Structure of the space of two-qubit gates, perfect entanglers and quantum control

Jiri Vala

Department of Mathematical Physics, National University of Ireland, Maynooth
and
School of Theoretical Physics, Dublin Institute for Advanced Studies

Outline

Two-qubit gates as the group SU(4)
- local invariants
- Cartan decomposition and three-torus
- Weyl chamber and local equivalence classes
- local equivalence classes of perfect entanglers

Geometric theory and applications:
Phys. Rev. A **67**, 042313 (2003)
Phys. Rev. Lett. **91**, 027903 (2003)
Phys. Rev. A **69**, 042309 (2004)
Phys. Rev. Lett. **93**, 020502 (2004)

Optimal control applications

Optimal control applications:
Phys. Rev. A **84**, 042315 (2011)
& a work in progress

Two-qubit gates as a metric space
- metric and invariant volume
- how large are control targets?
- what is the volume of the space of perfect entanglers?

Metric properties and applications:
P. Watts et al., submitted (2013)

I. Introduction and geometric theory

Jun Zhang
Jiri Vala
K. Birgitta Whaley
Shankar Sastry

Geometric theory and applications:

Phys. Rev. A **67**, 042313 (2003)
Phys. Rev. Lett. **91**, 027903 (2003)
Phys. Rev. A **69**, 042309 (2004)
Phys. Rev. Lett. **93**, 020502 (2004)

Two-qubit gates

Unitary operators acting on the state of two quantum bits

$$U : \mathcal{H}^4 \to \mathcal{H}^4$$

form the group of four-by-four unitary matrices $U(4)$:

$$U(4) = U(1) \otimes SU(4)$$

where U(1) is a global phase and $SU(4)$ is the group of four-by-four unitary matrices with unit determinant.

Examples: in the standard computational basis: $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$U_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad U_{H\otimes H} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad U_{CPHASE} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad U_Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}$$

$$U_{DCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & i & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix} \qquad U_{\sqrt{SWAP}} = \frac{e^{-i\pi/8}}{\sqrt{2}}\begin{pmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1+i \end{pmatrix} \qquad U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$SU(4)$ group and $su(4)$ algebra

$SU(4)$ group $\qquad e^{\sum_{ij}\theta_{ij}T_{ij}} \quad \longleftarrow \quad \sum_{ij}\theta_{ij}T_{ij}$ $\qquad su(4)$ algebra

Generators:

$$T_{ij} = \frac{i}{2}\sigma_i^1 \otimes \sigma_j^2 = \frac{i}{2}\sigma_i^1\sigma_j^2$$

Example:

$$T_{x0} = \frac{i}{2}\sigma_x^1 \otimes I$$

$[T_{ij}, T_{kl}]$	T_{x0}	T_{y0}	T_{z0}	T_{0x}	T_{0y}	T_{0z}	T_{xx}	T_{xy}	T_{xz}	T_{yx}	T_{yy}	T_{yz}	T_{zx}	T_{zy}	T_{zz}
T_{x0} T_{y0} T_{z0}	0 T_{z0} $-T_{y0}$	$-T_{z0}$ 0 T_{x0}	T_{y0} $-T_{x0}$ 0	0 0 0	0 0 0	0 0 0	0 T_{zx} $-T_{yx}$	0 T_{zy} $-T_{yy}$	0 T_{zz} $-T_{yz}$	$-T_{zx}$ 0 T_{xx}	$-T_{zy}$ 0 T_{xy}	$-T_{zz}$ 0 T_{xz}	T_{yx} $-T_{xx}$ 0	T_{yy} $-T_{xy}$ 0	T_{yz} $-T_{xz}$ 0
T_{0x} T_{0y} T_{0y}	0 0 0	0 0 0	0 0 0	0 T_{0z} $-T_{0y}$	$-T_{0z}$ 0 T_{0x}	T_{0y} $-T_{0x}$ 0	0 T_{xz} $-T_{xy}$	$-T_{xz}$ 0 T_{xx}	T_{xy} $-T_{xx}$ 0	0 T_{yz} $-T_{yy}$	$-T_{yz}$ 0 T_{yx}	T_{yy} $-T_{yx}$ 0	0 T_{zz} $-T_{zy}$	$-T_{zz}$ 0 T_{zx}	T_{zy} $-T_{zx}$ 0
T_{xx} T_{xy} T_{xz} T_{yx} T_{yy} T_{yz} T_{zx} T_{zy} T_{zz}	0 0 0 T_{zx} T_{zy} T_{zz} $-T_{yx}$ $-T_{yy}$ $-T_{yz}$	$-T_{zx}$ $-T_{zy}$ $-T_{zz}$ 0 0 0 T_{xx} T_{xy} T_{xz}	T_{yx} T_{yy} T_{yz} $-T_{xx}$ $-T_{xy}$ $-T_{xz}$ 0 0 0	0 T_{xz} $-T_{xy}$ 0 T_{yz} $-T_{yy}$ 0 T_{zz} $-T_{zy}$	$-T_{xz}$ 0 T_{xx} $-T_{yz}$ 0 T_{yx} $-T_{zz}$ 0 T_{zx}	T_{xy} $-T_{xx}$ 0 T_{yy} $-T_{yx}$ 0 T_{zy} $-T_{zx}$ 0	0 T_{0z} $-T_{0y}$ T_{z0} 0 0 $-T_{y0}$ 0 0	$-T_{0z}$ 0 T_{0x} 0 T_{z0} 0 0 $-T_{y0}$ 0	T_{0y} $-T_{0x}$ 0 0 0 T_{z0} 0 0 $-T_{y0}$	$-T_{z0}$ 0 0 0 T_{0z} $-T_{0y}$ T_{x0} 0 0	0 $-T_{z0}$ 0 $-T_{0z}$ 0 T_{0x} 0 T_{x0} 0	0 0 $-T_{z0}$ T_{0y} $-T_{0x}$ 0 0 0 T_{x0}	T_{y0} 0 0 $-T_{x0}$ 0 0 0 T_{0z} $-T_{0y}$	0 T_{y0} 0 0 $-T_{x0}$ 0 $-T_{0z}$ 0 T_{0x}	0 0 T_{y0} 0 0 $-T_{x0}$ T_{0y} $-T_{0x}$ 0

Cartan decomposition of su(4)

$$su(4) = k \oplus p$$

$$[k,k] \subset k$$
$$[p,k] \subset p$$
$$[p,p] \subset k$$

$$k = span\{T_{x0}, T_{y0}, T_{z0}, T_{0x}, T_{0y}, T_{0z}\}$$

$$p = span\{T_{xx}, T_{xy}, T_{xz}, T_{yx}, T_{yy}, T_{yz}, T_{zx}, T_{zy}, T_{zz}\}$$

$[T_{ij}, T_{kl}]$	T_{x0}	T_{y0}	T_{z0}	T_{0x}	T_{0y}	T_{0z}	T_{xx}	T_{xy}	T_{xz}	T_{yx}	T_{yy}	T_{yz}	T_{zx}	T_{zy}	T_{zz}
T_{x0}	0	$-T_{z0}$	T_{y0}	0	0	0	0	0	0	$-T_{zx}$	$-T_{zy}$	$-T_{zz}$	T_{yx}	T_{yy}	T_{yz}
T_{y0}	T_{z0}	0	$-T_{x0}$	0	0	0	T_{zx}	T_{zy}	T_{zz}	0	0	0	$-T_{xx}$	$-T_{xy}$	$-T_{xz}$
T_{z0}	$-T_{y0}$	T_{x0}	0	0	0	0	$-T_{yx}$	$-T_{yy}$	$-T_{yz}$	T_{xx}	T_{xy}	T_{xz}	0	0	0
T_{0x}	0	0	0	0	$-T_{0z}$	T_{0y}	0	$-T_{xz}$	T_{xy}	0	$-T_{yz}$	T_{yy}	0	$-T_{zz}$	T_{zy}
T_{0y}	0	0	0	T_{0z}	0	$-T_{0x}$	T_{xz}	0	$-T_{xx}$	T_{yz}	0	$-T_{yx}$	T_{zz}	0	$-T_{zx}$
T_{0y}	0	0	0	$-T_{0y}$	T_{0x}	0	$-T_{xy}$	T_{xx}	0	$-T_{yy}$	T_{yx}	0	$-T_{zy}$	T_{zx}	0
T_{xx}	0	$-T_{zx}$	T_{yx}	0	$-T_{xz}$	T_{xy}	0	$-T_{0z}$	T_{0y}	$-T_{z0}$	0	0	T_{y0}	0	0
T_{xy}	0	$-T_{zy}$	T_{yy}	T_{xz}	0	$-T_{xx}$	T_{0z}	0	$-T_{0x}$	0	$-T_{z0}$	0	0	T_{y0}	0
T_{xz}	0	$-T_{zz}$	T_{yz}	$-T_{xy}$	T_{xx}	0	$-T_{0y}$	T_{0x}	0	0	0	$-T_{z0}$	0	0	T_{y0}
T_{yx}	T_{zx}	0	$-T_{xx}$	0	$-T_{yz}$	T_{yy}	T_{z0}	0	0	0	$-T_{0z}$	T_{0y}	$-T_{x0}$	0	0
T_{yy}	T_{zy}	0	$-T_{xy}$	T_{yz}	0	$-T_{yx}$	0	T_{z0}	0	T_{0z}	0	$-T_{0x}$	0	$-T_{x0}$	0
T_{yz}	T_{zz}	0	$-T_{xz}$	$-T_{yy}$	T_{yx}	0	0	0	T_{z0}	$-T_{0y}$	T_{0x}	0	0	0	$-T_{x0}$
T_{zx}	$-T_{yx}$	T_{xx}	0	0	$-T_{zz}$	T_{zy}	$-T_{y0}$	0	0	T_{x0}	0	0	0	$-T_{0z}$	T_{0y}
T_{zy}	$-T_{yy}$	T_{xy}	0	T_{zz}	0	$-T_{zx}$	0	$-T_{y0}$	0	0	T_{x0}	0	T_{0z}	0	$-T_{0x}$
T_{zz}	$-T_{yz}$	T_{xz}	0	$-T_{zy}$	T_{zx}	0	0	0	$-T_{y0}$	0	0	T_{x0}	$-T_{0y}$	T_{0x}	0

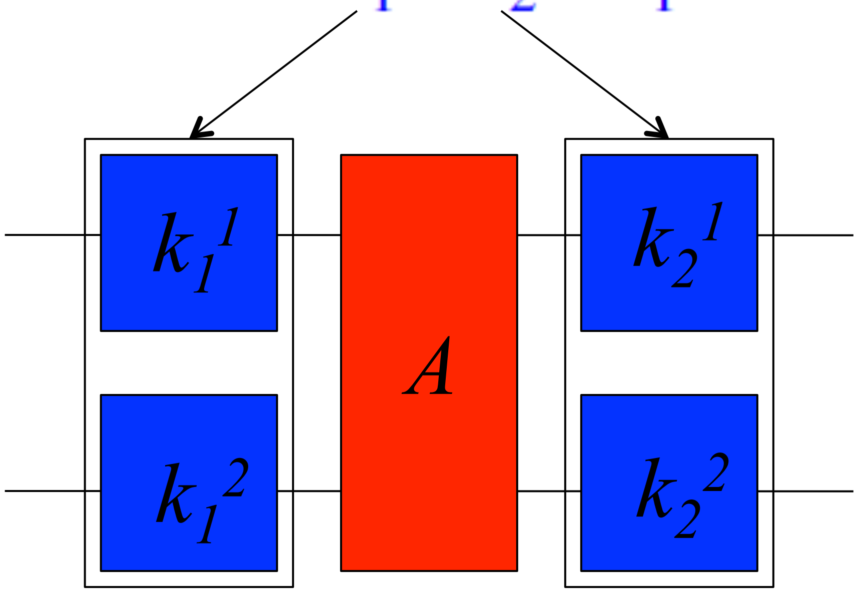
Cartan, maximal Abelian, subalgebra:

$$a = span\{T_{xx}, T_{yy}, T_{zz}\} = span\frac{i}{2}\{\sigma_x^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_z^1\sigma_z^2\} \subset p$$

Cartan decomposition of SU(4)

$U \in SU(4)$

$$U = k_1 \, A \, k_2 = k_1 \, e^{\frac{i}{2}\left(c_1 \, \sigma_x^1 \sigma_x^2 + c_2 \, \sigma_y^1 \sigma_y^2 + c_3 \, \sigma_z^1 \sigma_z^2 \right)} \, k_2$$



$k_1, k_2 \in SU(2) \otimes SU(2)$

Parameter counting:

$$6 \quad + \quad 3 \quad + \quad 6 \quad = \quad 15 \quad = \quad 4^2 - 1$$

If two gates have the same *A* in the Cartan decomposition, they are locally equivalent:

$$U_1 = k_1 U_2 k_2$$

Local equivalence and construction of local invariants

Two gates are locally equivalent if they differ only by local operations

$$U_1 = k_1 U_2 k_2 \qquad k_1,\ k_2 \in SU(2)\otimes SU(2)$$

Construction:

1) Cartan decomposition (fix: the standard computational basis)

$$U = k_1\, A\, k_2 = k_1\, e^{\frac{i}{2}\left(c_1\, \sigma_x^1\sigma_x^2 + c_2\, \sigma_y^1\sigma_y^2 + c_3\, \sigma_z^1\sigma_z^2 \right)}\, k_2$$

2) transformation into the Bell basis

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}$$

$$|00> \to \frac{1}{\sqrt{2}}(|00> + |11>)$$
$$|01> \to \frac{i}{\sqrt{2}}(|01> + |10>)$$
$$|10> \to \frac{1}{\sqrt{2}}(|01> - |10>)$$
$$|11> \to \frac{i}{\sqrt{2}}(|00> - |11>)$$

$$U_B = Q^\dagger U Q = Q^\dagger k_1 Q\; Q^\dagger A Q\; Q^\dagger k_2 Q = O_1 F O_2$$

$$O_1, O_2 \in SO(4) \qquad O_k^T O_k = I$$

$$F = Q^\dagger A Q = diag\left\{ e^{i\frac{c_1 - c_2 + c_3}{2}}, e^{i\frac{c_1 + c_2 - c_3}{2}}, e^{i\frac{c_1 + c_2 + c_3}{2}}, e^{i\frac{-c_1 + c_2 + c_3}{2}} \right\}$$

$$i/2\{\sigma_x^1\sigma_x^2, \sigma_y^1\sigma_y^2, \sigma_z^1\sigma_z^2\} \to i/2\{\sigma_z^1, -\sigma_z^2, \sigma_z^1\sigma_z^2\}$$

J. Makhlin, QIP, 1, 243 (2003)

J. Zhang, J. Vala, S. Sastry, K.B. Whaley
Phys. Rev. A 67, 042313 (2003)

Local equivalence and construction of local invariants

$$U_B = Q^\dagger U Q = Q^\dagger k_1 Q\, Q^\dagger A Q\, Q^\dagger k_2 Q = O_1 F O_2$$

3) elimination of the local part O_1

$$m = U_B^T U_B = O_2^T F O_1^T O_1 F O_2 = O_2^T F^2 O_2$$

$$O_k^T O_k = I$$

$$F^2 = diag\left\{e^{i(c_1-c_2+c_3)}, e^{i(c_1+c_2-c_3)}, e^{i(c_1+c_2+c_3)}, e^{i(-c_1+c_2+c_3)}\right\}$$

4) characteristic equation of m and elimination of O_2

$$\lambda^4 - \mathrm{tr}(m)\lambda^3 + \frac{1}{2}\left[\mathrm{tr}^2(m) - \mathrm{tr}\left(m^2\right)\right]\lambda^2 - \mathrm{tr}^*(m)\lambda + 1 = 0$$

F^2 determines the spectrum on the Makhlin matrix m: $\mathrm{tr}(m) = \mathrm{tr}(F^2)$

Local invariants

$$g_1 = \mathrm{Re}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_2 = \mathrm{Im}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_3 = \frac{\mathrm{tr}^2(m) - \mathrm{tr}\left(m^2\right)}{4}$$

J. Makhlin, QIP, 1, 243 (2003)

J. Zhang, J. Vala, S.Sastry, K.B. Whaley
Phys. Rev. A 67, 042313 (2003)

Local equivalence classes

Local invariants:

$$g_1 = \mathrm{Re}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_2 = \mathrm{Im}\left\{\frac{\mathrm{tr}^2(m)}{16}\right\},\ g_3 = \frac{\mathrm{tr}^2(m) - \mathrm{tr}\,(m^2)}{4}$$

Uniquelly characterize a class of gates that are equivalent up to local, single qubit, transformations; they define **local equivalence classes [*U*].**

Relation between the Cartan decomposition and local invariants:

$$\boldsymbol{\sigma(F^2)} = \{e^{i(c1-c2+c3)}, e^{i(c_1+c_2-c_3)}, e^{-i(c_1+c_2+c_3)}, e^{i(-c_1+c_2+c_3)}\}$$

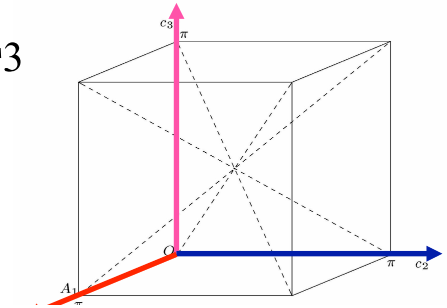
$$\begin{aligned} g_1 &= \frac{1}{4}\left[\cos(2c_1) + \cos(2c_2) + \cos(2c_3) + \cos(2c_1)\cos(2c_2)\cos(2c_3)\right] \\ g_2 &= \frac{1}{4}\sin(2c_1)\sin(2c_2)\sin(2c_3) \\ g_3 &= \cos(2c_1) + \cos(2c_2) + \cos(2c_3) \end{aligned}$$

Weyl chamber

Non-local factor A of the Cartan decomposition has the structure of three-torus

$$A = e^{\frac{i}{2}\left(c_1 \, \sigma_x^1 \sigma_x^2 + c_2 \, \sigma_y^1 \sigma_y^2 + c_3 \, \sigma_z^1 \sigma_z^2 \right)}$$

T^3



Local invariants

$$g_1 = \frac{1}{4}\left[\cos(2c_1) + \cos(2c_2) + \cos(2c_3) + \cos(2c_1)\cos(2c_2)\cos(2c_3)\right]$$

$$g_2 = \frac{1}{4}\sin(2c_1)\sin(2c_2)\sin(2c_3)$$

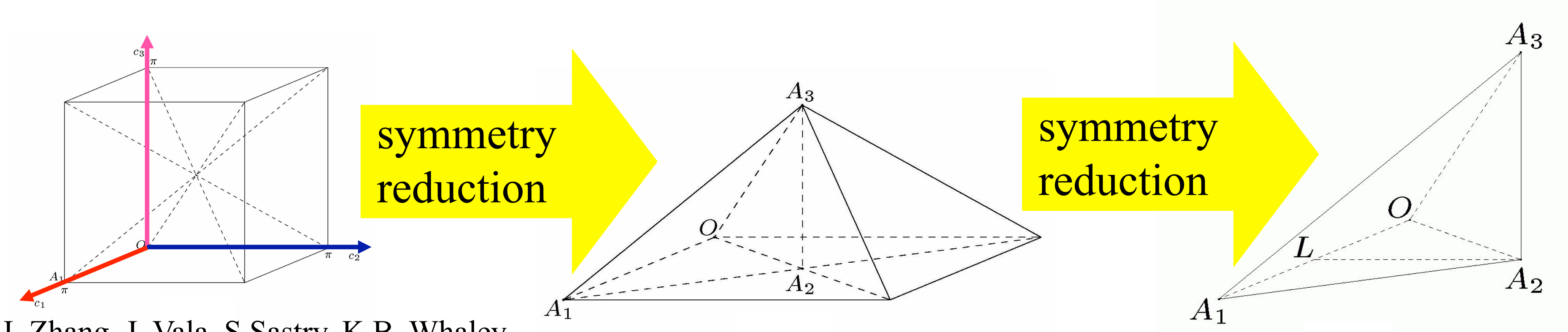
$$g_3 = \cos(2c_1) + \cos(2c_2) + \cos(2c_3)$$

$\{X \in \mathfrak{a}: c_1 - c_2 = 0\}$, $\{X \in \mathfrak{a}: c_1 + c_2 = \pi\}$

$\{X \in \mathfrak{a}: c_1 - c_3 = 0\}$, $\{X \in \mathfrak{a}: c_1 + c_3 = \pi\}$

$\{X \in \mathfrak{a}: c_2 - c_3 = 0\}$, $\{X \in \mathfrak{a}: c_2 + c_3 = \pi\}$

are invariant with interchanges of c_1, c_2, and c_3 with & without sign flips:



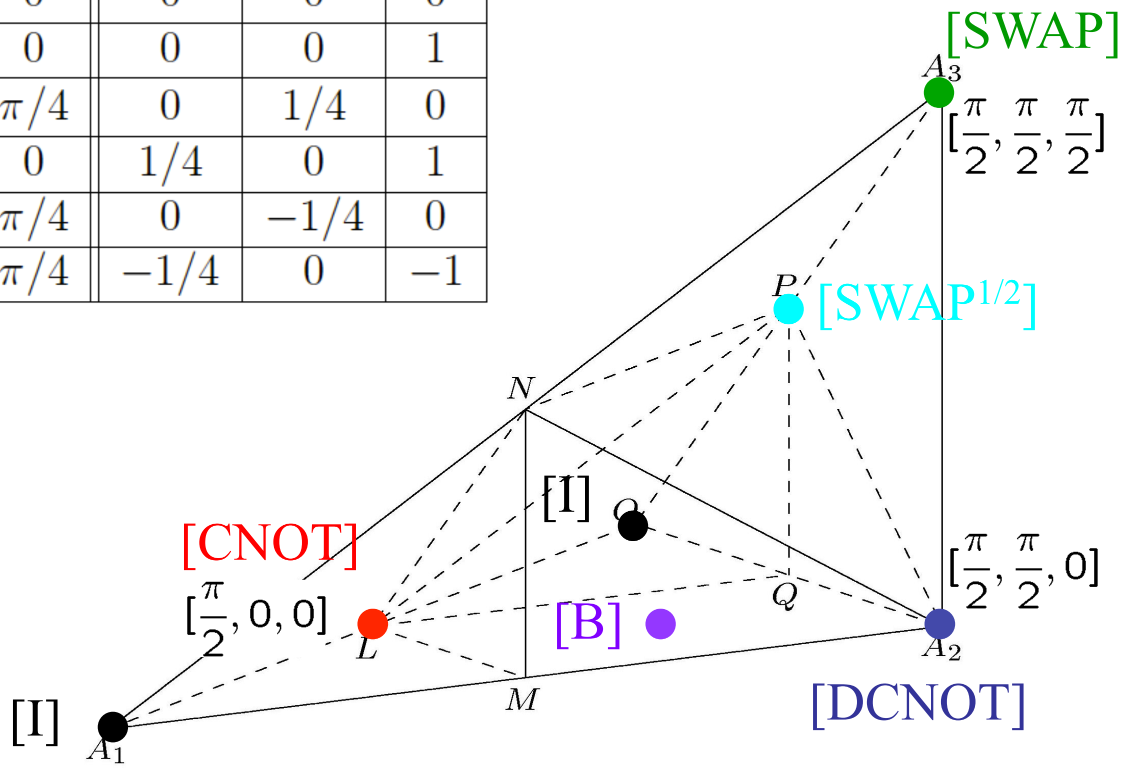
Weyl chamber

J. Zhang, J. Vala, S.Sastry, K.B. Whaley
Phys. Rev. A 67, 042313 (2003)

Examples

Each point inside of the Weyl chamber corresponds to one local equivalence class.
This is unique with except of the base of the Weyl chamber.

point (gate)	c_1	c_2	c_3	g_1	g_2	g_3
O, A_1 ([$\mathbb{1}$])	0, π	0	0	1	0	3
A_2 ([DCNOT])	$\pi/2$	$\pi/2$	0	0	0	-1
A_3 ([SWAP])	$\pi/2$	$\pi/2$	$\pi/2$	-1	0	-3
B ([B-Gate])	$\pi/2$	$\pi/4$	0	0	0	0
L ([CNOT])	$\pi/2$	0	0	0	0	1
P ([$\sqrt{\text{SWAP}}$])	$\pi/4$	$\pi/4$	$\pi/4$	0	$1/4$	0
Q, M	$\pi/4$, $3\pi/4$	$\pi/4$	0	$1/4$	0	1
N	$3\pi/4$	$\pi/4$	$\pi/4$	0	$-1/4$	0
R	$\pi/2$	$\pi/4$	$\pi/4$	$-1/4$	0	-1



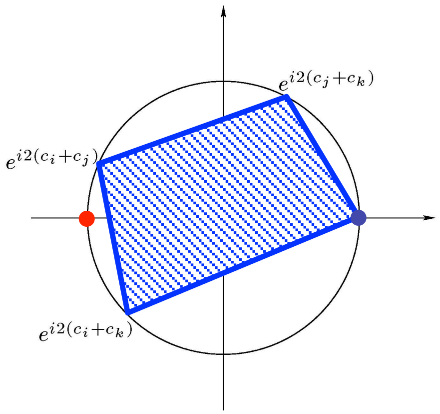
Perfect entanglers

Definition

A two qubit gate is called a **perfect entangler** if it can produce a maximally entangled state from a product state.

Theorem

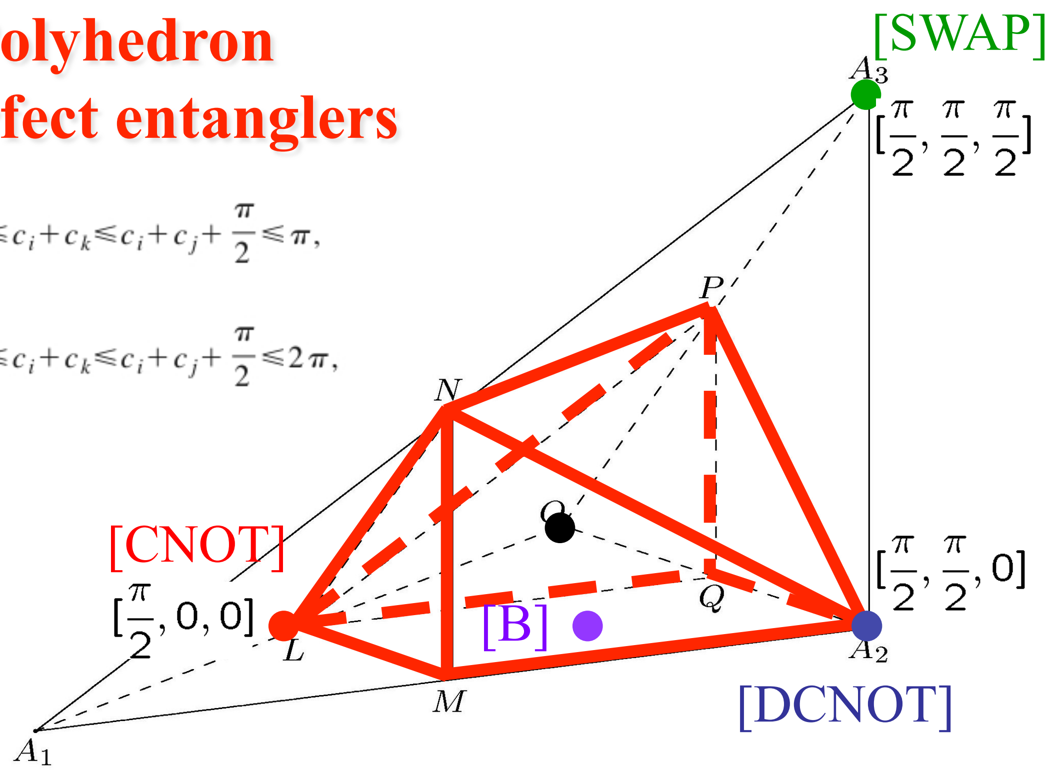
A two qubit gate U is a perfect entangler if and only if **the convex hull** of the eigenvalues of the Makhlin matrix $m(U)$ containes zero.



Polyhedron of perfect entanglers

$$\frac{\pi}{2} \leq c_i + c_k \leq c_i + c_j + \frac{\pi}{2} \leq \pi,$$

$$\frac{3\pi}{2} \leq c_i + c_k \leq c_i + c_j + \frac{\pi}{2} \leq 2\pi,$$



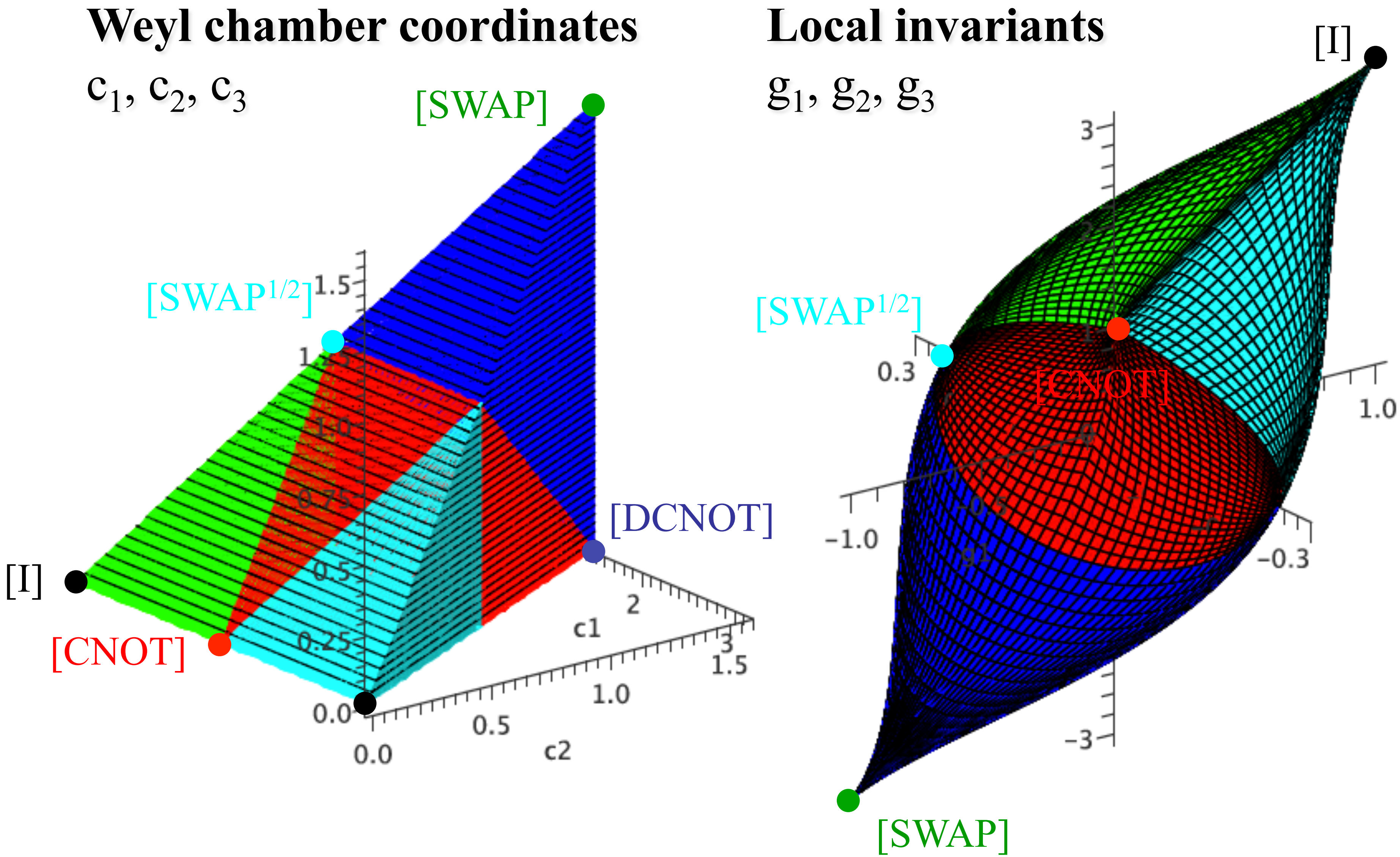
Examples

CNOT

σ[m(CNOT)] = {1, 1, -1, -1}

J. Zhang, J. Vala, S.Sastry, K.B. Whaley
Phys. Rev. A 67, 042313 (2003)

Weyl chamber and local equivalence classes



Generation of non-local gates: example

Josephson junction charge-coupled qubits

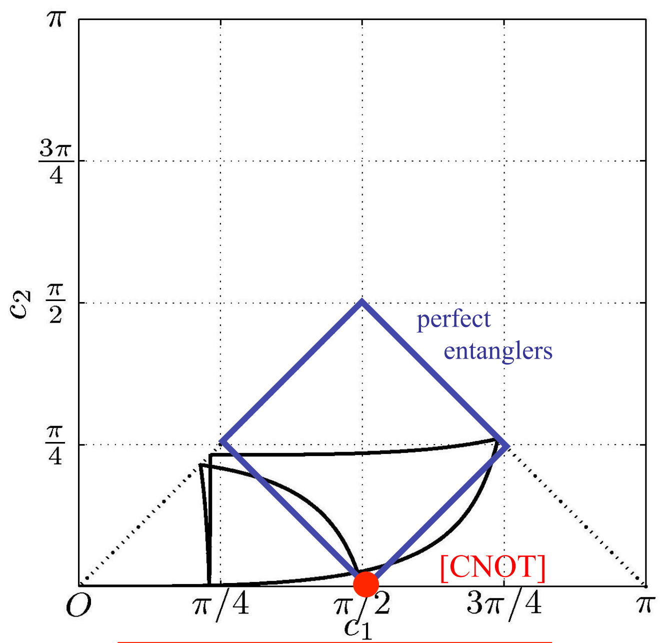
$$H = -(\alpha E_L/2)\,(\sigma_x^1 + \sigma_x^2) + \alpha^2 E_L\, \sigma_y^1\, \sigma_y^2$$

curvature translation

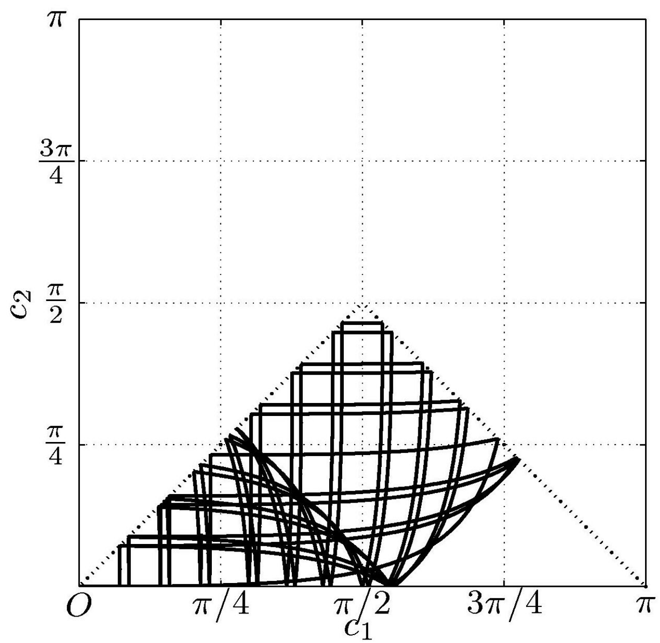
Weyl chamber trajectory:

$$c_1(t) = \alpha^2 E_L t - \omega(\alpha, t),$$
$$c_2(t) = \alpha^2 E_L t + \omega(\alpha, t),$$
$$c_3(t) = 0.$$

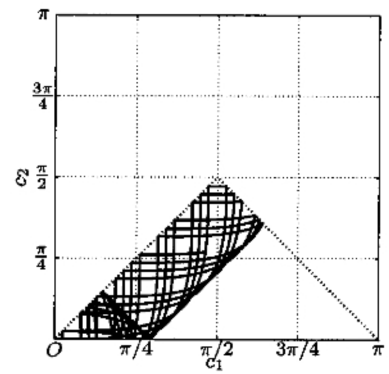
$\omega(\alpha,t) = tan^{-1}(\frac{\alpha^2+1}{2})$



$\alpha = 1.1991,\ t = 2.7309$



$\alpha = 1.1991,\ t = 20$



$\alpha = 0.5,\ t = 80$

Optimal control applications:
Phys. Rev. A **84**, 042315 (2011)
&
a work in progress

II. Optimal control applications

ulm university universität uulm

Tommaso Calarco
Matthias Mueller

UNI KASSEL VERSITÄT

Christianne Koch
Daniel Reich

K. Birgitta Whaley

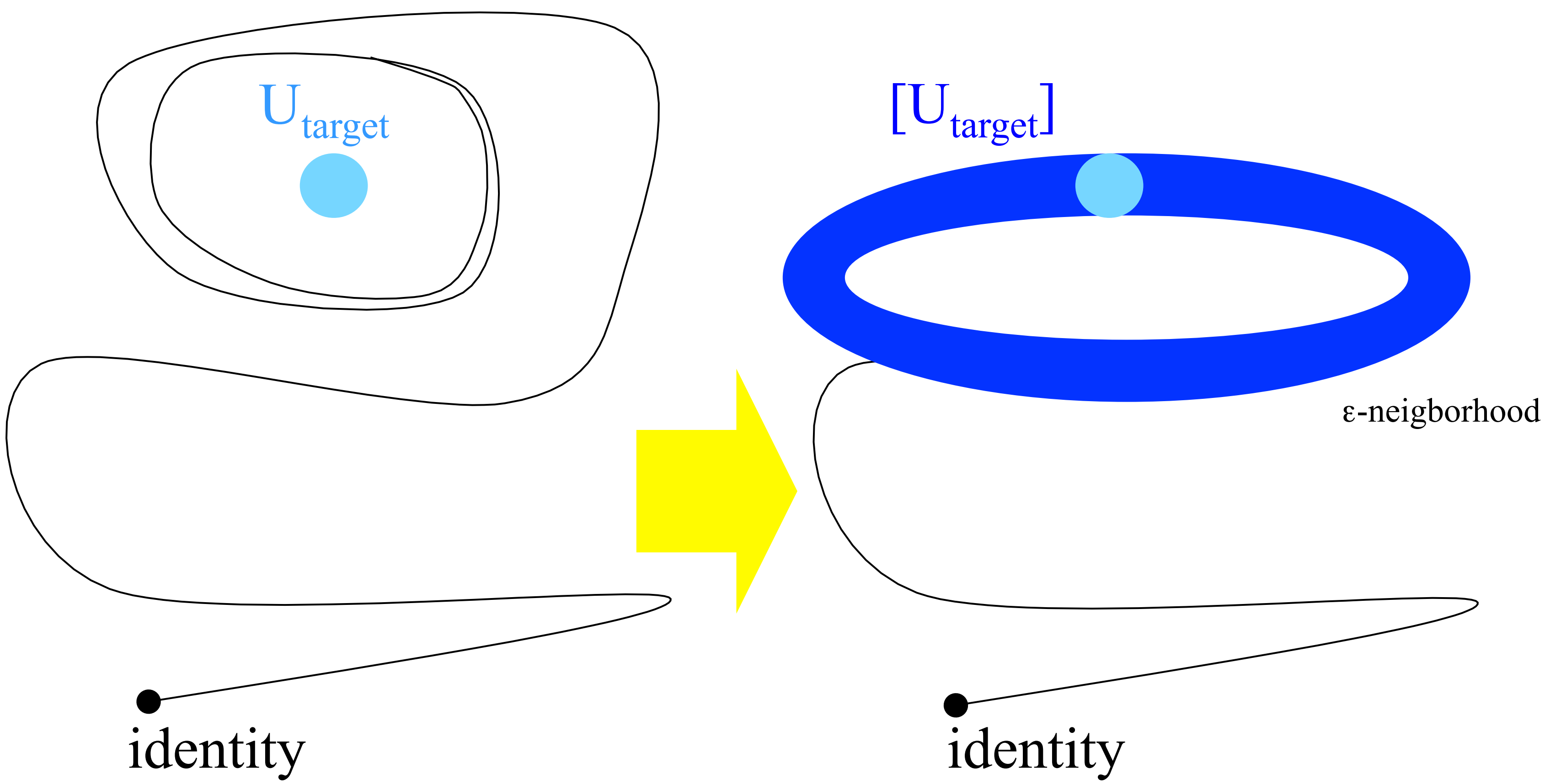
NUI MAYNOOTH
Ollscoil na hÉireann Má Nuad

Jiri Vala
Paul Watts

Haidong Yuan

Optimal control

Optimization target is defined not as a specific target gate U_{target}
but rather its local equivalence class $[U_{target}]$, i.e. $SU(2)$x$SU(2)$ orbit of U_{target} .



Optimal control

M.M. Muller et al.,
Phys. Rev. A 84, 042315 (2011).

Direct optimization functional

$$J_T^D = 1 - \frac{1}{N}\mathrm{Re}[\mathbf{Tr}\{\hat{\mathsf{O}}^{+}\hat{\mathsf{P}}_N\hat{\mathsf{U}}(T,0;\varepsilon)\hat{\mathsf{P}}_N\}]$$

with possible additional terms

$$J = J_T^D + g_a + g_b$$

finite pulse fluence

$$g_a = \lambda_a \int_0^T [\varepsilon(t) - \varepsilon_{\mathrm{ref}}(t)]^2 / S(t)dt$$

loss of population

$$g_b = \frac{\lambda_b}{NT}\int_0^T \sum_{m=1}^{N} \langle\varphi_m(t)|\hat{\mathsf{P}}_{\mathrm{avoid}}|\varphi_m(t)\rangle dt$$

Optimization functional based on local invariants

$$J_T^{LI} = \Delta g_1^2 + \Delta g_2^2 + \Delta g_3^2 + 1 - \frac{1}{N}\mathbf{Tr}\{\hat{\mathsf{U}}_{T,N}\hat{\mathsf{U}}_{T,N}^{+}\}$$

Krotov iteration

$$\varepsilon^{(i+1)}(t) = \varepsilon^{(i)}(t) + \frac{S(t)}{\lambda_a}\mathrm{Im}\left\{\sum_{k=1}^{N}\langle\chi_k^{(i)}(t)|\frac{\partial\hat{\mathsf{H}}}{\partial\varepsilon}^{(i+1)}|\varphi_k^{(i+1)}(t)\rangle + \frac{1}{2}\sigma(t)\sum_{k=1}^{N}\langle\Delta\varphi_k^{(i+1)}(t)|\frac{\partial\hat{\mathsf{H}}}{\partial\varepsilon}^{(i+1)}|\varphi_k^{(i+1)}(t)\rangle\right\}$$

Case I: Effective spin model

Trapped polar molecules with $^2\Sigma_{1/2}$ electronic ground state subject to a near resonant microwave driving inducing strong dipole-dipole coupling:

$$\hat{H}_{\mathrm{eff}}(t) = \frac{\hbar|\Omega(t)|}{8} \sum_{i,j=1}^{4} \hat{\sigma}_i A_{ij}(x_0,t)\hat{\sigma}_j$$

SrF molecules in optical lattice with a=300 nm and 15GHz microwave fields with different polarizations

$$\hat{H}(t) = \hat{H}_0 + S(t)\hat{H}_1$$

M.M. Muller et al.,
Phys. Rev. A 84, 042315 (2011).

drift

$$\hat{H}_0 = \begin{pmatrix} 5.711 & 0.324 & 0.324 & 0 \\ 0.324 & -1.840 & 1.054 & 0 \\ 0.324 & 1.054 & 1.840 & 0 \\ 0 & 0 & 0 & -2.030 \end{pmatrix}$$

control

$$\hat{H}_1 = S(t) \begin{pmatrix} -153.65 & 0 & 0 & 3.906 \\ 0 & 153.65 & 16.085 & 0 \\ 0 & 16.085 & 153.65 & 0 \\ 3.906 & 0 & 0 & -153.65 \end{pmatrix}$$

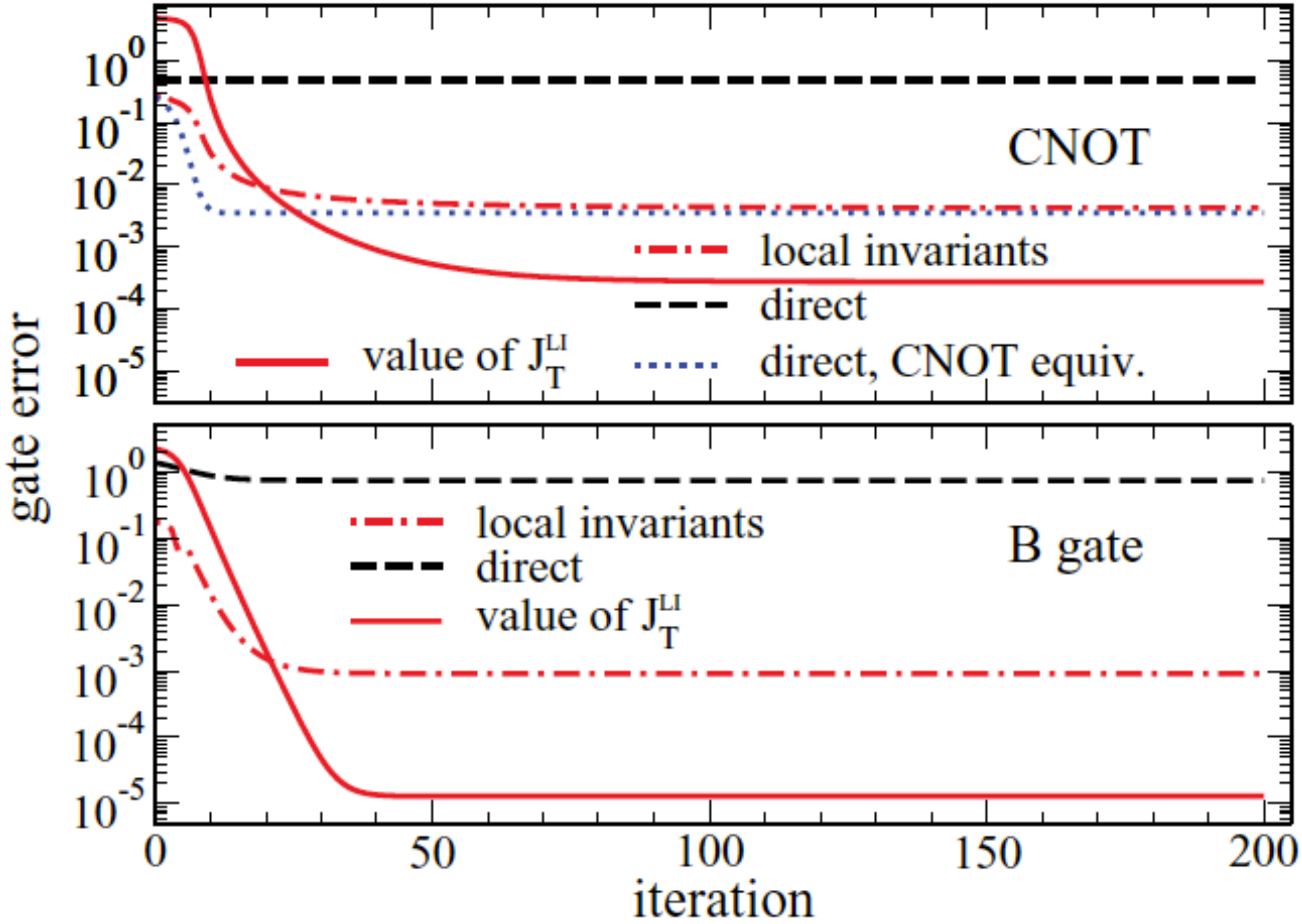
$S(t)$ → envelope function

cw field:
CNOT: $\delta = 1.2$ kHz and $\Omega = 590$ kHz
B-gate: $\delta = 1.2$ kHz and $\Omega = 4.74$ MHz

pulsed field:
CNOT: $\delta = 50$ kHz and $\Omega_0 = 1.81$ MHz
B-gate: $\delta = 84$ kHz and $\Omega_0 = 1.81$ MHz

Case I: Results

M.M. Muller et al.,
Phys. Rev. A 84, 042315 (2011).



Case II: Rydberg gate with trapped neutral atoms

Atoms of ^{87}Rb trapped by optical tweezers with a non-local gate implemented by simultaneous near resonant two-photon transition to Rydberg states

$|0\rangle = |5s_{1/2}, F = 2, M_F = 2\rangle$ $|i\rangle = |5p_{1/2}, F = 2, M_F = 2\rangle$

$|1\rangle = |5s_{1/2}, F = 1, M_F = 1\rangle$ $|r\rangle = |58d_{3/2}, F = 3, M_F = 3\rangle$

M.M. Muller et al., Phys. Rev. A 84, 042315 (2011).

The Hamiltonian for a single trapped atom in RWA

$$\hat{H}_j^{(1)}(t) = |0\rangle\langle 0| \otimes (\hat{T} + V_{\text{trap}}(\hat{x}_j)) + |1\rangle\langle 1| \otimes (\hat{T} + V_{\text{trap}}(\hat{x}_j)) + |i\rangle\langle i| \otimes \left(\hat{T} + \frac{\delta_R}{2}\right) + |r\rangle\langle r| \otimes \left(\hat{T} + \frac{\delta_B}{2}\right) + \frac{\Omega_R(t)}{2}(|0\rangle\langle i| + |i\rangle\langle 0|) \otimes \mathbb{1}_{\hat{x}_j} + \frac{\Omega_B(t)}{2}(|i\rangle\langle r| + |r\rangle\langle i|) \otimes \mathbb{1}_{\hat{x}_j}$$

$\omega_R = 795$ nm
$\omega_B = 474$ nm
$\delta_R = 2\pi.600$ MHz
$\delta_B = 0$
$\Omega_{R,0} = \Omega_{B,0} = 2\pi.260$ MHz

The total two-atom Hamiltonian

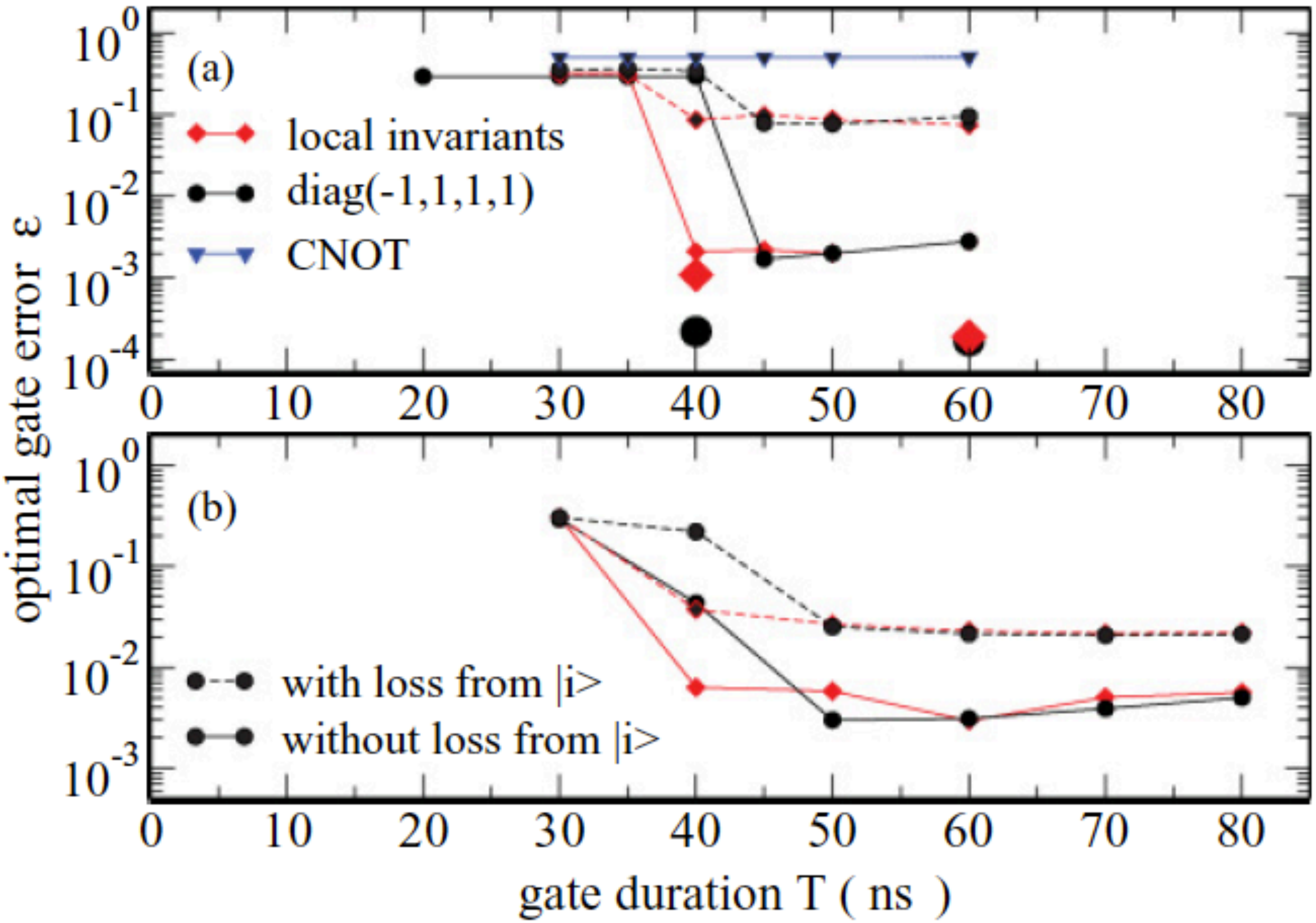
$$\hat{H}^{(2)}(t) = \hat{H}_1^{(1)}(t) \otimes \mathbb{1}_{4,2} \otimes \mathbb{1}_{\hat{x}_2} + \mathbb{1}_{4,1} \otimes \hat{H}_2^{(1)}(t) \otimes \mathbb{1}_{\hat{x}_1} + |rr\rangle\langle rr| \otimes \frac{u_0}{\hat{r}^3}$$

Rydberg-Rydberg interaction

50MHz at $r_0 = 4\mu$m, i.e. 10 ns in $|rr\rangle$ to pick the phase π

Case II: Results

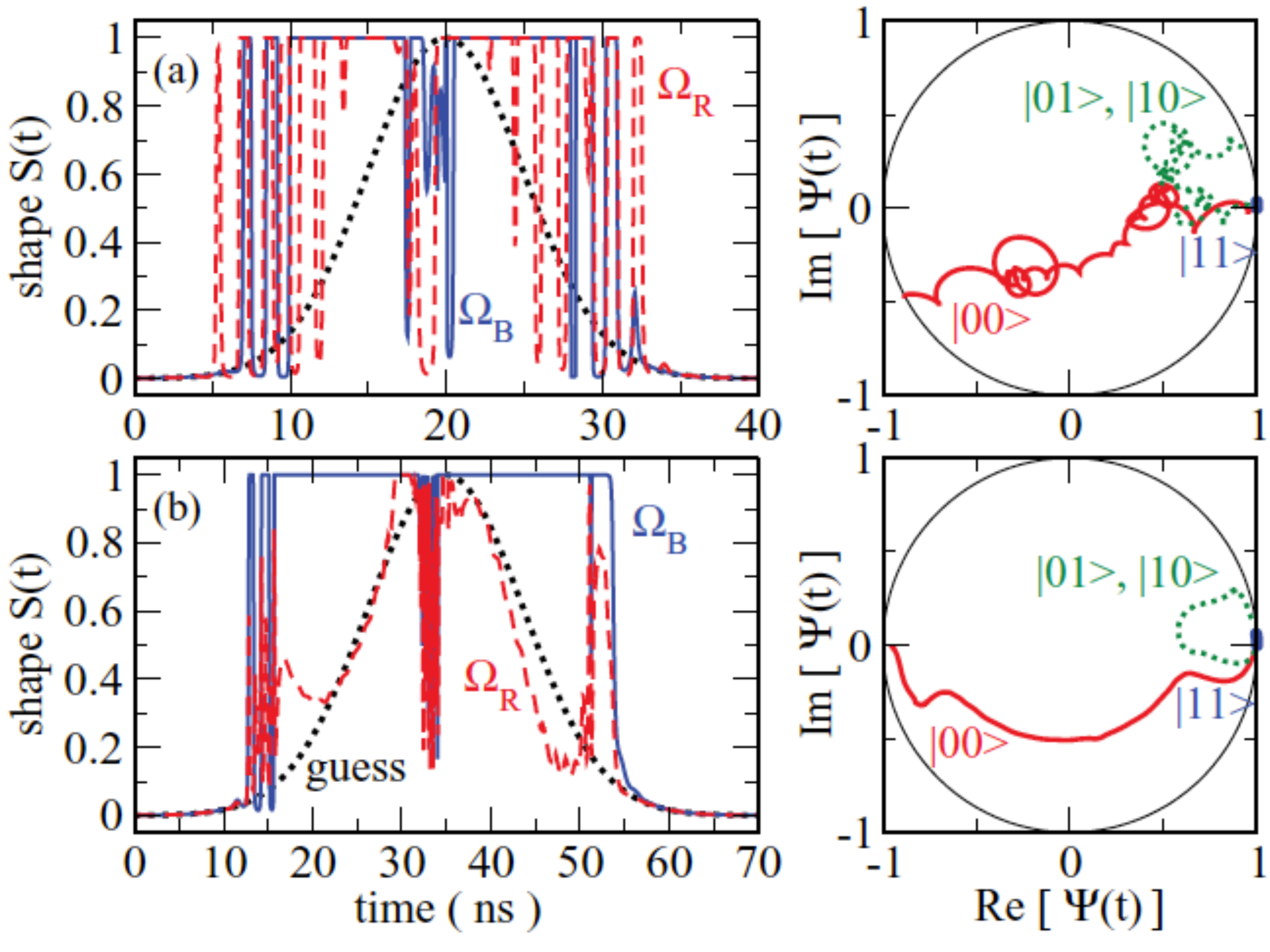
M.M. Muller et al.,
Phys. Rev. A 84, 042315 (2011)



Case II: Effect of spontaneous emission

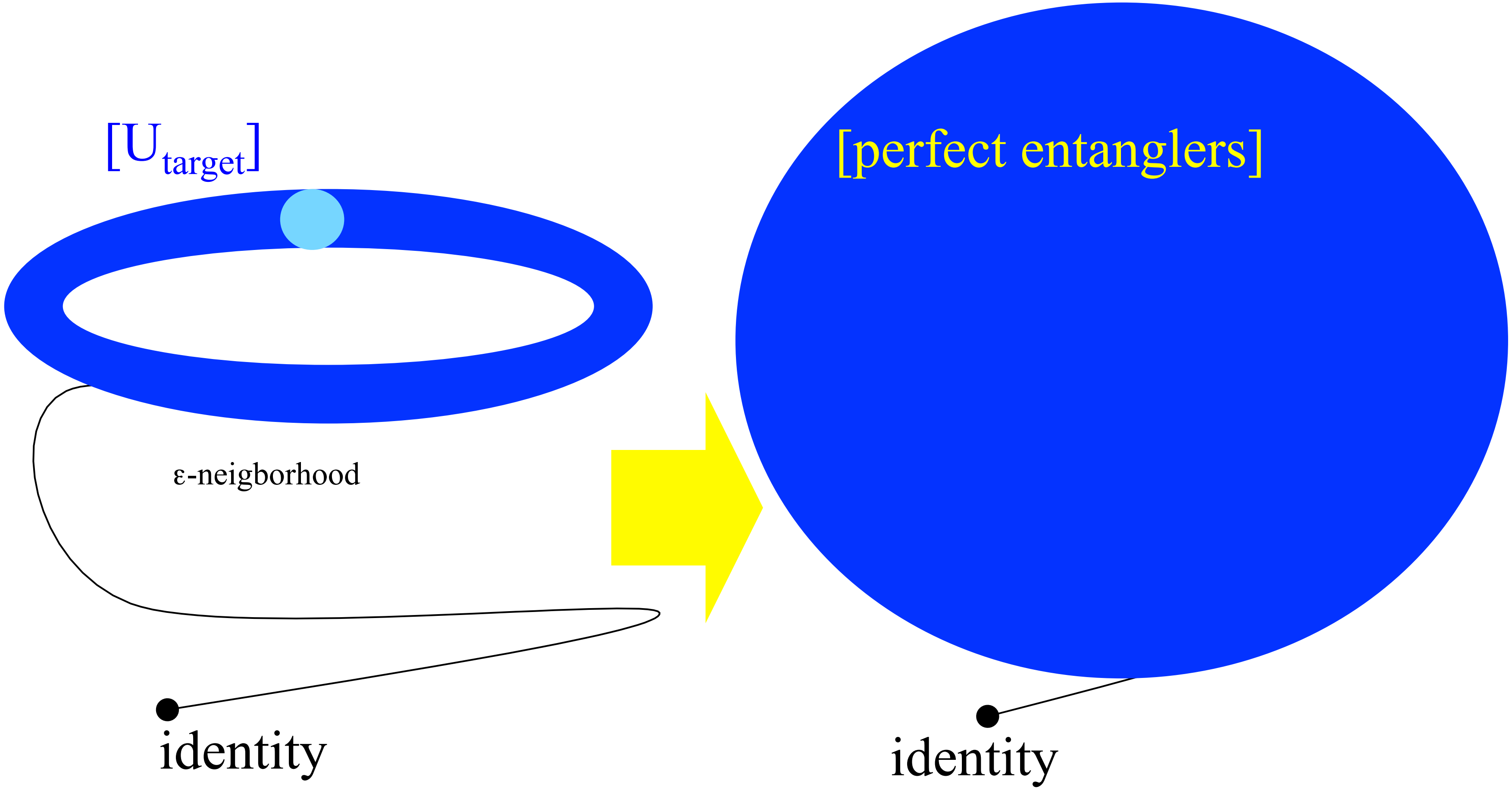
M.M. Muller et al.,
Phys. Rev. A 84, 042315 (2011)

spontaneous
emission



Perfect entanglers

Optimization target is defined not as a specific local equivalence class $[U_{target}]$, i.e. $SU(2)\times SU(2)$ orbit of U_{target}, but **the full set of perfect entanglers**



Optimal control: perfect entanglers

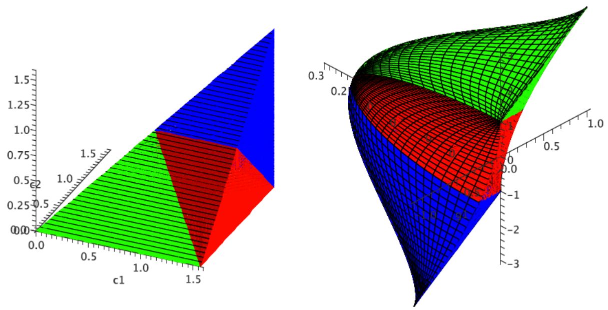
Direct optimization functional

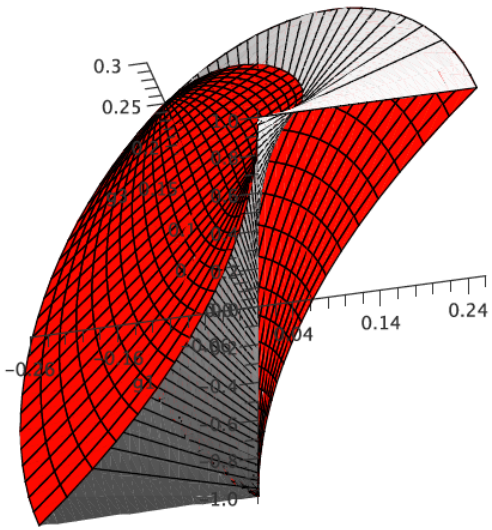
$$J_T^D = 1 - \frac{1}{N}\mathrm{Re}[\mathbf{Tr}\{\hat{\mathsf{O}}^{+}\hat{\mathsf{P}}_N\hat{\mathsf{U}}(T,0;\varepsilon)\hat{\mathsf{P}}_N\}]$$

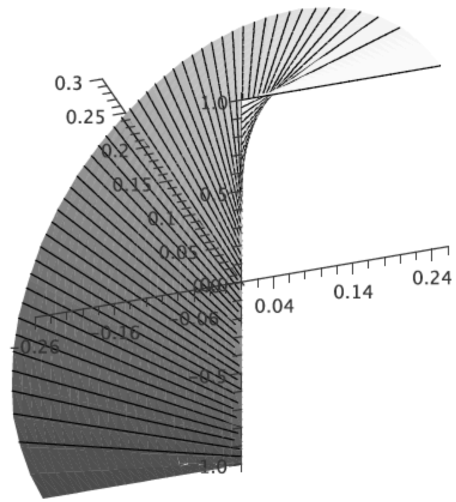
is to be based on the function

$$\mathcal{D}(U) \;=\; g_3\sqrt{g_1^2+g_2^2}-g_1$$

which goes to zero when the evolution operator reaches perfect entanglers







Numerical experiment: Tommaso Calarco (first results),
Christiane Koch (in progress)

III. Two-qubit gates as a metric space

NUI MAYNOOTH
Ollscoil na hÉireann Má Nuad

Jiri Vala
Paul Watts
Maurice O'Connor

Metric properties and applications:
P. Watts et al., submitted (2013)

Decomposition and parametrisation of $SU(4)$

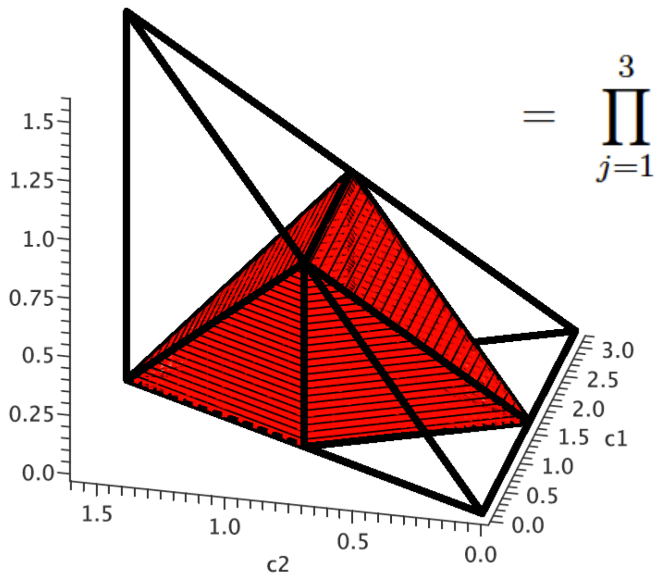
$$U = k_1 A k_2$$

1) Local part

$$k\left(\vec{\alpha},\vec{\beta}\right) = \exp\left(-\frac{i}{2}\vec{\alpha}\cdot\vec{\sigma}\right)\otimes\exp\left(-\frac{i}{2}\vec{\beta}\cdot\vec{\sigma}\right)$$
$$= \left[I\cos\left(\frac{\alpha}{2}\right) - i\hat{\alpha}\cdot\vec{\sigma}\sin\left(\frac{\alpha}{2}\right)\right]\otimes\left[I\cos\left(\frac{\beta}{2}\right) - i\hat{\beta}\cdot\vec{\sigma}\sin\left(\frac{\beta}{2}\right)\right]$$

2) Non-local part

$$A\left(c_1,c_2,c_3\right) = \exp\left(-\frac{i}{2}\sum_{j=1}^{3} c_j\sigma_j\otimes\sigma_j\right)$$
$$= \prod_{j=1}^{3}\left[I\otimes I\cos\left(\frac{c_j}{2}\right) - i\sigma_j\otimes\sigma_j\sin\left(\frac{c_j}{2}\right)\right]$$



Restriction to the Weyl chamber:

$$0 \le c_3 \le c_2 \le c_1 \le \frac{\pi}{2} \qquad \&$$

$$\frac{\pi}{2} < c_1 < \pi,\ 0 \le c_3 \le c_2 < \pi - c_1$$

Parameters:

$$x = \left(x^1,\ldots,x^{15}\right) = \left(\alpha_1,\theta_1,\phi_1,\beta_1,\lambda_1,\xi_1,\alpha_2,\theta_2,\phi_2,\beta_2,\lambda_2,\xi_2,c_1,c_2,c_3\right)$$

Invariant measure

Calculation of the Haar measure for SU(4) and its Cartan subalgebra:

We start with the Maurer-Cartan form

$$\Theta \quad := \quad U^{-1}\mathrm{d}U$$

which can be rewritten in terms of Lia algebra generators and coordinate 1-forms

$$\Theta \quad = \quad -iE^{A}{}_{\mu}\,(x)\,T_A\mathrm{d}x^{\mu}$$

where Θ is an NxN matrix whose determinant gives us the Haar measure

$$\mathrm{d}\mu \quad = \quad \frac{|\det E(x)|\,\mathrm{d}^N x}{\int_M |\det E\,(x')|\,\mathrm{d}^N x'}$$

The results for SU(2):

$$\Theta_{SU(2)} \quad = \quad e^{i\vec{\alpha}\cdot\vec{\sigma}/2}\mathrm{d}e^{-i\vec{\alpha}\cdot\vec{\sigma}/2} \quad = \quad -\frac{i}{2}\sum_i \zeta^i\,(\vec{\alpha})\,\sigma_i$$

$$\mathrm{d}\mu_{SU(2)}(\alpha,\theta,\phi) \quad = \quad \frac{1}{8\pi^2}\sin^2\left(\frac{\alpha}{2}\right)\sin\theta\,\mathrm{d}\alpha\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi$$

Haar measure for SU(4)

$$\mathrm{d}\mu = \mathrm{d}\mu_{SU(2)}\left(\vec{\alpha}_1\right) \wedge \mathrm{d}\mu_{SU(2)}\left(\vec{\beta}_1\right) \wedge \mathrm{d}\mu_{SU(2)}\left(\vec{\alpha}_2\right) \wedge \mathrm{d}\mu_{SU(2)}\left(\vec{\beta}_2\right) \wedge \mathrm{d}\mu_{\mathcal{A}}\left(c_1, c_2, c_3\right)$$

Local parts:

$$\mathrm{d}\mu_{SU(2)}(\alpha, \theta, \phi) = \frac{1}{8\pi^2} \sin^2\left(\frac{\alpha}{2}\right) \sin\theta \, \mathrm{d}\alpha \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$$

Non-local part:

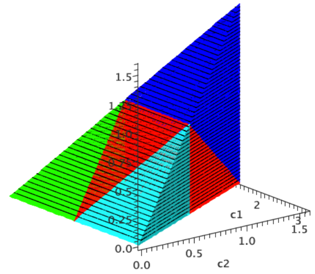
$$\begin{aligned}
\mathrm{d}\mu_{\mathcal{A}}\left(c_1, c_2, c_3\right) &= \frac{48}{\pi} \left|\sin\left(c_1 + c_2\right) \sin\left(c_1 - c_2\right) \sin\left(c_1 + c_3\right) \sin\left(c_1 - c_3\right)\right. \\
&\quad \left.\times \sin\left(c_2 + c_3\right) \sin\left(c_2 - c_3\right)\right| \mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3, \\
&= M_{\mathcal{A}}\left(c_1, c_2, c_3\right) \mathrm{d}c_1 \wedge \mathrm{d}c_2 \wedge \mathrm{d}c_3
\end{aligned}$$

The full SU(4):

$$\begin{aligned}
\mathrm{d}\mu &= \frac{3}{256\pi^9} \prod_{i=1}^{2} \left[\sin^2\left(\frac{\alpha_i}{2}\right) \sin\theta_i \sin^2\left(\frac{\beta_i}{2}\right) \sin\lambda_i\right] \\
&\quad \times \prod_{1 \le j < k \le 3} \left[\sin\left(c_j + c_k\right) \sin\left(c_j - c_k\right)\right] \mathrm{d}^{15}x.
\end{aligned}$$

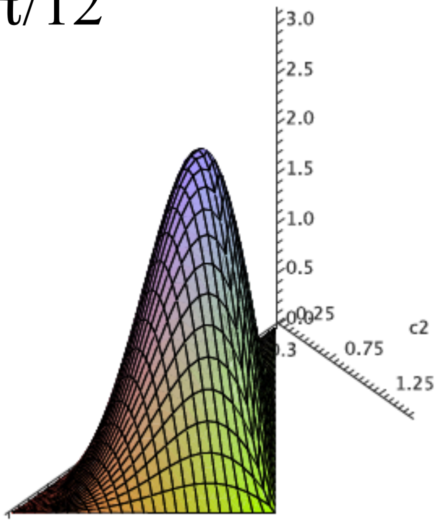
What is the size of control targets?

Control targets in the Weyl chamber can be defined as a small neigborhood of the local equivalence class, e.g. a cube of the volume $V_{Wc} = a^3$.

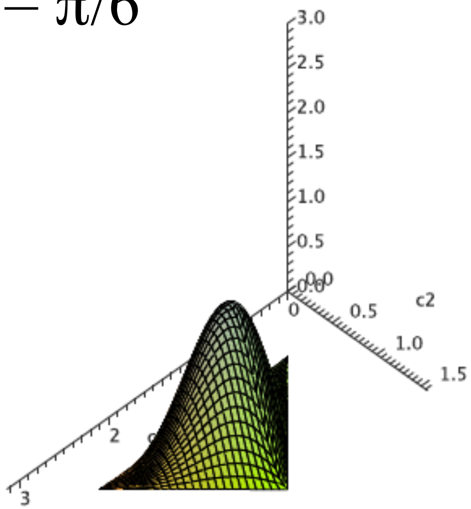


The corresponding invariant volume in the full SU(4) depends on the location in the Weyl chamber

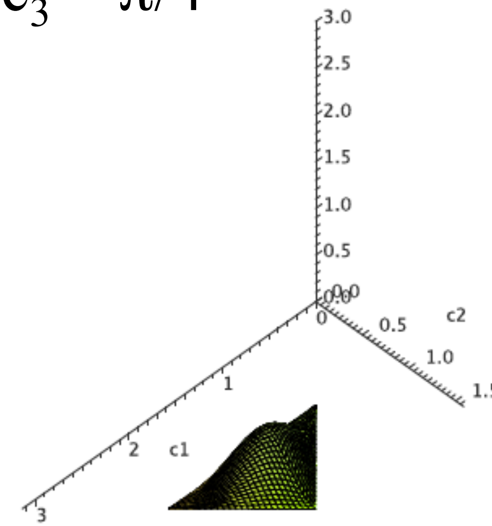
$c_3 = \pi/12$



$c_3 = \pi/6$



$c_3 = \pi/4$



Examples

$$V(\mathcal{U}) = \int_{(SU(2)\otimes SU(2))\times(SU(2)\otimes SU(2))\times\mathcal{U}} \mathrm{d}\mu = \int_{\mathcal{U}} \mathrm{d}\mu_{\mathcal{A}}$$

$[\mathbb{1}]$ at $(0,0,0)$

$$a^9/40\pi + O(a^{11})$$

[SWAP] at $(\pi/2, \pi/2, \pi/2)$

$$a^9/40\pi + O(a^{11})$$

$[\sqrt{\text{SWAP}}]$ at $(\pi/4, \pi/4, \pi/4)$

$$8a^6/5\pi + O(a^8)$$

[B-gate] at $(\pi/2, \pi/4, 0)$

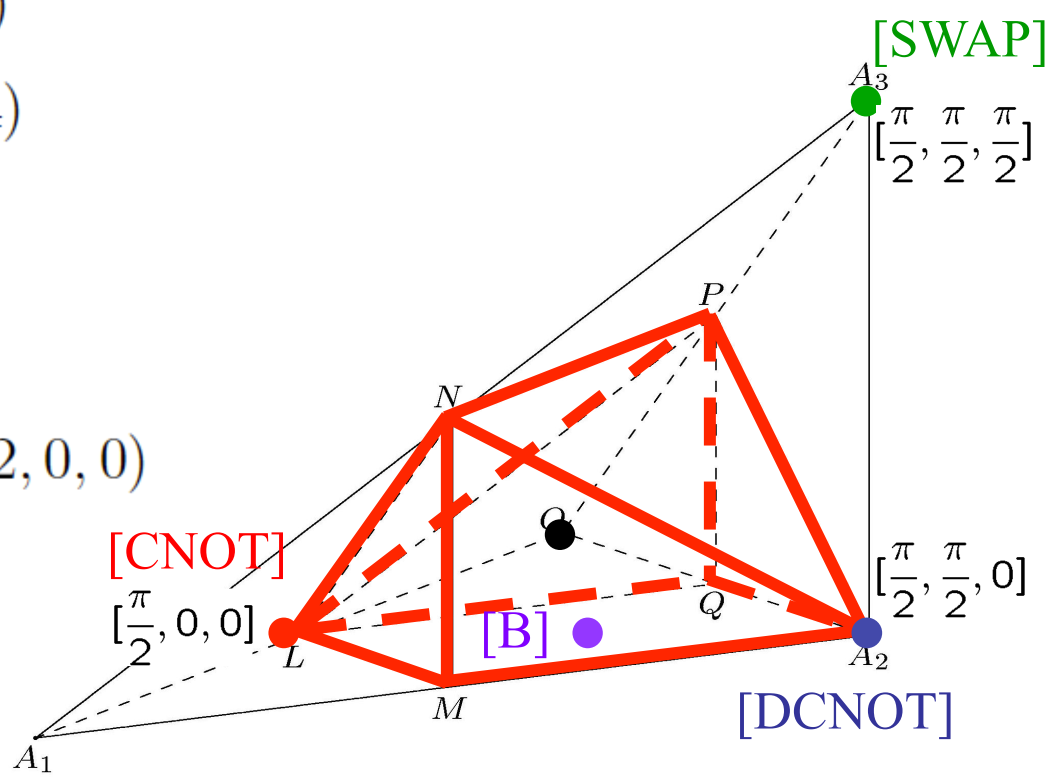
$$12a^3/\pi + O(a^5)$$

[CNOT]/[CPHASE] at $(\pi/2, 0, 0)$

$$4a^5/\pi + O(a^7)$$

[DCNOT] at $(\pi/2, \pi/2, 0)$

$$4a^5/\pi + O(a^7)$$

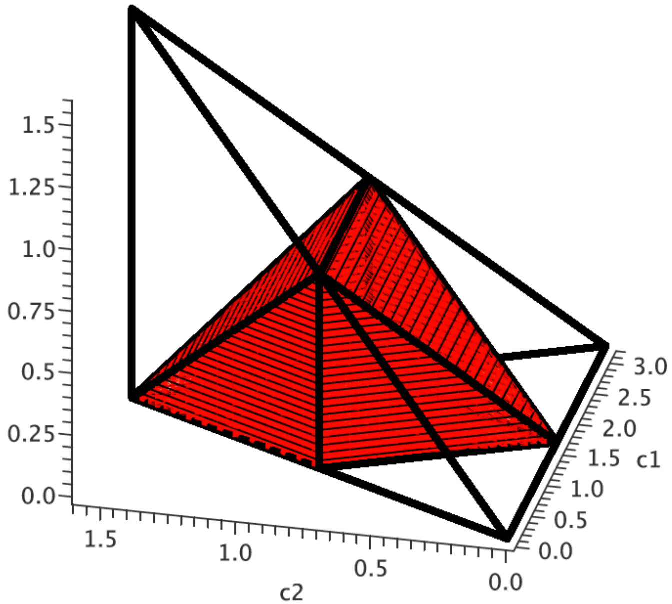


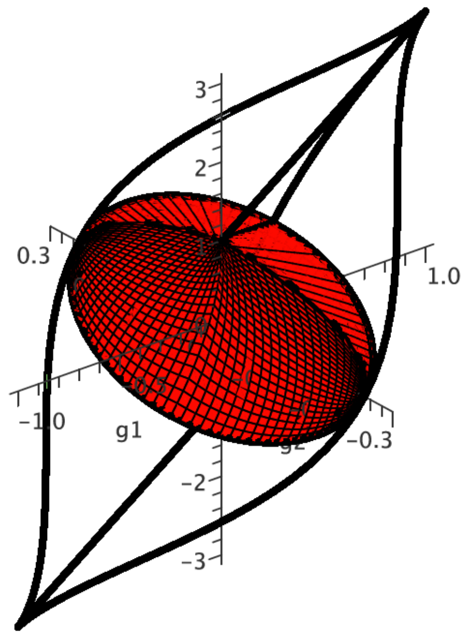
Invariant volume of perfect entanglers

The perfect entanglers occupy a half of the volume of the Weyl chamber but the invariant volume of the perfect entanglers in the full SU(4) is the integral

$$V(\mathcal{U}) = \int_{(SU(2)\otimes SU(2))\times(SU(2)\otimes SU(2))\times\mathcal{U}} \mathrm{d}\mu = \int_{\mathcal{U}} \mathrm{d}\mu_{\mathcal{A}}$$

over the perfect entanglers in the Weyl chamber



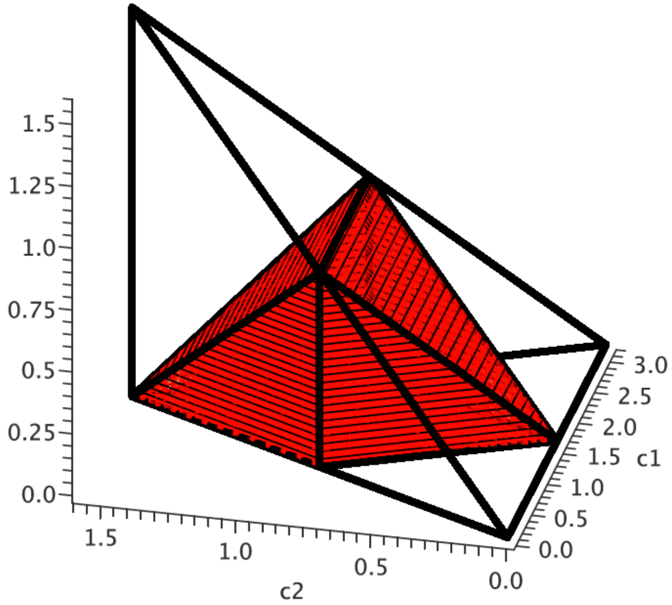


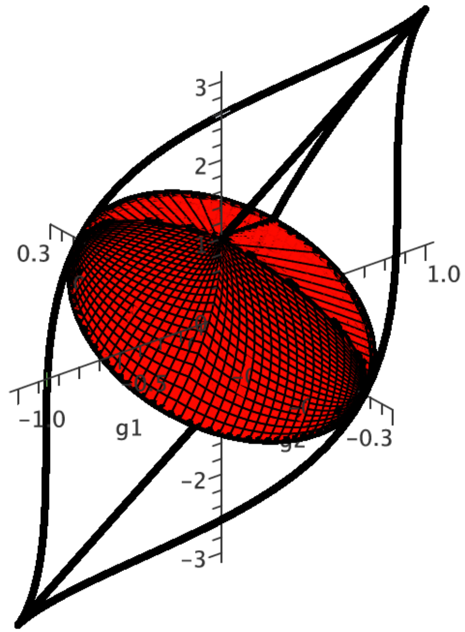
Invariant volume of perfect entanglers

The perfect entanglers occupy a half of the volume of the Weyl chamber but the invariant volume of the perfect entanglers in the full SU(4) is the integral

$$V(\mathcal{U}) = \int_{(SU(2)\otimes SU(2))\times(SU(2)\otimes SU(2))\times\mathcal{U}} \mathrm{d}\mu = \int_{\mathcal{U}} \mathrm{d}\mu_{\mathcal{A}}$$

over the perfect entanglers in the Weyl chamber





Perfect entanglers occupy over 84% of the total volume of SU(4)

Targeting perfect entanglers as clay pigeon shooting

Conclusions

Geometric theory of two-qubit gates

- provides powerful representation of two-qubit local equivalence classes;
- allows insights into structure and properties of perfect entanglers;
- gives intuitive picture of two-qubit quantum evolution;
- enables analytical construction of two-qubit quantum circuits;
- leads to new gates (B – gate) and implementations.

Optimal control applications

- relaxing constraints on the optimization target relaxes constraints on physical interactions, optimization process and implementation;
- optimization to a given local equivalence class converges faster and more reliably;
- optimization to the set of perfect entanglers promises to maximize entanglement generation, preliminary results are quite encouraging

Metric properties

- derived expressions for the invariant length element and volume in the representation particularly suitable for quantum information processing;
- true size of optimization targets; the largest in the center of the Weyl chamber;
- perfect entanglers are (almost) everywhere!

K. Birgitta Whaley
Shankar Sastry
Jun Zhang

Jiri Vala
Paul Watts
Maurice O'Connor

Freie Universität Berlin

Christianne Koch
Daniel Reich

ulm university universität ulm

Tommaso Calarco
Matthias Mueller

Haidong Yuan

sfi science foundation ireland

ICHEC Irish Centre for High-End Computing

IRCSET