Commutativity of Adiabatic Elimination and Instantaneous Feedback for a General Class of Quantum Feedback Networks with Markovian Nodes

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(joint work with J. Gough, Aberystwyth University)

Associated paper

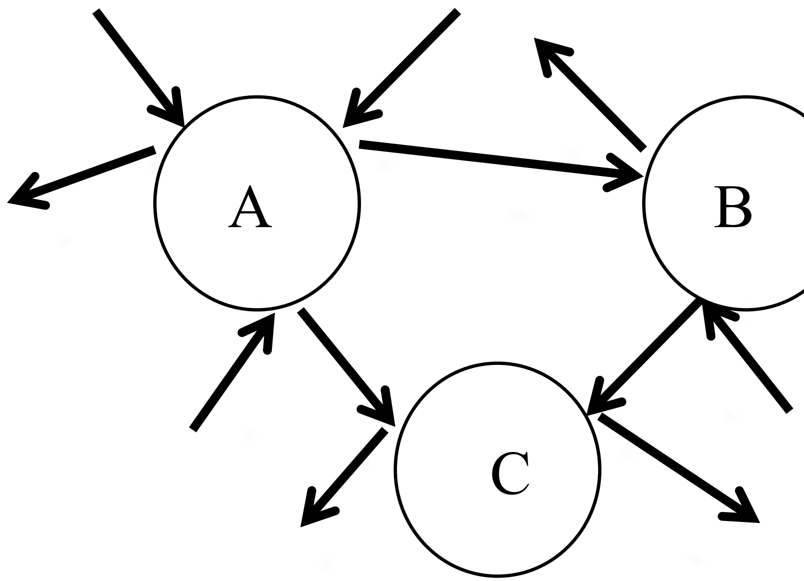
Talk based on the paper:

H. I. Nurdin and J. E. Gough, “On structure preserving transformations of the Itō generator matrix for model reduction of quantum feedback networks,” Royal Society Philosophical Transactions A 370, pp. 5422-5436, Nov. 2012.

Outline

- Overview of instantaneous feedback and adiabatic elimination.
- Instantaneous feedback and adiabatic elimination as instances of Schur complementation.
- Main results.
- Concluding remarks.

Dynamical quantum networks with Markovian components



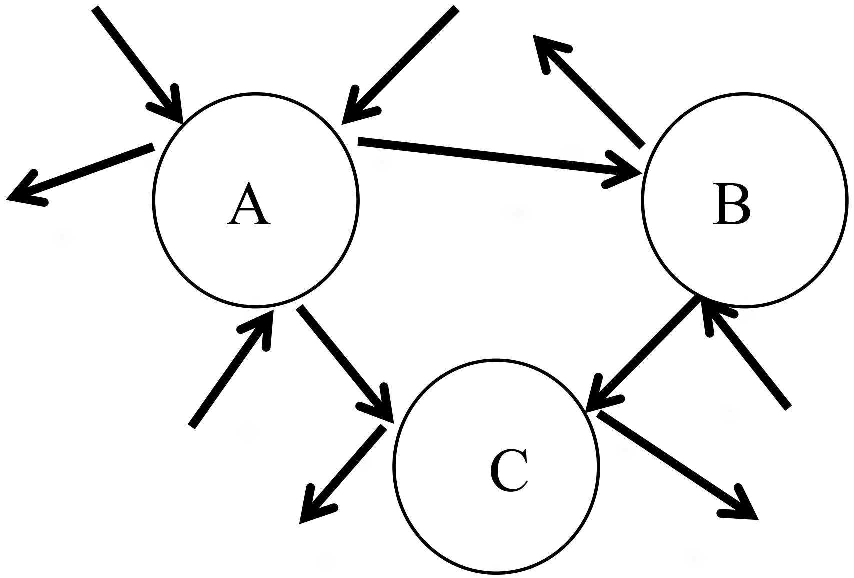
A dynamical quantum network with open Markov quantum system nodes A, B, and C. Arrows indicate optical bosonic fields: incoming optical fields (pointing in) and outgoing optical fields (pointing out). Equal number of incoming and outgoing fields at each node.

Markovian components

- Each isolated node assumed to be an open Markov quantum system.
- Tracing out optical fields coupled to an isolated node, the density operator of the system evolves according to the Markovian quantum master equation:

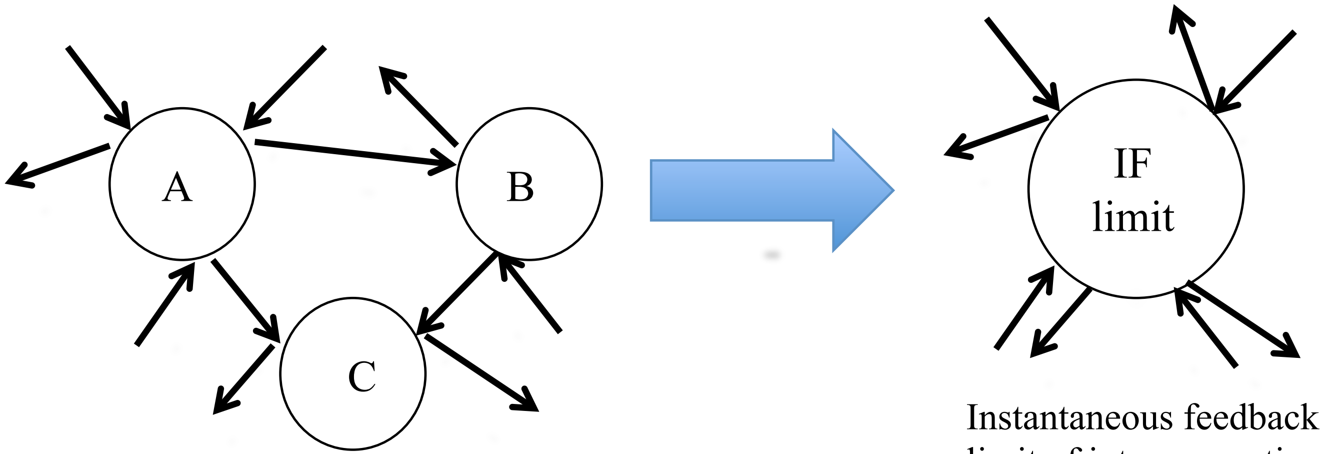
$$\dot{\rho}(t) = -i[H,\rho(t)] + \sum_{k=1}^{m}\left(L_j\rho(t)L_j^* - \frac{1}{2}\left\{L_j^*L_j,\rho(t)\right\}\right)$$

Finite time delays



There is a finite non-zero propagation delay for the optical field to travel from one node to another. Due to this delay the network itself is no longer Markov.

Instantaneous feedback (IF) limit



Original quantum network

Non-Markov quantum network

Instantaneous feedback limit of interconnecting optical fields

Single node Markov

- Is essentially a model reduction operation from a non-Markov network model to a (more tractable) Markov model.

Adiabatic elimination (AE)

- Often systems have dynamics on two well-separated time scales: slow and fast dynamics.
- In the limit of infinite separation of time scales, fast dynamics are removed retaining only the slow → known in physics as *adiabatic elimination*.
- [Bouten, van Handel and Silberfarb, *J. Func. Analysis*, 254, 2008] gives a rigorous treatment of adiabatic elimination of open Markov quantum systems in a quite general setting.

Adiabatic elimination (AE)

- Adiabatic elimination has proven to be extremely useful:
  - Model reduction by reducing the system Hilbert space to the smaller Hilbert space of the slow dynamics.
  - Approximate engineering of "exotic" types of two or more body couplings, e.g.,
    - Field-quadrature measurements (Wiseman and Milburn).
    - Continuous parity measurement operators (Kerckhoff, Bouten, Silberfarb and Mabuchi).
    - Arbitrary linear couplings between oscillator modes and traveling optical fields (Nurdin, James and Doherty).

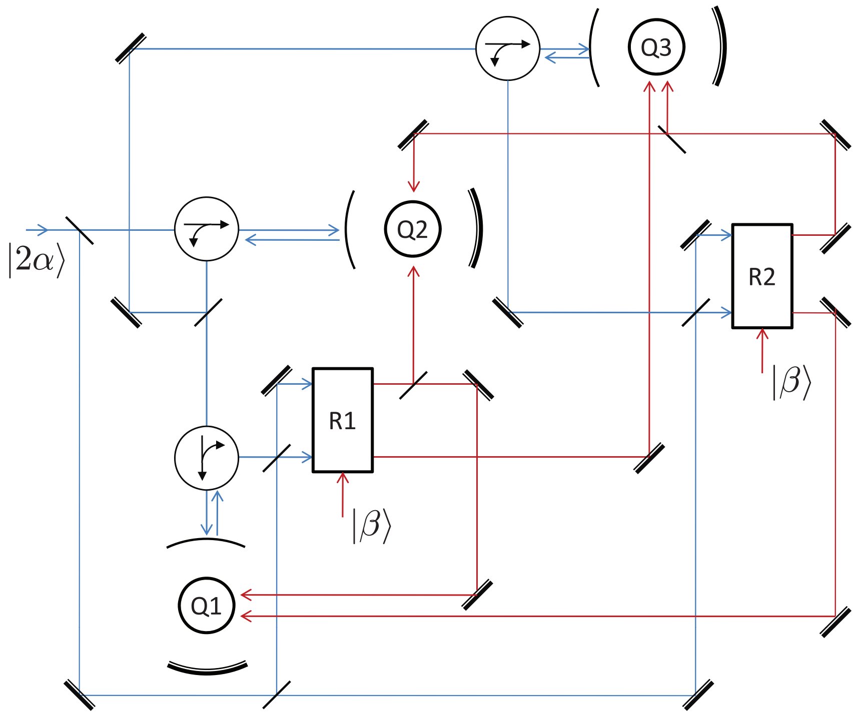
Unification

- Can these two seemingly distinct model reduction operations on a quantum feedback network be unified?

Previous work

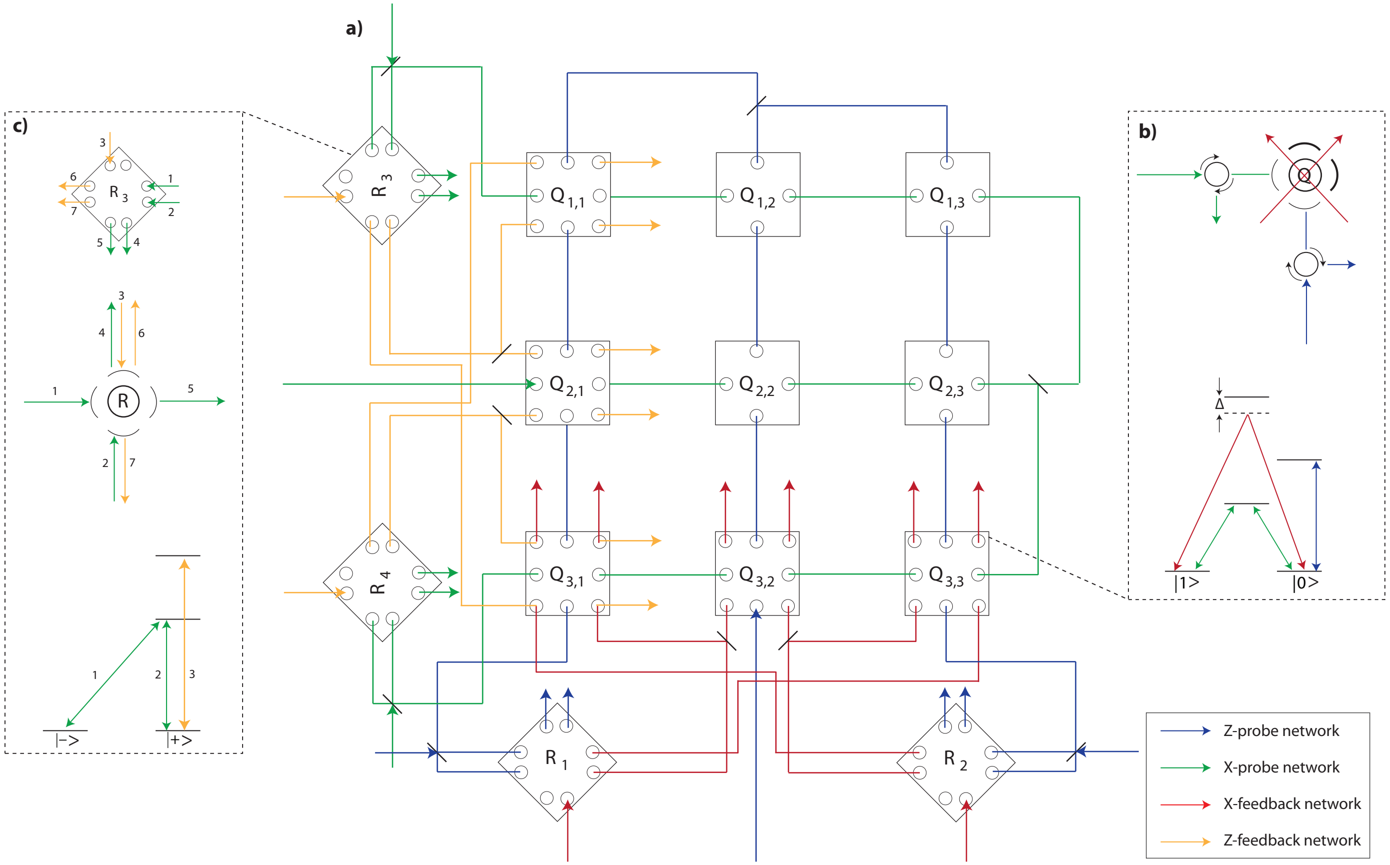
- It has been shown that they can be unified for a special class of quantum feedback networks with fast oscillators [Gough, Nurdin & Wildfeuer, *J. Math. Phys.* 51(12), 2010].
- Also, under some conditions, for this special class the two operations *can be commuted*.
- This talk: extension of these results to a much more general class of quantum feedback networks with Markovian components.

Motivation: Complex coherent-feedback networks



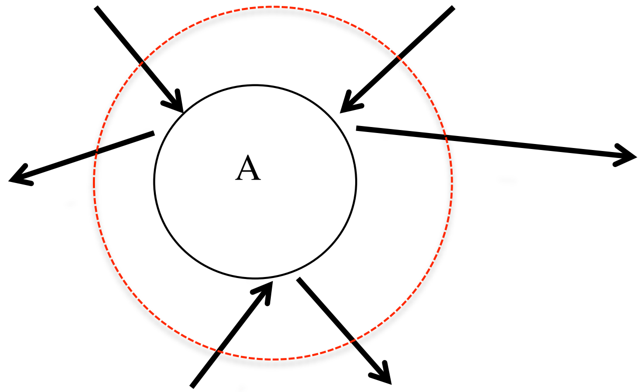
- Coherent three qubit bit-flip QEC network [Kerckhoff *et al*, PRL 105, 040502, 2010].

Motivation: Complex coherent-feedback networks



- Coherent nine qubit Bacon-Shor QEC network [Kerckhoff *et al*, NJP 13, 055022, 2010].

QSDEs for open Markov quantum systems



Open Markov quantum system + optical fields = closed system

The joint system + fields evolve according a unitary propagator $U(t)$ ($U(t)^*U(t) = U(t)U(t)^* = I$) that solves a (right) quantum stochastic differential equation (Hudson and Parthasarathy, Gardiner and Collett):

$$dU(t) = U(t)\left(\mathrm{tr}((N - I)^T d\Lambda(t)) + LdA(t) + dA(t)^* M + \underbrace{(iH - \frac{1}{2}LL^*)}_{=K} dt \right)$$

Three parameters N, K, L. $K + K^* = -LL^*$, N is unitary, and $M = -NL^*$

The Itō generator matrix

$$dU(t) = U(t)\left(\mathrm{tr}((N-I)^T d\Lambda(t)) + L dA(t) + dA(t)^* M + K dt\right)$$

- Assume elements of K, L, M, N and their adjoints have a common invariant domain $\mathcal{D}$.
- QSDE coefficients can be assembled into an Itō generator matrix $\mathbf{G}$, defined as

$$\mathbf{G} = \begin{bmatrix} K & L \\ M & N-I \end{bmatrix}$$

The Schur complement

- The Schur complement *M*/*A* of a block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

(possibly linear operator entries having a common invariant domain) with *A* invertible is

$$M/A = D - CA^{-1}B$$

Instantaneous feedback connection

- Partition the fields into internal (i) and external (e) fields. Internal fields will be interconnected with one another.
- Suppose that L, M and N have been indexed according to the partitioning as

N_i

$$L = \begin{bmatrix} L_\mathrm{i} & L_\mathrm{e} \end{bmatrix},\ M = \begin{bmatrix} M_\mathrm{i} \\ M_\mathrm{e} \end{bmatrix},\ N = \begin{bmatrix} N_\mathrm{ii} & N_\mathrm{ie} \\ N_\mathrm{ei} & N_\mathrm{ee} \end{bmatrix}$$

N_e

Instantaneous feedback as Schur complementation

- Partition the Itō generator matrix of the pre-interconnected network as:

$$\mathbf{G} = \begin{bmatrix} K & L_{\rm i} & L_{\rm e} \\ -N_{\rm i}L^{*} & N_{\rm ii} - I & N_{\rm ie} \\ -N_{\rm e}L^{*} & N_{\rm ei} & N_{\rm ee} - I \end{bmatrix}$$

- Interconnecting internal fields and taking instantaneous feedback limit along these fields, the network Itō generator matrix becomes [Gough and James, *Commun. Math. Phys.*, 287, 2009]: $\mathcal{F}\mathbf{G} : \mathbf{G} \mapsto \mathbf{G}/(N_{\rm ii} - I)$

$$= \begin{bmatrix} K & L_{\rm e} \\ -N_{\rm e}L^{*} & N_{\rm ee} - I \end{bmatrix} - \begin{bmatrix} L_{\rm i} \\ N_{\rm ei} \end{bmatrix} (N_{\rm ii} - I)^{-1} \begin{bmatrix} -N_{\rm i}L^{*} & N_{\rm ie} \end{bmatrix}$$

Instantaneous feedback as Schur complementation

- $\mathbf{G}/(N_{\text{ii}} - I)$ is again an Itō generator matrix.
- Thus $\mathcal{F}$ is a structure preserving transformation of Itō generator matrices to Itō generator matrices: $\mathbf{G} \rightarrow \mathbf{G}/(N_{\text{ii}} - I)$.

Pre-adiabatic elimination

$$dU^{(k)}(t) = U^{(k)}(t)\Big(\mathrm{tr}((N^{(k)} - I)^T d\Lambda(t)) + L^{(k)} dA(t) + dA(t)^* M^{(k)} + K^{(k)} dt\Big)$$

$$N^{(k)} = N \qquad Y + Y^* = -FF^*$$

$$L^{(k)} = kF + G \qquad A + A^* = -(FG^* + GF^*)$$

$$M^{(k)} = -NL^{(k)*} \qquad B + B^* = -GG^*$$

$$K^{(k)} = k^2 Y + kA + B \qquad NN^* = N^* N = I$$

- Slow subspace is $\mathfrak{h}_s = \ker(Y)$ and $\mathfrak{h}_f = \mathfrak{h}_s^\perp$.
- Let P_s denote the projection onto the slow subspace and P_f onto the fast.

Pre-adiabatic elimination

- Assumption 1: The following structural assumptions hold (here $X_{ab} = P_a X P_b$, $a, b = \mathrm{s}, \mathrm{f}$)
  - $\mathcal{D}$ is invariant under P_s.
  - $$Y = \begin{bmatrix} Y_\mathrm{ff} & 0 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} A_\mathrm{ff} & A_\mathrm{sf} \\ A_\mathrm{fs} & 0 \end{bmatrix}, \; F = \begin{bmatrix} F_\mathrm{ff} & F_\mathrm{fs} \\ 0 & 0 \end{bmatrix}$$
  - Y_ff is invertible on $P_\mathrm{f}\mathcal{D}$.
- Condition 2: $\hat{L}_\mathsf{f} = \hat{N}_\mathsf{sf} = \hat{N}_\mathsf{fs} = 0$ with

$$\begin{aligned} \hat{L}_\mathsf{f} &= G_\mathsf{sf} - A_\mathsf{sf} Y_\mathsf{ff}^{-1} F_{fs}; \quad \hat{N}_\mathsf{sf} = N_\mathsf{sf} + N_\mathsf{sf} F_\mathsf{ff}^* Y_\mathsf{ff}^{-1} F_\mathsf{ff} \\ \hat{N}_\mathsf{fs} &= N_\mathsf{fs} + N_\mathsf{ff} F_\mathsf{ff}^* Y_\mathsf{ff}^{-1} F_\mathsf{fs} \end{aligned}$$

Adiabatic elimination as Schur complement

- Introduce the extended Itō generator matrix $\mathbf{G}_E$:

$$\mathbf{G}_E = \begin{bmatrix} B & A_{\mathrm{sf}} & G \\ P_{\mathrm{f}}A & Y_{\mathrm{ff}} & P_{\mathrm{f}}F \\ -NG^* & -NF^*P_{\mathrm{f}} & N-I \end{bmatrix}$$

- Also, introduce the adiabatic elimination operator $\mathcal{A}$

$$\mathcal{A}\mathbf{G}^{(k)} = \hat{\mathbf{G}} = P_{\mathrm{s}}(\mathbf{G}_E / Y_{\mathrm{ff}})P_{\mathrm{s}}\Big|_{\mathcal{H}_s};\ \mathbf{G}^{(k)} = \begin{bmatrix} k^2Y + kA + B & kF + G \\ -kNF^* - kG^* & N - I \end{bmatrix}$$

- Under Assumption 1 and Condition 2, $\mathcal{A}\mathbf{G}^{(k)}$ is again an Itō generator matrix. In this case $\mathcal{A}$ is also a structure preserving transformation (of Itō generator matrices to Itō generator matrices).

Adiabatic elimination as Schur complement

- Let

$$\mathcal{A}\mathbf{G}^{(k)} = \hat{\mathbf{G}} = \begin{bmatrix} \hat{K} & \hat{L} \\ \hat{M} & \hat{N} - I \end{bmatrix},$$

then

$$\hat{K} = B_{\mathrm{ss}} - A_{\mathrm{sf}} Y_{\mathrm{ff}}^{-1} A_{\mathrm{fs}}$$

$$\hat{L} = G_{\mathrm{ss}} - A_{\mathrm{sf}} Y_{\mathrm{ff}}^{-1} F_{\mathrm{fs}}$$

$$\hat{M} = \sum_{b=\mathrm{s,f}} N_{\mathrm{s}b} (-G_{\mathrm{s}b}^{*} + F_{\mathrm{f}b}^{*} Y_{\mathrm{ff}}^{-1} A_{\mathrm{fs}})$$

$$\hat{N} = N_{\mathrm{ss}} + \sum_{c=\mathrm{s,f}} G_{\mathrm{s}c} F_{\mathrm{f}c}^{*} Y_{\mathrm{ff}}^{-1} F_{\mathrm{fs}}$$

Adiabatic elimination as Schur complement

- From

$$\hat{\mathbf{G}} = \begin{bmatrix} \hat{K} & \hat{L} \\ \hat{M} & \hat{N} - I \end{bmatrix}$$

define the corresponding QSDE

$$d\hat{U}(t) = \hat{U}(t)\Big(\mathrm{tr}((\hat{N} - I)^T d\Lambda(t)) + \hat{L}dA(t) + dA(t)^* \hat{M} + \hat{K}dt\Big)$$

- Condition 3: For any n dimensional complex vectors α and β, $P_s\mathcal{D}$ is a core for the operator

$$\mathcal{L}^{(\alpha\beta)} = \alpha^* \hat{N}\beta + \alpha^* \hat{M} + \hat{L}\beta + \hat{K} - \frac{|\alpha|^2 + |\beta|^2}{2}$$

Adiabatic elimination as Schur complementation

Theorem 1 [Bouten, van Handel & Silberfarb] Suppose that Assumption 1 holds, and Conditions 2 and 3 are satisfied. If the QSDE for $U^{(k)}(t)$ has a unique solution that extends to a contraction co-cycle on $\mathfrak{h} \otimes \mathfrak{F}$ for all $k > 0$, and the QSDE for $\hat{U}(t)$ has a unique solution that extends to a unitary co-cycle on $\mathfrak{h}_\mathsf{s} \otimes \mathfrak{F}$, then

$$\lim_{k\to\infty} \sup_{0\le t\le T} \|U^{(k)}(t)^*\phi - \hat{U}(t)^*\phi\| = 0, \quad \forall \phi \in \mathfrak{h}_\mathsf{s} \otimes \mathfrak{F},$$

for each fixed $T \ge 0$.

Adiabatic elimination followed by instantaneous feedback

- Partition $\mathbf{G}_E$ according to input and external components

$$\mathbf{G}_E = \begin{bmatrix} B & A_{\mathsf{sf}} & G_{\mathsf{i}} & G_{\mathsf{e}} \\ A_{\mathsf{f}} & Y_{\mathtt{ff}} & F_{\mathtt{f}\mathsf{i}} & F_{\mathtt{f}\mathsf{e}} \\ -N_{\mathsf{i}}G^* & -N_{\mathsf{i}}F_{\mathtt{f}}^* & N_{\mathsf{ii}} - I & N_{\mathsf{ie}} \\ -N_{\mathsf{e}}G^* & -N_{\mathsf{e}}F_{\mathtt{f}}^* & N_{\mathsf{ei}} & N_{\mathsf{ee}} - I \end{bmatrix},$$

Lemma 1 If Assumption 1 and Condition 2 are satisfied, and $N_{\mathsf{ii}}+N_{\mathsf{i}}F_{\mathtt{f}}^*Y_{\mathtt{ff}}^{-1}F_{\mathtt{f}\mathsf{i}}-I$ is invertible, then

$$P_{\mathsf{s}}\big((\mathbf{G}_E/Y_{\mathtt{ff}})/(N_{\mathsf{ii}} + N_{\mathsf{i}}F_{\mathtt{f}}^*Y_{\mathtt{ff}}^{-1}F_{\mathtt{f}\mathsf{i}} - I)\big)P_{\mathsf{s}} = \mathcal{F}\mathcal{A}\mathbf{G}^{(k)},$$

where $F_{\mathtt{f}\mathsf{i}} = P_{\mathtt{f}}F_{\mathsf{i}}$.

Instantaneous feedback followed by adiabatic elimination

Lemma 2 Suppose that Assumption 1 is satisfied, $N_{\mathsf{ii}} - I$ is invertible, $\ker(Y + F_{\mathsf{i}}(N_{\mathsf{ii}} - I)^{-1}F_{\mathsf{i}}) = \mathfrak{h}_{\mathtt{s}}$, and there exists an operator $\hat{Y}^{-}$ such that $\hat{Y}^{-}, \hat{Y}^{-*}$ have $\mathcal{D}$ as a common invariant domain and $\hat{Y}\hat{Y}^{-} = \hat{Y}^{-}\hat{Y} = P_{\mathtt{f}}$, where $\hat{Y} = Y + F_{\mathsf{i}}(N_{\mathsf{ii}} - I)^{-1}F_{\mathsf{i}}$. Then

$$\mathcal{AF}\mathbf{G}^{(k)} = P_{\mathtt{s}}((\mathbf{G}_E/(N_{\mathsf{ii}} - I))/(Y_{\mathtt{ff}} + F_{\mathtt{f}\mathsf{i}}(N_{\mathsf{ii}} - I)^{-1}N_{\mathsf{i}}F_{\mathtt{f}}^{*}))P_{\mathtt{s}} \,|_{\mathfrak{h}_{\mathtt{s}}} .$$

Successive Schur complementation rule

$$M = \begin{bmatrix} M_{A,A} & M_{A,B} & M_{A,C} \\ M_{B,A} & M_{B,B} & M_{B,C} \\ M_{C,A} & M_{C,B} & M_{C,C} \end{bmatrix} \qquad M_{B\cup C, B\cup C}$$

Lemma 9 (Successive complementation rule): *Suppose that A, B, C is a partition of the index set $\mathfrak{I}$ (that is, A, B, C are disjoint nonempty subsets whose union is $\mathfrak{I}$) then, whenever the generalized Schur complements are well-defined, we have the rule*

$$\begin{aligned} M/M_{B\cup C, B\cup C} &= (M/M_{C,C})/(M/M_{C,C})_{B,B} \\ &= (M/M_{B,B})/((M/M_{B,B}))_{C,C}. \end{aligned} \tag{26}$$

Commutativity of adiabatic elimination and instantaneous feedback

Theorem 2 Suppose that the conditions of Lemmata 1 and 2 are satisfied. Then

$$\mathcal{A}\mathcal{F}\mathbf{G}^{(k)} = \mathcal{F}\mathcal{A}\mathbf{G}^{(k)}.$$

Furthermore, if in addition

1. Condition 3 (the core condition) is satisfied.
2. $\mathcal{F}\mathbf{G}^{(k)}$ corresponds to a QSDE that has a unique solution that extends to a contraction co-cycle on $\mathfrak{h} \otimes \mathfrak{F}$,
3. The coefficients of $\mathcal{F}\mathcal{A}\mathbf{G}^{(k)}$ satisfy Condition 3 (in lieu of $\hat{K}, \hat{L}, \hat{M}, \hat{N}$) and the associated QSDE has a unique solution that extends to a unitary co-cycle on $\mathfrak{h}_{\mathbf{s}} \otimes \mathfrak{F}$,

then the instantaneous feedback and adiabatic elimination operations can be commuted. That is, applying adiabatic elimination followed by instantaneous feedback or, conversely, applying instantaneous feedback followed by adiabatic elimination yields the same QSDE and this QSDE has a unique solution that extends to a unitary co-cycle on $\mathfrak{h}_{\mathbf{s}} \otimes \mathfrak{F}$.

Concluding remarks

- Adiabatic elimination and instantaneous feedback can be viewed as transformations of Ito generator matrices to themselves.
- Both operations can be unified as Schur complements (w.r.t. to different sub-blocks) of a common matrix of operators: the extended Ito generator matrix $\mathbf{G}_E$.
- Under certain conditions, adiabatic elimination and instantaneous feedback can be interchanged/ commuted to yield a unique reduced QSDE model.

Thank you for listening!

A final detail … (1)

- Generally, before instantaneous feedback on a quantum network, there is a collection of C independent and unconnected open quantum system components (but possibly initialized in an *entangled* state) with Ito generator matrices $\mathbf{G}_j^{(k)}$, depending on the same scaling parameter k.
- If each component can be adiabatically eliminated and has $\mathcal{A}\mathbf{G}_j^{(k)}$ after elimination, is simultaneous adiabatic elimination of the collection = the collection of adiabatically eliminated components?

A final detail … (2)

- The problem is that the kernel of the sum $Y_1 + Y_2 + \ldots + Y_C$ may not coincide with the tensor product of the kernel spaces of each of the Y_j, but can be larger than the latter.
- This can also be treated, necessary and sufficient as well simpler sufficient only conditions can be written to guarantee that the two kernels do coincide.