discrete and continuous environments ⇒ time-induced renormalization of system-environment interaction
discrete environments/local interactions $\Rightarrow$ quasi-locality of quantum dynamics
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continuous environments/non-local interactions $\Rightarrow$ quasi-finite resolution of quantum dynamics
Outline

- discrete environments/local interactions $\Rightarrow$ quasi-locality of quantum dynamics

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- correspondence between discrete and continuous environments
discrete environments/local interactions \Rightarrow \textit{quasi-locality of quantum dynamics}

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correspondence between discrete and continuous environments

\Rightarrow \textit{time-induced renormalization of system-environment interaction}
Quasi-Locality of quantum dynamics
discrete environments: microscopic model

\[ \hat{H} = \hat{H}_S + \sum_{i=1}^{N_{int}} \hat{\Phi}_{SB}^{i} + \sum_{i \leq j = 1}^{N_{B}} \hat{\Phi}_{B}^{ij} \]

with \( \hat{\Phi}_{ij}^{\mu} = \sum_{\dim(B(H_i)) - 1} \mu = 0 \sum_{\dim(B(H_j)) - 1} \nu = 0 \) \( \hat{O}_\mu^i \hat{O}_\nu^j \) and \( N_{int}^S \leq N_S ; N_{int}^B \leq N_B \to \infty \).

Our goal is to truncate the sums over the environmental DOF in a well-defined manner \( \Rightarrow \) we need to quantify the influence of the DOF upon each other \( \Rightarrow \) we need to introduce a metric.
discrete environments: microscopic model

**system-bath Hamiltonian**

\[ \hat{H} = \hat{H}_S + \sum_{i=1}^{N_S^{\text{int}}} \sum_{j=1}^{N_B^{\text{int}}} \hat{\Phi}_{ij}^{SB} + \sum_{i \leq j=1}^{N_B} \hat{\Phi}_{ij}^{B} \]

with \( \hat{\Phi}_{ij} = \sum_{\mu=0}^{\dim(B(H_i))} \sum_{\nu=0}^{\dim(B(H_j))} J_{ij}^{\mu\nu} \hat{O}_i^{\mu} \hat{O}_j^{\nu} \) and \( N_S^{\text{int}} \leq N_S; N_B^{\text{int}} \leq N_B \to \infty. \)

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⇒ we need to quantify the influence of the DOF upon each other
⇒ we need to introduce a metric.
Hamiltonian-graph correspondence

Hilbert space:

\[ H_{\text{tot}} \iff \text{graph } G(N, E) \]

\[ N = \{ \text{nodes} : N_S + N_B \} \]

\[ E = \{ \text{edges} : J_{\mu\nu}^{ij} \neq 0 \text{ for any } \mu, \nu \} \]

Adjacency matrix:

\[ A \text{ with } A_{ij} = \begin{cases} 1 & \text{if } J_{\mu\nu}^{ij} \neq 0 \text{ for any } \mu, \nu \\ 0 & \text{if } J_{\mu\nu}^{ij} = 0 \end{cases} \]

Example:

physical lattice = one dimensional chain

\[ H = \sum_i J \hat{O}_i \hat{O}_{i+1} \]
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Hilbert space: $H_{tot} \iff \text{graph}$

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![One dimensional chain diagram]
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weight matrix on $G$: $J_{ij} = (\sum_{\mu\nu} [J_{ij}^{\mu\nu}]^2)^{1/2}$ with $i,j = 1, \cdots, N_S + N_B$

(remove indices of internal degrees of freedom)
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**metric $d$ on $G$: shortest path between two nodes**

$$d(i, j) := \min\{n \in \mathbb{N}_0 : [A^n]_{i,j} \neq 0\}$$
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$\Rightarrow$ reorder bath DOF according to their distance from the system S

$$\hat{H} = \sum_{d=0}^{\infty} \left( \hat{h}_d + \hat{h}_{d,d+1} \right) \quad (\hat{h}_0 = \hat{H}_S, \hat{h}_{01} = \hat{H}_{SB})$$

$\hat{h}_d$ interactions within same layer, $\hat{h}_{d,d+1}$ between two successive layers
Generic system operator $\hat{A}_S$ evolves as

$$\hat{A}_S(t) = e^{i\hat{H}t} \hat{A}_S e^{-i\hat{H}t} = \hat{A}_S + \sum_{d=1}^{\infty} \frac{(-it)^d}{d!} \hat{C}_d$$

with $\hat{C}_d = [\hat{H}, \hat{C}_{d-1}]$ and $\hat{C}_0 = \hat{A}_S$. 
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$\hat{C}_d$ has non-vanishing commutators only with terms in $\hat{H}$ up to $h_{d,d+1}$.
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$\Rightarrow$ at $n_{th}$ perturbative order

$$\hat{C}_n = [\hat{H}, \hat{C}_{n-1}] \equiv [\hat{H}_n, \hat{C}_{n-1}] \quad \text{with} \quad \hat{H}_n = \sum_{d=0}^{n-1} \left( \hat{h}_d + \hat{h}_{d,d+1} \right)$$

the truncation of the full generator $\hat{H}$ to the first $n$ layers of the graph.
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**truncation of $\hat{H}$ ≡ truncation of perturbative expansion**
system dynamics

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the truncation of the full generator $\hat{H}$ to the first $n$ layers of the graph.

truncation of $\hat{H} \equiv$ truncation of perturbative expansion

DOF at distance $n$ from S contribute only from the $n_{th}$ perturbative order

⇒ system dynamics appreciably affected only when corresponding perturbative term is non-negligible.
Error made by replacing $\hat{H}$ by the truncated generator $= \text{remainder of the series}$

$$\|\hat{A}_S(t) - \hat{A}_S^n(t)\| \leq \|\hat{A}_S\| \sum_{d=n+1}^{\infty} \frac{(2t\mathcal{O})^d}{d!} \sum_{i,j \in \mathcal{I}_d} [J^d]_{ij}$$

where $\mathcal{O} = \max_{(i,j) \in N;\mu,\nu} \|\hat{O}_i^\mu \hat{O}_j^\nu\|$ and $\mathcal{I}_d = \{i \in N : d(s, i) \leq d\}$ the set of DOF at distance at most $d$ from $S$. 

with $v = 2\mathcal{O}\bar{c}^2 \|J\| e^B$ Bath DOF outside of the effective light cone give only an exponentially vanishing contribution to $\hat{A}_S(t)$. The full bath is needed only in the limit of infinite time.
Error made by replacing $\hat{H}$ by the truncated generator $= \text{remainder of the series}$

$$
\left\| \hat{A}_S(t) - \hat{A}_S^n(t) \right\| \leq \left\| \hat{A}_S \right\| \sum_{d=n+1}^{\infty} \frac{(2t\mathcal{O})^d}{d!} \sum_{i,j \in I_d} [J^d]_{ij}
$$

where $\mathcal{O} = \max_{(i,j) \in \mathbb{N}; \mu, \nu} \left\| \hat{O}_i^\mu \hat{O}_j^{\nu} \right\|$ and $I_d = \{i \in \mathbb{N} : d(s, i) \leq d\}$ the set of DOF at distance at most $d$ from $S$.

if each DOF interacts with a finite number of other DOF (**local finiteness**) $\Rightarrow \sum_{i,j \in I_d} [J^d]_{ij} \leq (\bar{c}^2 \|J\|)^d$ ($\bar{c} = \text{maximum vertex degree of G}$) $\Rightarrow$ the sum can be bounded
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$\Rightarrow$ Lieb-Robinson bound

$$
\left\| \hat{A}_S(t) - \hat{A}_S^n(t) \right\| \leq \left\| \hat{A}_S \right\| e^{-(n-\nu t)}
$$

with $\nu = 2\mathcal{O}\bar{c}^2 \| J \| e$
Error made by replacing $\hat{H}$ by the truncated generator $= \text{remainder of the series}$

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$$\left\| \hat{A}_S(t) - \hat{A}_S^n(t) \right\| \leq \left\| \hat{A}_S \right\| e^{-(n-\nu t)} \quad (1)$$

with $\nu = 2\mathcal{O} \bar{c}^2 \| J \| e$

Bath DOF outside of the effective light cone give only an exponentially vanishing contribution to $\hat{A}_S(t)$. The full bath is needed only in the limit of infinite time.
Quasi-finite resolution of quantum dynamics
Giulia Gualdi

Renormalization approach to open quantum system dynamics

Hamiltonian of a central system interacting with a continuous environment

\[ \hat{H} = \hat{H}_S + \int_{0}^{x_{\text{max}}} J(x) \left( \hat{c}_x + \hat{c}_x^\dagger \right) dx + 2 \int_{0}^{x_{\text{max}}} \int_{0}^{x_{\text{max}}} K\left( |x - x'| \right) \left[ \hat{c}_x \hat{c}_{x'} + \hat{c}_{x'}^\dagger \hat{c}_x^\dagger \hat{c}_{x'} + \hat{c}_{x'}^\dagger \hat{c}_x \hat{c}_{x'}^\dagger \right] dx dx' + \int_{0}^{x_{\text{max}}} g(x) \hat{c}_x \hat{c}_x^\dagger \ dx, \]

\( x = \text{relevant bath variable}, \ x_{\text{max}} < \infty \) finite cut-off, and \( \hat{O}_{IS} = \text{generic system operator}, \) \( \|c\| = \max_{x \in [0, x_{\text{max}}]} \|\hat{c}_x\| < \infty \) problem: \( S \) interacts with all bath DOF which all may interact among themselves (non-local interactions) \( \implies \) graph with all nodes at distance 1 from \( S \) \( \implies \) in general no LR bound

\(^1\) general applicable bound for continuous environments using the idea of the 'surrogate Hamiltonian' (Baer, Kosloff, JCP 106, 8862 (1997))
Hamiltonian of a central system interacting with a continuous environment

\[ \hat{H} = \hat{H}_S + \hat{O}_S^I \int_0^{x_{\text{max}}} J(x) (\hat{c}_x + \hat{c}_x^\dagger) \, dx \]

\[ + 2 \int_0^{x_{\text{max}}} \int_x^{x_{\text{max}}} K(|x - x'|) \left[ c_x c_{x'}^\dagger + c_{x'}^\dagger c_x \right] \, dx \, dx' + \int_0^{x_{\text{max}}} g(x) \hat{c}_x \hat{c}_x^\dagger \, dx, \]

where:
- \( x = \) relevant bath variable,
- \( x_{\text{max}} < \infty \) finite cut-off,
- \( \| c \| = \max_{x \in [0, x_{\text{max}}]} \| \hat{c}_x \| < \infty \)

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the surrogate generator

Giulia Gualdi

Renormalization approach to open quantum system dynamics
the surrogate generator

Sequence of $n$ sampling points: $\{x_i\}_{i=0}^{n-1}$, in $[0, x_{max}]$, with $x_i < x_{i+1}$
Sequence of \( n \) sampling points: \( \{x_i\}_{i=0}^{n-1} \), in \([0, x_{\text{max}}]\), with \( x_i < x_{i+1} \)

\[ \Rightarrow \text{partition } P_n = \{\delta x_i\} \text{ with } \delta x_i = x_{i+1} - x_i, \quad |P_n| = \max_{i<n}(\delta x_i) \]
Sequence of $n$ sampling points: $\{x_i\}_{i=0}^{n-1}$, in $[0, x_{max}]$, with $x_i < x_{i+1}$

$\Rightarrow$ partition $P_n = \{\delta x_i\}$ with $\delta x_i = x_{i+1} - x_i$, $|P_n| = \max_{i\leq n}(\delta x_i)$

$\Rightarrow$ sequence of partitions $\{P_n\}$ with $|P_{n+1}| < |P_n|$
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$\Rightarrow$ sequence of Hamiltonians $\{\hat{H}_{P_n}\}$ with

$$
\hat{H}_{P_n} = \hat{H}_S + \hat{O}_S^I \sum_{i=0}^{n-1} \tilde{J}_i (\hat{c}_i + \hat{c}_i^\dagger) + \\
2 \sum_{i<j=0}^{n-1} \tilde{K}_{ij} \left[ \hat{c}_i^\dagger \hat{c}_j + \hat{c}_i^\dagger \hat{c}_j + \hat{c}_i^\dagger \hat{c}_i \hat{c}_j \hat{c}_j^\dagger \right] + \\
\sum_{i=0}^{n-1} \tilde{g}_i \hat{c}_i^\dagger \hat{c}_i,
$$

$\hat{c}_i = \hat{c}_{x_i}$, $\tilde{J}_i = J(x_i)\delta x_i$, $\tilde{K}_{ij} = K(|x_i - x_j|)\delta x_i \delta x_j$, and $\tilde{g}_i = g(x_i)\delta x_i$ rescaled couplings at the $n$ sampling points
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+ \sum_{i=0}^{n-1} \tilde{g}_i \hat{c}_i^\dagger \hat{c}_i,
\]

$\hat{c}_i = \hat{c}_{x_i}, \ \tilde{J}_i = J(x_i)\delta x_i, \ \tilde{K}_{ij} = K(|x_i - x_j|)\delta x_i \delta x_j,$ and $\tilde{g}_i = g(x_i)\delta x_i$ rescaled couplings at the $n$ sampling points

truncation + rescaling $\Rightarrow \hat{H}_{P_n} = $ Riemann sums built on $P_n$ approximating $\hat{H}$ $\Rightarrow \lim_{n \to \infty} \hat{H}_{P_n} = \hat{H}$
finite-resolution of quantum dynamics

\[ |\hat{A}_S(t) - \hat{A}_{H P_n}(t)| \leq R_1(P_n) + R_2(P_n) \]

\[ R_1(P_n) \] is the error made in assuming \( \hat{H} \) and \( \hat{H} P_n \) to commute.

\[ R_2(P_n) \] is the distance between \( \hat{A}_S \) and its evolution under \( \hat{H} - \hat{H} P_n \).

For \( t < \infty \) the system cannot resolve the full continuum of environmental modes ⇒ within arbitrary accuracy a surrogate description can be used and infinitely close bath modes can be dropped.
finite-resolution of quantum dynamics

error made by time evolving $\hat{A}_S$ using $\hat{H}_{P_n}$ instead of $\hat{H}$

$$\mathcal{R}(P_n) = \|\hat{A}_S(t) - \hat{A}_S^{H_{P_n}}(t)\| \leq R_1(P_n) + R_2(P_n)$$
finite-resolution of quantum dynamics

error made by time evolving $\hat{A}_S$ using $\hat{H}_{P_n}$ instead of $\hat{H}$

$$R(P_n) = \|\hat{A}_S(t) - \hat{A}_S^{HP_n}(t)\| \leq R_1(P_n) + R_2(P_n)$$

- $R_1(P_n)$ = error made in assuming $\hat{H}$ and $\hat{H}_{P_n}$ to commute
  ⇒ at finite $t$ vanishes for $n \to \infty$ due to the convergence of Riemann sums as $t^2 \| [\hat{H}, \hat{H}_{P_n}] \|$
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finite-resolution of quantum dynamics

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quasi-finite resolution of quantum dynamics

\[
\| \hat{A}_S(t) - \hat{A}_S^{H_{P_n}}(t) \| \leq R_1(P_n) + \| \hat{A}_S \| \left( e^{2\| \hat{H}-\hat{H}_{P_n} \| t} - 1 \right),
\]

\[\text{Giulia Gualdi} \quad \text{Renormalization approach to open quantum system dynamics}\]
finite-resolution of quantum dynamics

error made by time evolving $\hat{A}_S$ using $\hat{H}_P$ instead of $\hat{H}$

$$\mathcal{R}(P_n) = \|\hat{A}_S(t) - \hat{A}^{H_P}(t)\| \leq R_1(P_n) + R_2(P_n)$$

- $R_1(P_n)$ = error made in assuming $\hat{H}$ and $\hat{H}_P$ to commute
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quasi-finite resolution of quantum dynamics

$$\|\hat{A}_S(t) - \hat{A}^{H_P}(t)\| \leq R_1(P_n) + \|\hat{A}_S\| \left( e^{2\|\hat{H} - \hat{H}_P\|t} - 1 \right),$$

for $t < \infty$ the system cannot resolve the full continuum of environmental modes

\(\Rightarrow\) within arbitrary accuracy a surrogate description can be used and infinitely close bath modes can be dropped.
few observations
few observations

- upper bounds to the error made by replacing the full generator, $\hat{H}$, by an effective one, $\hat{H}_n$ or $\hat{H}_{P_n}$
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  In some specific cases, tighter model-dependent bounds can be derived
  (Burrell, Osborne PRL 99, 167201 (2007)) and for certain classes of initial states, the scaling with
time can be dramatically reduced (Hastings PRB 77, 144302 (2008); Eisert, Cramer, Plenio RMP 82 (2010))
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- Extension of the bounds to $k$-linear interactions straightforward
few observations

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- Extension of the bounds to $k$-linear interactions straightforward

Correspondence between discrete and continuous environments
**discrete bath**

infinite layers $\Rightarrow$ truncation
<table>
<thead>
<tr>
<th>discrete bath</th>
<th>continuous bath</th>
</tr>
</thead>
<tbody>
<tr>
<td>infinite layers $\Rightarrow$ truncation</td>
<td>single layer $\Rightarrow$ truncation + rescaling</td>
</tr>
</tbody>
</table>

System-bath coupling $J_{SB}$ of the paths needed by the system to explore all of the environment.

Continuous environments:

$$J_{SB} = \int_{0}^{x_{\text{max}}} J(x) \, dx \rightarrow \text{finite}$$

Discrete environments:

$$J_{SB} = \sum_{n=0}^{\infty} \sum_{j: \text{d}(s,j) = n} [J_n]_{sj} \rightarrow \text{local finiteness}$$

Local finiteness $\Leftrightarrow$ finite cut-off

Infinitely long paths $\Leftrightarrow$ infinitely close modes

The dynamics in Hilbert space remains unaffected since any rescaling of the coupling matrix is cancelled out by a corresponding rescaling of time.
discrete bath
infinite layers $\Rightarrow$ truncation

continuous bath
single layer $\Rightarrow$ truncation + rescaling

**system-bath coupling** $= \text{weight } J_{SB}$ of the paths needed by the system to explore all of the environment.
**discrete bath**

infinite layers $\Rightarrow$ truncation

**continuous bath**

single layer $\Rightarrow$ truncation + rescaling

**system-bath coupling** = weight $\mathcal{J}_{SB}$ of the paths needed by the system to explore all of the environment.

- **continuous environments:** $\mathcal{J}_{SB} = \int_0^{x_{\text{max}}} J(x)\,dx \rightarrow$ finite because the support of the integral is finite

- **discrete environments:**
  
  $\mathcal{J}_{SB} = \sum_{n=0}^{\infty} \sum_{j:\text{d}(s_j,j) = n} J_n s_j \rightarrow$ local finiteness $\Rightarrow$ $\mathcal{J}_{SB}$ can be made finite by rescaling the coupling matrix i.e. by penalizing longer paths (the dynamics in Hilbert space remains unaffected since any rescaling of the coupling matrix is cancelled out by a corresponding rescaling of time)
discrete bath
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**local finiteness** $\Leftrightarrow$ **finite cut-off**
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infinite layers ⇒ truncation

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**local finiteness ↔ finite cut-off**

infinitely long paths ↔ infinitely close modes
Dynamical renormalization
time naturally induces a dynamical renormalization over the system-bath interaction

⇒ the bounds provide a recursive update rule for the effective generators
time naturally induces a dynamical renormalization over the system-bath interaction

⇒ the bounds provide a recursive update rule for the effective generators

discrete environment:
time naturally induces a dynamical renormalization over the system-bath interaction

⇒ the bounds provide a recursive update rule for the effective generators

discrete environment:

bound + accuracy = number of bath modes as function of time 
\( n = n(t) \)

running coupling =
\[
\tilde{J}(n(t)) = \sum_{d=0}^{n(t)} \sum_{j:d(s,j)=d} [\tilde{J}^d]_{sj}
\]
(effective system-bath coupling)

renormalization flow =
\[
\lim_{t\to\infty} \tilde{J}(n(t)) = \tilde{J}_{SB}
\]
time naturally induces a dynamical renormalization over the system-bath interaction

⇒ the bounds provide a recursive update rule for the effective generators

continuous environment:
time naturally induces a dynamical renormalization over the system-bath interaction

⇒ the bounds provide a recursive update rule for the effective generators

continuous environment:

bound + accuracy = number of bath modes/partition mesh as function of time $|P_n| = |P_{n(t)}|$

running coupling =

$\mathcal{J}(P_{n(t)}) = \sum_{i \in P_{n(t)}} J(x_i) \delta x_i$

(effective system-bath coupling = Riemann sums)

renormalization flow =

$\lim_{t \to \infty} \mathcal{J}(P_{n(t)}) = \mathcal{J}_{SB}$

(convergence of Riemann sums)
computational cost & co.

Renormalization approach to open quantum system dynamics

⇒ Suzuki-Trotter
⇒ efficient simulation on quantum computer (polynomial in $t$ and number of effective DOF)

classical computer: still in principle exponential resources in number of effective DOF (state needs to be stored)
⇒ in general need further controlled restrictions of the size of the effective Hilbert (e.g. $t$-DMRG, restriction of excitation subspaces, etc)

controllable approx ⇒ exponential scaling (in # of effective DOF)
uncontrollable approx ⇒ constant scaling (no bath DOF)
computational cost & co.

- **from infinite to finite Hilbert space:**
  - ⇒ Suzuki-Trotter
  - ⇒ efficient simulation on quantum computer (polynomial in $t$ and number of effective DOF)
Computational Cost & Co.

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- **uncontrollable approx** $\Rightarrow$ constant scaling (no bath DOF)
Giulia Gualdi
Renormalization approach to open quantum system dynamics

The reduced dynamics of an arbitrary open quantum system can be obtained reliably and accurately, employing a finite-dimensional effective Hamiltonian. The required renormalizability condition, locality of the interactions for discrete environments and finite support of the interactions for continuous environments, is generally fulfilled (quasi-locality + quasi-finite resolution) of quantum dynamics = worst case computational cost of truncation-based algorithms for non-Markovian dynamics + a priori certification of accuracy vs computational complexity.

(a many-body remark) A generalized notion of approximate locality holds also for non-local interactions: renormalizability seems a more general concept than locality.

The question: general strategies to prolong convergence times? (e.g. more general effective Hilbert space truncation schemes, characterization of the portion of Hilbert space explored by the open system dynamics, embedding in a secondary Markovian bath, etc...)

What I did not talk about (but I’d be happy to discuss):
efficient characterization of quasi-unitary quantum operations
QOC functionals for open system dynamics
quantum process tomography/quantum device characterization
summary & outlook

- **time-induced dynamical renormalization of S-E interaction**

  ⇒ the reduced dynamics of an arbitrary open quantum system can be obtained reliably and accurately, employing a finite-dimensional effective Hamiltonian.

- **Required renormalizability condition**
  - Locality of the interactions for discrete environments
  - Finite support of the interactions for continuous environments

- **Renormalizability seems a more general concept than locality**
  - Many-body remark: A generalized notion of approximate locality holds also for non-local interactions.

- **Questions for prolonging convergence times**
  - More general effective Hilbert space truncation schemes
  - Characterization of the portion of Hilbert space explored by the open system dynamics
  - Embedding in a secondary Markovian bath, etc...

- **What I did not talk about (but I'd be happy to discuss):**
  - Efficient characterization of quasi-unitary quantum operations
  - QOC functionals for open system dynamics
  - Quantum process tomography/quantum device characterization
time-induced dynamical renormalization of S-E interaction

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(quasi-locality + quasi-finite resolution) of quantum dynamics = worst case computational cost of truncation-based algorithms for non-Markovian dynamics + a priori certification of accuracy vs computational complexity.
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(question: general strategies to prolong convergence times?
( e.g. more general effective Hilbert space truncation schemes, characterization of the portion of Hilbert space explored by the open system dynamics, embedding in a secondary Markovian bath, etc...)
what I did not talk about (but I’d be happy to discuss):

efficient characterization of quasi-unitary quantum operations

- QOC functionals for open system dynamics
- quantum process tomography/quantum device characterization
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