Control of Complex Quantum Systems
February 19, 2013

Advances in Dynamical Quantum Error Suppression

Lorenza Viola
Dept. Physics & Astronomy
Dartmouth College

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Dartmouth College

'Quantum Firmware Collaboration'

Michael J. Biercuk
U. Sydney

Amir Yacoby
Harvard

[Hendrik Bluhm
U. Aachen]

The premise: DQES

Dynamical Quantum Error Suppression = Non-dissipative QEC:
Open-loop Hamiltonian engineering based on unitary control operations.

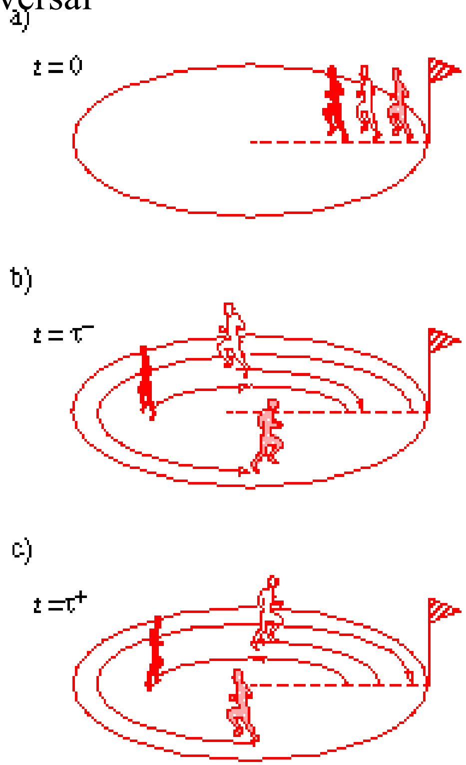
Simplest setting: Multipulse decoherence control for quantum memory ⇒ Dynamical Decoupling

LV & Lloyd, PRA

Key principle: Time-scale separation ⇒ Coherent averaging of interactions

Paradigmatic example: Spin echo ⟷ Effective time-reversal

Hahn 1950.



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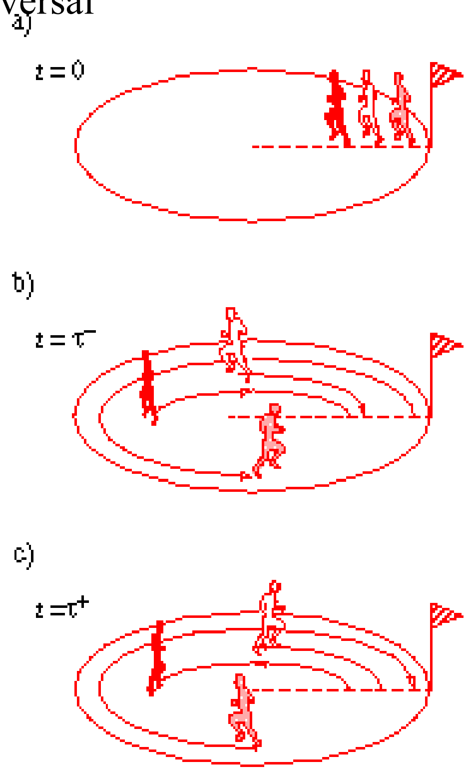


Key features: Non-Markovian open quantum system dynamics

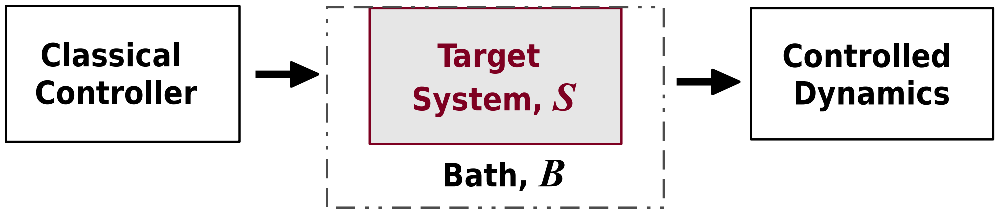
(1) Error component may include coupling to *quantum* bath

(2) Error suppression is enforced *perturbatively*

$$\frac{\tau_{control}}{\tau_c} \sim \omega_c \tau_{control}$$ *small* parameter

(3) Error suppression/control synthesis are achieved *without* requiring quantitative knowledge of error sources [basic difference wrto optimal control theory approaches]

Control-theoretic setting



- Target system exposed to noise due to a *quantum or classical* environment /bath

$$H = \overbrace{[H_{S,g} + H_{S,err}]}^{\text{Pure-system}} \otimes \boldsymbol{I}_B + \boldsymbol{I}_B \otimes \overbrace{H_B}^{\text{Pure-bath}} + H_{SB} \equiv \sum_a S_a \otimes B_a$$

  - System operators $\{S_a\}$ form Hermitian operator basis, with $S_0 = \boldsymbol{I}_S$ *and* $S_{a \neq 0}$ *traceless.*
  - Bath operators $\{B_a\}$ are bounded but otherwise arbitrary [possibly *unknown*].
    Classical bath limit [stochastic field] is formally recovered for $B_\alpha = \boldsymbol{I}_S$ and $B_0 = 0$.

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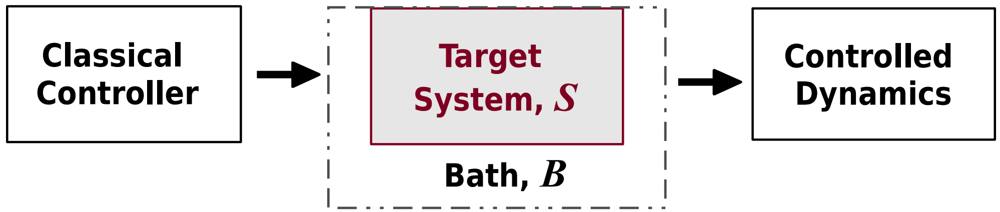

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    Classical bath limit [stochastic field] is formally recovered for $B_\alpha = \boldsymbol{I}_S$ and $B_0 = 0$.

- Environment B is *uncontrollable*. Controller acts on *system* only,

$$H_{tot}(t) \equiv H + H_{ctrl}(t), \qquad H_{ctrl,\boldsymbol{0}}(t) = \sum_m \left(H_m \otimes \boldsymbol{I}_B\right) h_m(t) \quad \longleftarrow \text{Control inputs}$$

$$U_0(t) = T\exp\left\{-i\int_0^t ds\,\left[H_{ctrl,0}(s) + H_{S,g}\right]\right\} \quad \longleftarrow \text{'Toggling frame' propagator}$$

  - Universal control on S may or may not require a non-zero pure-system [*drift*] Hamiltonian.
  - Control capabilities are typically restricted, and themselves *noisy* [systematic + random].

DQES: Overview and challenges

- DQES theory has diversified into a [still growing...] number of related directions – Broadly categorized based on control objective:

  1. Arbitrary state preservation ⇒ DQES theory for quantum memory
     - ✔ Pulsed dynamical decoupling – 'Bang-bang' limit/instantaneous pulses
     - ✔ Pulsed dynamical decoupling – Bounded control/'fat' pulses
     - ✔ Continuous time-dependent modulation

  2. Quantum gate synthesis ⇒ DQES theory for quantum computation
     - ✔ Hybrid DD-QC schemes – BB resources w or w/o encoding
     - ✔ Dynamically corrected gates – Bounded control only
     - ✔ Optimal control/convex optimization approaches

- DQES has been validated in a variety of proof-of-principle experiments in different systems – Emerging method of choice for resource-efficient physical-layer decoherence control...

Recent DD/DCG experiments [partial list...]

YEAR	PROTOCOL	QUBIT SYSTEM	NOISE
2009	CPMG, UDD, LODD	Be^+ trapped ion	Engineered phase noise
	CPMG, UDD	ESR in crystals	Natural dephasing
	XY4	Polarization qubit	Engineered depolarization
2010	CPMG, UDD	Atomic ensemble	Collisional dephasing
	CPMG, UDD, CDD	Double QD	Spin-bath dephasing
	CPMG, UDD, PDD	NV-center	Spin-bath dephasing/pulse errors
	CPMG, XY4	NV-center	Spin-bath dephasing
	CPMG, XY4, CDD	ESR in Si:P	Spin-bath dephasing/pulse err
2011	Hahn, CPMG	Double QD	Spin-bath dephasing
	Hahn, CPMG	NV-centers	Spin-bath dephasing
	CPMG, UDD	Ca^+ trappedions	Natural dephasing
	CPMG, XY4, CDD	Solid-state NMR	Spin-bath dephasing
	CPMG, UDD, PDD	Flux qubit	Natural phase noise
2012	CPMG	Rare-earth crystal	Spin-bath dephasing
	Hahn, XY4	ESR in Si:P	Spin-bath/defect dephasin
	DCGs	Be trapped ion	Laser-frequency jitter

⋮

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- DQES has been validated in a variety of proof-of-principle experiments in different systems – Emerging method of choice for resource-efficient physical-layer decoherence control...

- Going *beyond* proof-of principle [inevitably] entails more 'complex' control scenarios ⇒

Key challenge:
To systematically address and incorporate practical [*system and control*] constraints.

Sample Problem 1: High-fidelity Long-time Low-latency Quantum Memory

Khodjasteh, Sastrawan, Hayes, Green, Biercuk & LV,
Nature Commun., submitted.

Setting and control objective

- Simplest DD scenario: Single purely-dephasing qubit controlled by perfect π pulses
  - DD sequence of duration T_p specified in terms of pulse pattern $p = \{t_j\}, j = 1,\ldots,n$
  - Qubit coherence decays as $e^{-\chi_p}$, with DD error at $t = T_p$ determined by spectral overlap

$$\chi_p = \int_0^{\infty} \frac{S(\omega)}{2\pi\omega^2} F_p(\omega) d\omega$$

Power spectrum x **Filter Function (FF)**

$$S(\omega) \propto \omega^s f(\omega, \omega_c) \qquad F_p(\omega) \propto (\omega\tau)^{2(\alpha_p+1)}$$

$$\tau \equiv min(t_{j+1} - t_j)$$ **Minimum 'switching time'**

- The larger the order of error suppression α_p, the higher the degree of error cancellation, as long as $\omega_c\tau$ is perturbatively small.

Uhrig, PRL **98** 2007; Cywinski&al, PRB **77** (2008);
Khodjasteh, Erdelyi & LV, PRA **83** (2011).

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- Goal: Achieve high fidelity over desired storage time T_s $\Rightarrow$ Straightforward *in principle...*
  - Simply use high-order DD sequence with $T_s = T_p$, e.g. Uhrig DD, $\tau \equiv t_1 = O(T_p/n^2)$

Quantum memory requirements

- In *practical* quantum-memory applications, high fidelity should be delivered while

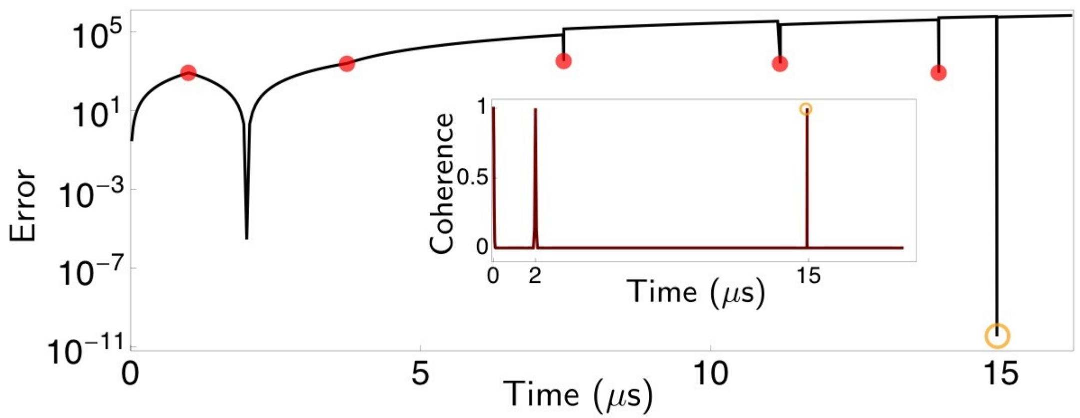
  (1) allowing arbitrarily long storage time T_s;
  (2) minimizing latency for information retrieval;
  (3) operating under technological [timing and sequencing] control limitations.

- Interconnected issues to address:

  - Perturbative DD (CDD, UDD...) is *not viable* if min switching time is constrained – for fixed $\tau > 0$, a max storage time exists, beyond which increasing α_p no longer helps...

  - Numerical DD (BADD...) is *not viable* – search complexity grows exponentially with T_s...

  - Mid-sequence interruptions are *not permitted* – min access latency is set by T_p...



UDD5
GaAs S-T spin qubit

$$S(\omega) \propto \omega^{-2} e^{-\omega^2/\omega_c^2}$$

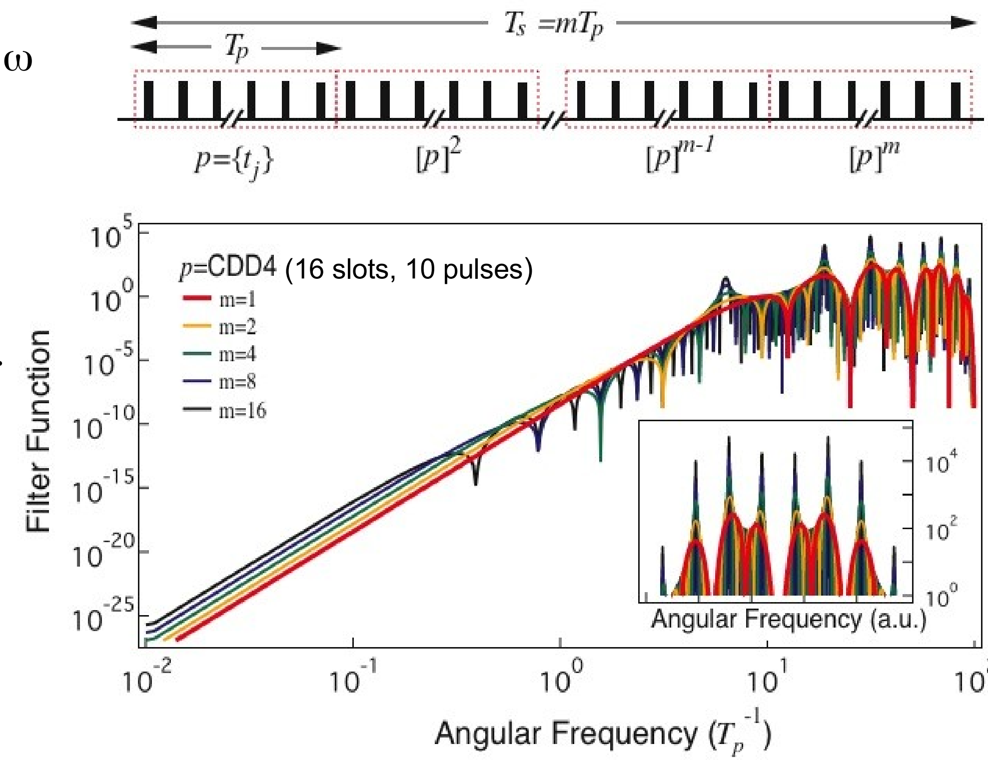
$$\omega_c / 2\pi = 10\,kHz$$

$$T_2^* \approx 35\,ns\,,\ \tau = 1\,\mu s$$

Quantum memory via repetition

- <u>Strategy</u>: Periodically repeat a high-order DD sequence

$$\chi_{[p]^m} = \int_0^{\infty} \frac{S(\omega)}{2\pi\omega^2} \frac{\sin^2(m\omega T_p/2)}{\sin^2(\omega T_p/2)} F_p(\omega)\, d\omega$$



- Long-time coherence determined by α_p, s, and high-frequency contributions at 'resonating frequencies' $\omega_{res} \equiv k\ 2\pi/mT_p$.

- Assume a *hard spectral cutoff*, then

$$\chi_{[p]^\infty} = \int_0^{\omega_c} \frac{S(\omega)}{4\pi\omega^2} \frac{F_p(\omega)}{\sin^2(\omega T_p/2)}\, d\omega$$

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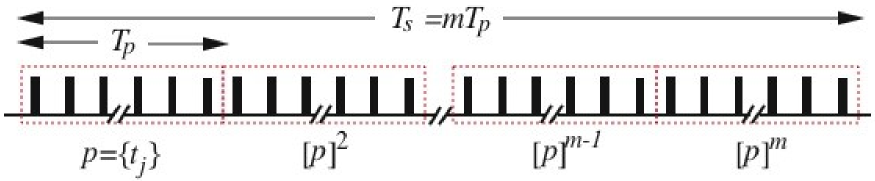

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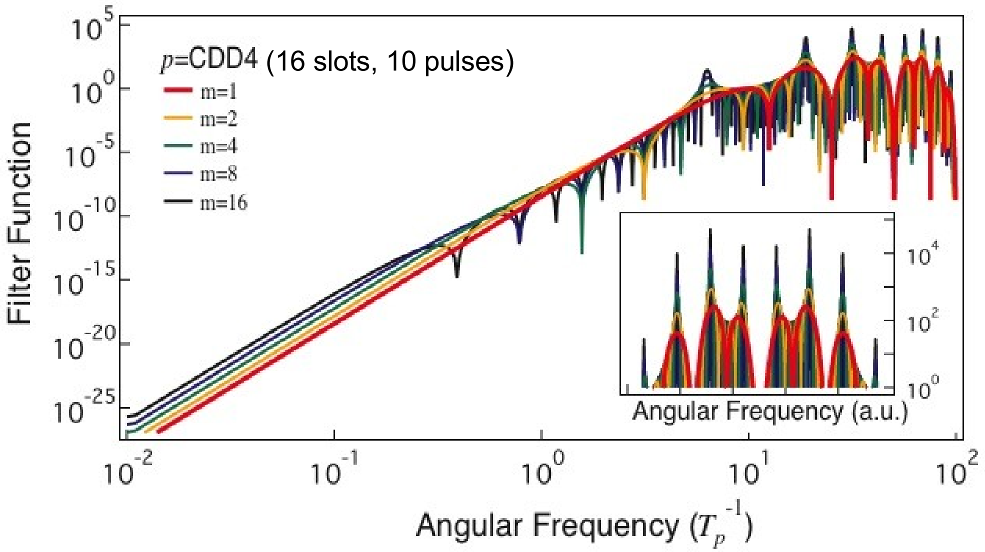


- A *coherence plateau* may be engineered by judicious selection of a base sequence:

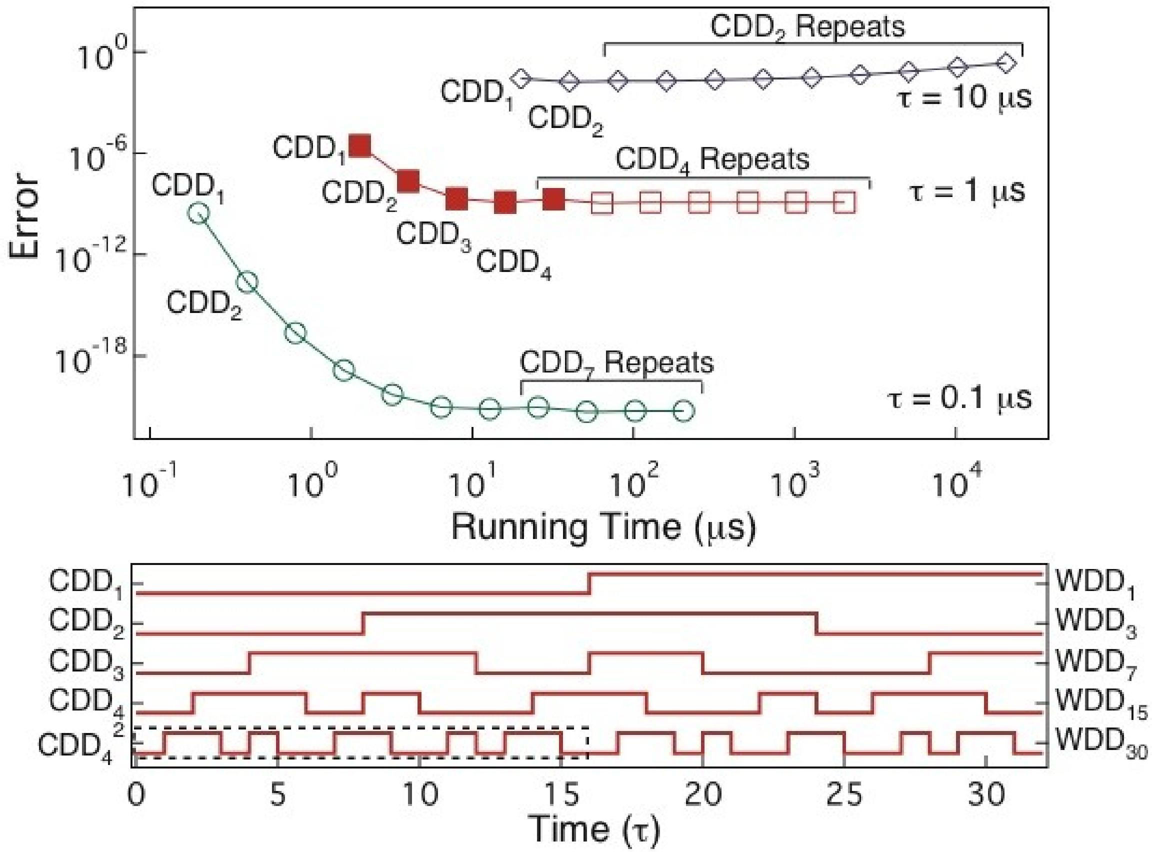
$$s + 2\alpha_p > 1, \quad \omega_c T_p < 2\pi$$

- Guaranteed high fidelity throughout long storage times, with latency capped at $T_p \ll T_s$.

Quantum memory via repetition: Results

- A direct search up to $t = T_s$ is viable by restricting to *digital* DD sequences in the 'Walsh family'
  - Minimize *sequencing complexity*. Number of Walsh DD sequences: $(1/2)T_s/\tau$ vs. $2^{T_s/\tau}$

Hayes, Khodjasteh, LV & Biercuk, PRA **84** (2011).



- A periodic sequence structure emerges *naturally* for sufficiently long storage time.

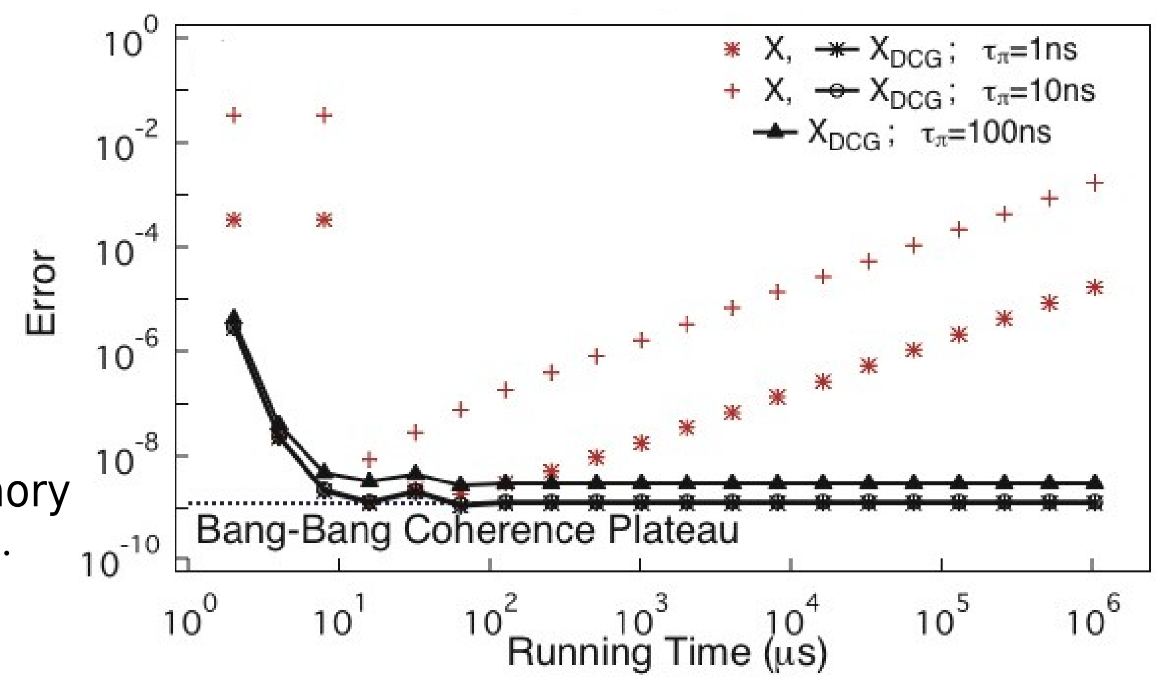
Realistic effects: Pulse errors and soft cutoffs

- In reality, pulses are imperfect, noise need not have a hard cutoff nor be purely dephasing...
  A coherence plateau with finite yet exceptionally long duration may still be engineered.

  - Pulse-length errors may be included using *multi-axis FF formalism*. New plateau conditions:

$$s + 2\alpha_p > 1, \quad s + 2\alpha_{pul} > 1, \quad \omega_c T_p < 2\pi$$

  - Coherence plateau can be restored by replacing 'primitive' with 'error-corrected' pulses.
    Storage times in excess of $1s$ $(\approx T_1)$ at plateau error rates of 10^{-9} with realistic noise spectrum



- Results extend to *multi-qubit* memory under independent dephasing noise.

Sample Problem 2: Automated Dynamically Corrected Quantum Gates

Khodjasteh, Bluhm & LV, *Phys. Rev. A* **86**, 042329 (2012).

Setting and control objective

- <u>Goal:</u> To suppress evolution due to unwanted Hamiltonians, $H_{err} = H_{S,err} + H_B + H_{SB} + H_{ctrl,err}$
  - Intended evolution ⇒ Ideal gate propagator over duration T:

$$U^{ideal}(T) = U_0(T) \otimes \boldsymbol{I}_B \equiv Q \otimes \boldsymbol{I}_B = T\exp\left\{-i \int_0^T ds \left[H_{ctrl,0}(s) + H_{S,g}\right] \otimes \boldsymbol{I}_B\right\}$$

  - Actual evolution ⇒ Total gate propagator over duration T:

$$U^{actual}(T) = T\exp\left\{-i \int_0^T ds \left[H_{ctrl}(s) + H_{S,g} + H_{err}\right]\right\} \equiv Q \exp\left(-i E_{Q[T]}\right)$$

$$\exp\left(-i E_{Q[T]}\right) = T\exp\left\{-i \int_0^T ds \left[U_0(s)^\dagger H_{err} U_0(s)\right]\right\}$$

Error action operator

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$$\exp(-i E_{Q[T]}) = T\exp\{-i \int_0^T ds \, [U_0(s)^\dagger H_{err} U_0(s)]\}$$

Error action operator

- **Fact:** The norm of the error action operator 'modulo pure-bath terms' upper-bounds the distance [fidelity loss] between the intended and the actual system evolution:

Lidar, Zanardi & Khodjasteh, PRA **78** 2008;
Khodjasteh & LV, PRL **102** 2009

$$mod_B(E) \equiv E - \frac{\boldsymbol{I}_S}{d_S} \otimes trace_S(E)$$

$$1 - f_{Uhlman}(T) \leqslant \| \rho_S^{ideal}(T) - \rho_S^{actual}(T) \|_1 \leqslant \| mod_B(E_{Q[T]}) \|_{op} \equiv \text{EPG}$$

   - Fidelity loss in DQES is reduced by *minimizing EPG.*

DCG synthesis ⇔ Seek control modulation such that effect of H_{err} is perturbatively canceled.

Analytical DCGs: Assumptions

- System assumptions: Driftless – System Hamiltonian is zero/not needed for complete control,

$$H_{S,g} = 0,\ H_S \equiv H_{S,err} \Rightarrow \exp(-i E_{Q[T]}) = T\!exp\{-i \int_0^T ds\, [U_0(s)^\dagger H_{err} U_0(s)]\}$$

$$U_0(t) = T\!exp\{-i \int_0^t ds\ H_{ctrl,0}(s)\}$$

- Control assumptions: Access to universal set of 'primitive' control Hamiltonians – e.g.

$$\{h_x(t)\sigma_x^{(i)},\ h_y(t)\sigma_y^{(i)},\ h_{zz}(t)\sigma_z^{(i)} \otimes \sigma_z^{(j)}\},\quad i,j = 1,\ldots,N,$$ subject to

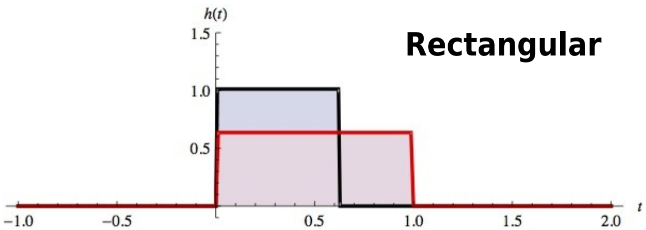
(C1) Finite-power and bandwidth constraints – *Bounded* amplitude, fixed *minimum* gate duration

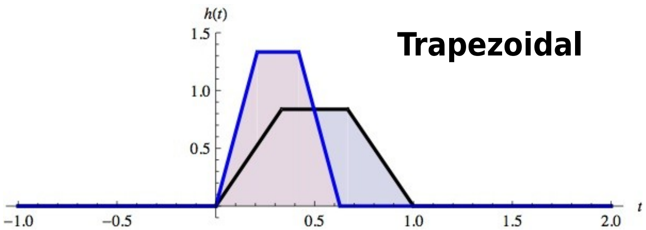
(C2) Perfect control – Bath coupling is the *only* error source

(C3) Stretchable control profiles – *Same* primitive gate achievable with different 'speeds'

$$h(t) \rightarrow \frac{1}{r} h\left(\frac{t}{r}\right)$$

$$E_{Q[T]} \rightarrow r\ E_{Q[rT]}$$





Stretching gives controllable relationship between EPGs of different gat

Analytical DCGs: Constructions

- If target gate $Q = I$ (NOOP), a solution is given by Eulerian DD. LV & Knill, PRL **90** 2003.

  To effect non-trivial Q, identify two combinations of primitive gates that have *same* [first-order] EPG

$$Q_* = Q \exp(-i E_Q), \quad I_Q = \exp(-i E_Q)$$

[First-order] 'balance pair'

- Modified Eulerian construction: Implement control path starting at I on *'augmented' Cayley graph* $\Rightarrow$

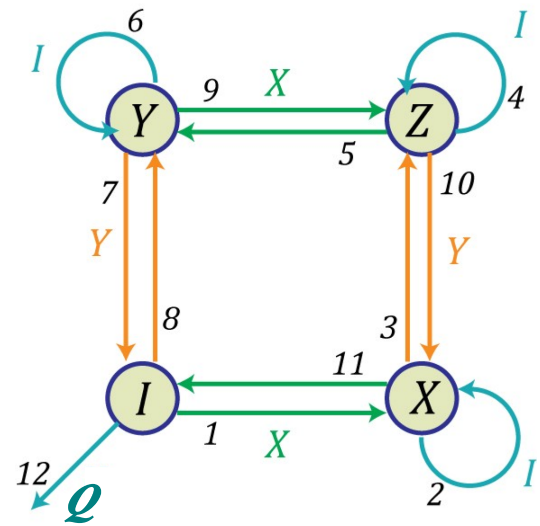
  (i) To non-identity vertex, attach edge labeled by I_Q

  (ii) To identity vertex, attach edge labeled by Q_*

$$E_{DCG} = E_{EDD} + \sum_{i=1}^{G} U_{g_i}^{\dagger} E_Q U_{g_i} + E_{DCG}^{[2+]}$$

Total first-order error vanishes as long as the primitive gate errors and E_Q obey DD condition $\Rightarrow$

$$\| mod_B(E_{DCG}) \| = \| mod_B(E_{DCG}^{[2+]}) \| = O(\|H_{err}\|^2)$$



Euler path: $X \; I \; Y \; I \; X \; I \; Y \; Y \; X \; Y \; X \; Q$

Khodjasteh & LV, PRL **102** 2009; PRA **80** 2009

- Significantly smaller error compared to 'direct switching'.
- Higher-order cancellation achievable by concatenation.

Khodjasteh, Lidar & LV, PRL **104** 2010.

DCGs in the lab: Trapped ions

Hayes, Khodjasteh, LV & Biercuk
Hayes et al

- Recently implemented MØlmer-Sørensen composite gate sequences can be interpreted as DCGs under a simple error model...

$$U_Q(t) = \exp[S_N(\alpha(t)a^\dagger - \alpha(t)^* a)]\,Q\,, \quad Q = \exp[-i\Phi(t)\,S_N^2]$$ Spin-dependent gate

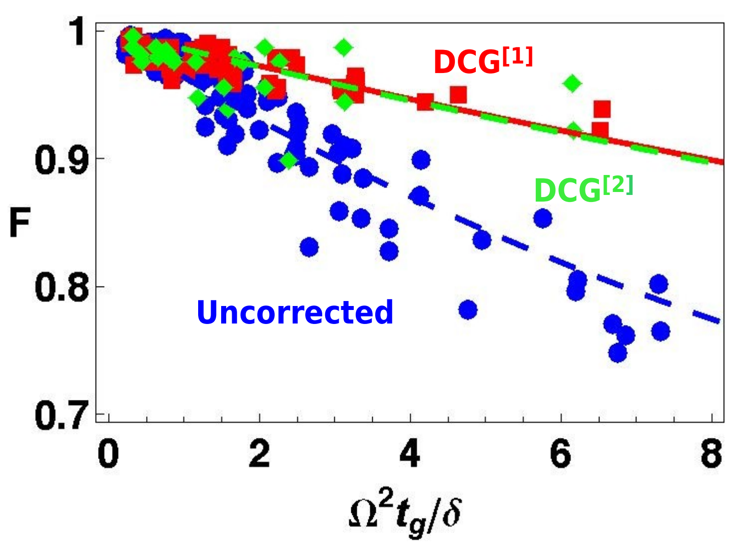
$$\alpha(t) = \frac{\Omega}{2}\int_0^t \exp[-i(\delta+\Delta)s]ds$$ Δ = Detuning error $\ll \delta$

Gate relies on disentangling spin and motional degrees of freedom at $t_g = j2\pi/\delta$ =
Residual spin-motional entanglement results in error action $-iE_Q(t) = S_N(\alpha(t)a^\dagger - \alpha(t)^* a)$

- Key simplifications:
  (1) Target gate commutes with spin-flips (X gate);
  (2) Error action anti-commutes with X gate ⇒

$$X\,Q\exp(-iE_Q)\,X\,Q\exp(-iE_Q) =$$
$$Q^2\,X\exp(-iE_Q)\,X\exp(-iE_Q) = Q^2$$

- First-order implementation of gate Q^2 ⇒
  Iterate to achieve higher-order suppression...



Automated DCGs in a nutshell

- Problem: Control requirements are still *too stringent* for many lab settings
  - Control fields are themselves imperfect/noisy...
  - Stretchable control profiles need not be available...
  - Complete control often relies on internal system Hamiltonian...

Analytic DCG constructions become ineffective (

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- Goal: To synthesize 'aDCGs' that cancel *both* [non-Markovian] decoherence *and* control errors, without relying on stretching and allowing for/exploiting internal drift.

Strategy: Relax portability and exploit control knowledge to achieve [non-competing] objectives of gate synthesis and error cancellation ⇒ multi-objective minimization problem:

$$F(\{x_i\}) = dist\left(Q\,, Texp\{-i\int_0^T ds\,[H_{ctrl,0}(s) + H_{S,g}]\}\right)$$

Primitive gate synthesis

$$B_j G_j(\{x_i\}) = \left\| \partial\, mod_B\left(E^{(\vec{\delta})}_{Q[T]}\right)/\partial\delta_j \right\|_{\vec{\delta}=0}$$

[First-order] error cancellation/ 'sensitivity minimization'

- Solving for $F = 0 = G_j$, for all j, yields gate implementation that is insensitive to the error parameters δ_j ⇒ Robust control solution as long as errors are perturbatively small.
- Complete controllability of target system essential to justify existence of aDCG solution

Case study: Single-triplet spin qubit

- Model Hamiltonian in the logical singlet-triplet basis $\{|S\rangle, |T_0\rangle\}$:

$$H(t) = [B + \delta B(t)] \frac{\sigma_x}{2} + J(t) \frac{\sigma_z}{2}, \qquad J(t) = J_0(t)[1 + \delta J(t)]$$

Known static magnetic field gradient [drift]

Hyperfine-induced error Hamiltonian [additive noise]

Exchange splitting control, subject to [multiplicative] voltage noise

  - *Quasi-static Gaussian approximation* is an adequate starting point for both noise sources:

$$\sqrt{\langle \delta B^2 \rangle} \equiv \sigma_{\delta B}, \quad \sigma_{\delta B/2\pi} \leq 0.15 \text{ MHz}, \qquad \sqrt{\langle \delta J^2 \rangle} \equiv \sigma_{\delta J}, \quad \sigma_{\delta J} \leq 0.02$$

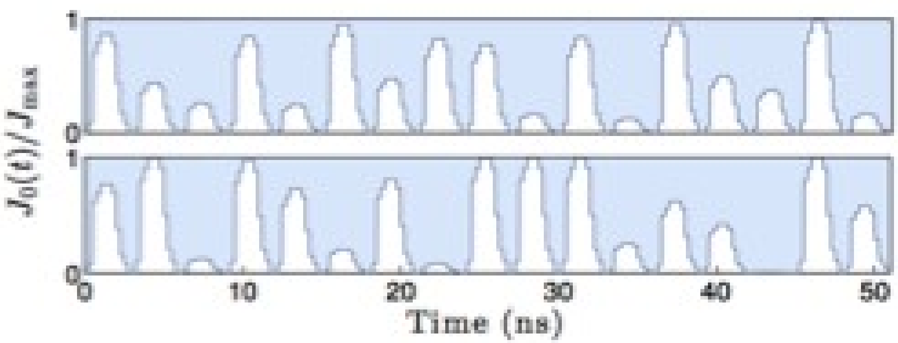
Foletti et al, Nature Phys. 2009; Bluhm et al, PRL 2010.

  - Zeeman drift term is crucial for universality. Available exchange control is constrained in magnitude and sign [stretching *not* an option!]:

$$B/2\pi \in [0.03, 0.2] \text{ GHz}, \qquad 0 < J_0/2\pi \leq J_{max}/2\pi = 0.3 \text{ GHz},$$

- Each aDCG consists of a sequence of n pulses characterized by a *fixed* [digitized] profile with *fixed* duration, τ = 3 ns, compatible with current horizontal temporal resolutions.
  - Control variables = Pulse amplitudes, $\{x_i\} \equiv \{h_i\}$. Error sources 'marked' by $(\delta B, \delta J)$.
  - Weighted objective function: $O(\{h_i\}) \equiv F + \lambda_1 G^{(\delta B)} + \lambda_2 G^{(\delta J)}$

- aDCG synthesis: Search problem solved by off-the-shelf Matlab routines (FMINCON)



$Q = \pi/8$ gate $B/2\pi = 0.1$ GHz, $J_{max} = 0.3$ GHz,

$Q =$ Hadamard gate

ADCG implementations using $n = 17$ pulses

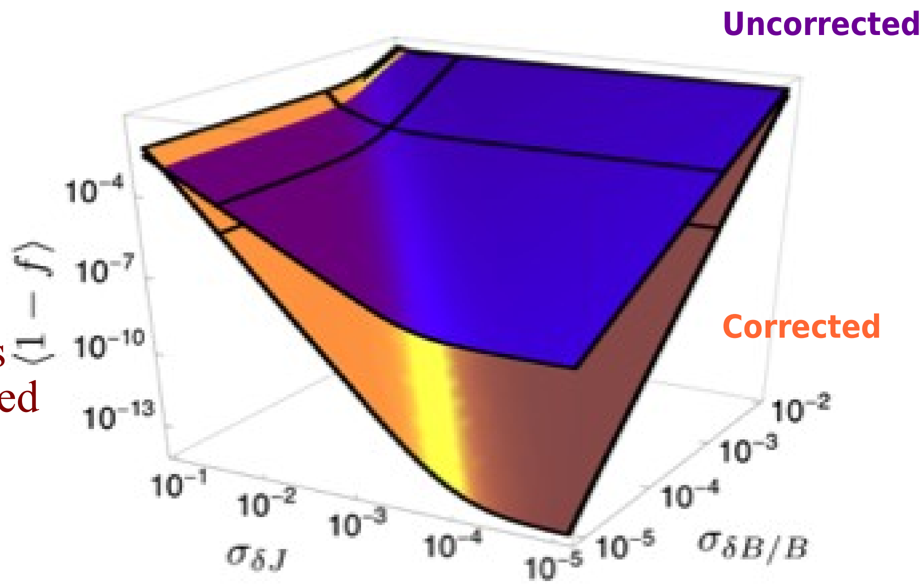
- Same aDCG sequence applies to *fully quantum* spin-bath model.

- aDCG performance: Evaluate average fidelity

$$\langle 1-f \rangle = 1 - \frac{1}{2} \left\langle \left| Tr\left(Q^{\dagger} Q_{DCG}^{(\delta B, \delta J)}\right) \right| \right\rangle_{\delta B, \delta J}$$

- aDCGs are simultaneously robust against [non-Markovian] gradient field fluctuations and voltage noise to leading order provided

$$\sigma_{\delta B/B} + \sigma_{\delta J} \lesssim 0.1$$



Conclusion and outlook

- DQES has the potential to reduce memory and gate errors below the level required by accuracy threshold for non-Markovian quantum error correction.
  - Systematic/quantitative comparison between DCGs and *composite pulses*?...

    Kabytayev, Green, LV, Biercuk and Brown, in preparation.
  - DQES with *continuous* driving fields?...

    Fanchini & al, arXiv:1005.1666; Chaudhry & Gong, arXiv:1110.4695; Jones, Ladd & Fong, arXiv:1205.2402.

- Plenty of room exists for tailoring and/or improving DQES constructions and for optimizing performance under specific system and/or control assumptions.
  - ✔ Single-qubit setting:
    - Complexity/convergence/landscape of aDCG solution... Impose *time-optimality* on top?
  - ✔ Many-qubit setting:
    - Better exploitation of *locality and sparsity* of physical error models...

      De & Pryadko, arXiv:1209.2764.
    - Impact and role of *correlation* effects... DQES for *noise spectroscopy*?

- Dedicated experimental realizations/benchmarking of DQES schemes can continue to validate theoretical insights and identify key trade-offs/practical constraints to address.