Optimal transfer of population for quantum systems

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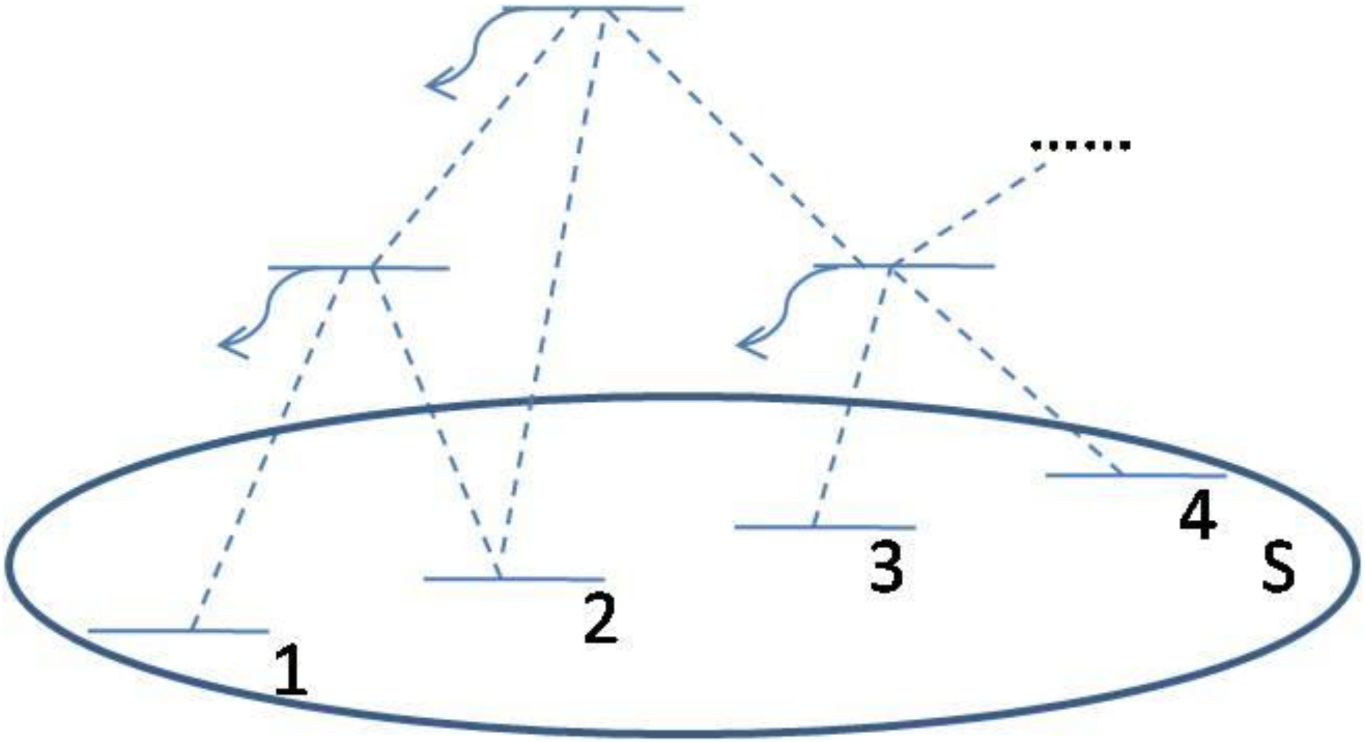
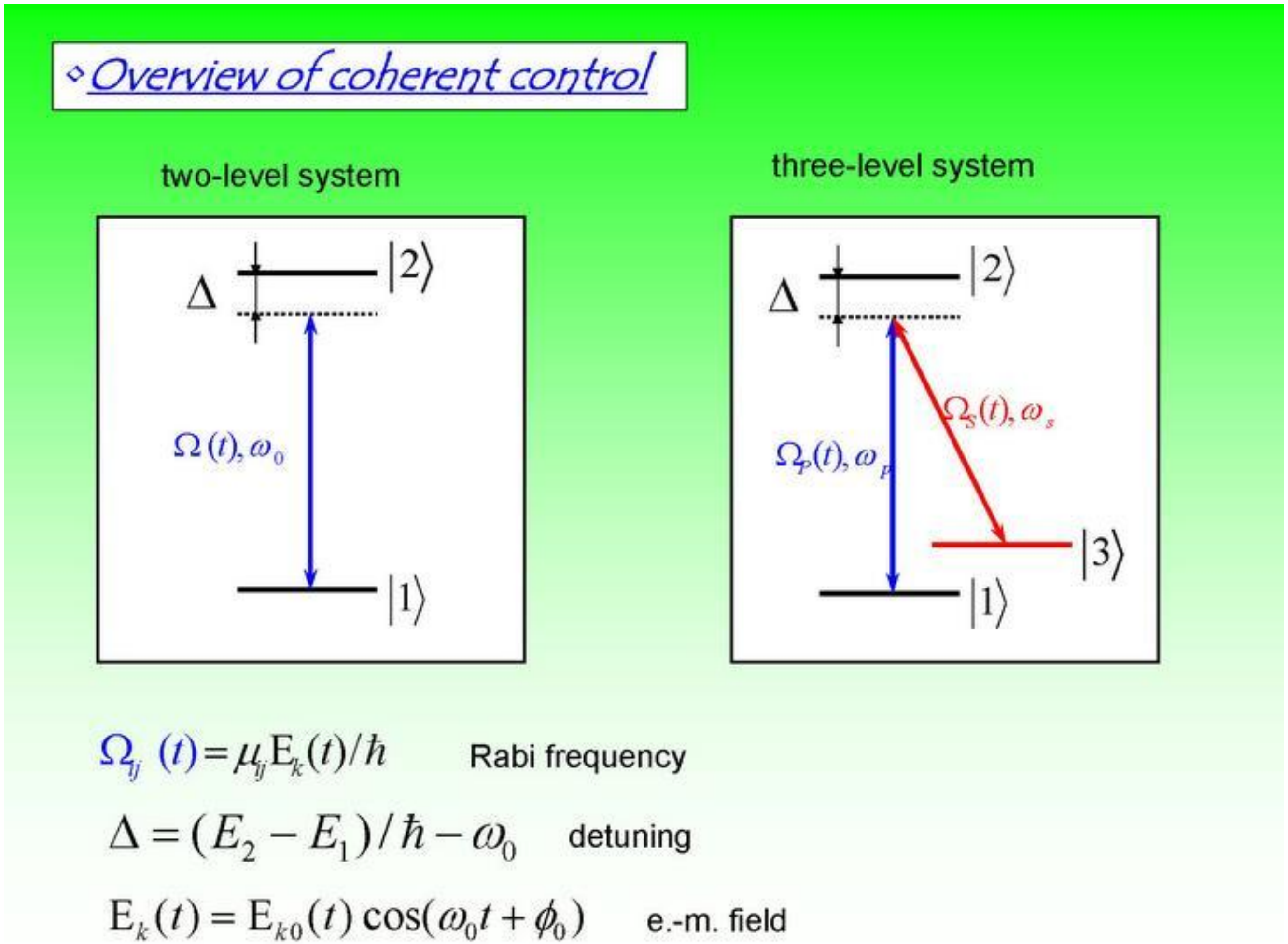
QUESTION



COHERENT CONTROL ON POPULATION TRANSFER



Adiabatic transfer

Adiabatic Basis, dressed states

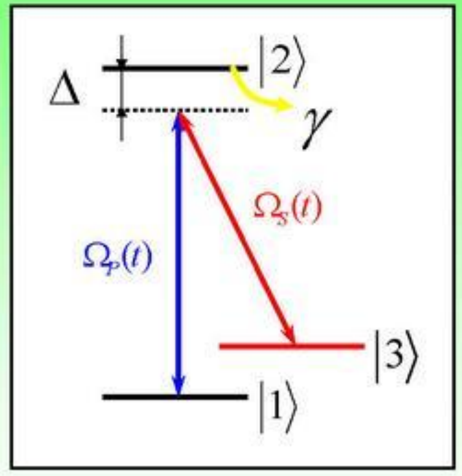
•Time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\Psi(t) = \hat{H}(t)\Psi(t)$$

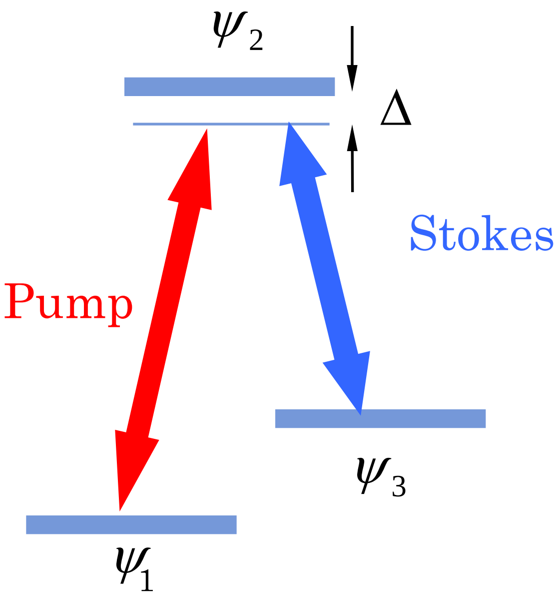
$$\Psi(t) = a_1(t)|1\rangle + a_2(t)|2\rangle + a_3(t)|3\rangle$$

In the rotating wave approximation(RWA)

$$\hat{H}(t) = -\frac{\hbar}{2}\begin{pmatrix} 0 & \Omega_P(t) & 0 \\ \Omega_P(t) & 2\Delta & \Omega_S(t) \\ 0 & \Omega_S(t) & 0 \end{pmatrix}$$



Three level atom- Λ configuration



$$H(t)=\frac{\hbar}{2}\begin{bmatrix}0 & \Omega_p & 0\\ \Omega_p & 2\Delta & \Omega_s\\ 0 & \Omega_s & 0\end{bmatrix}$$

$\phi_-(t),\phi_0(t),\phi_+(t)-$

adiabatic basis (eigenvectors of H(t))

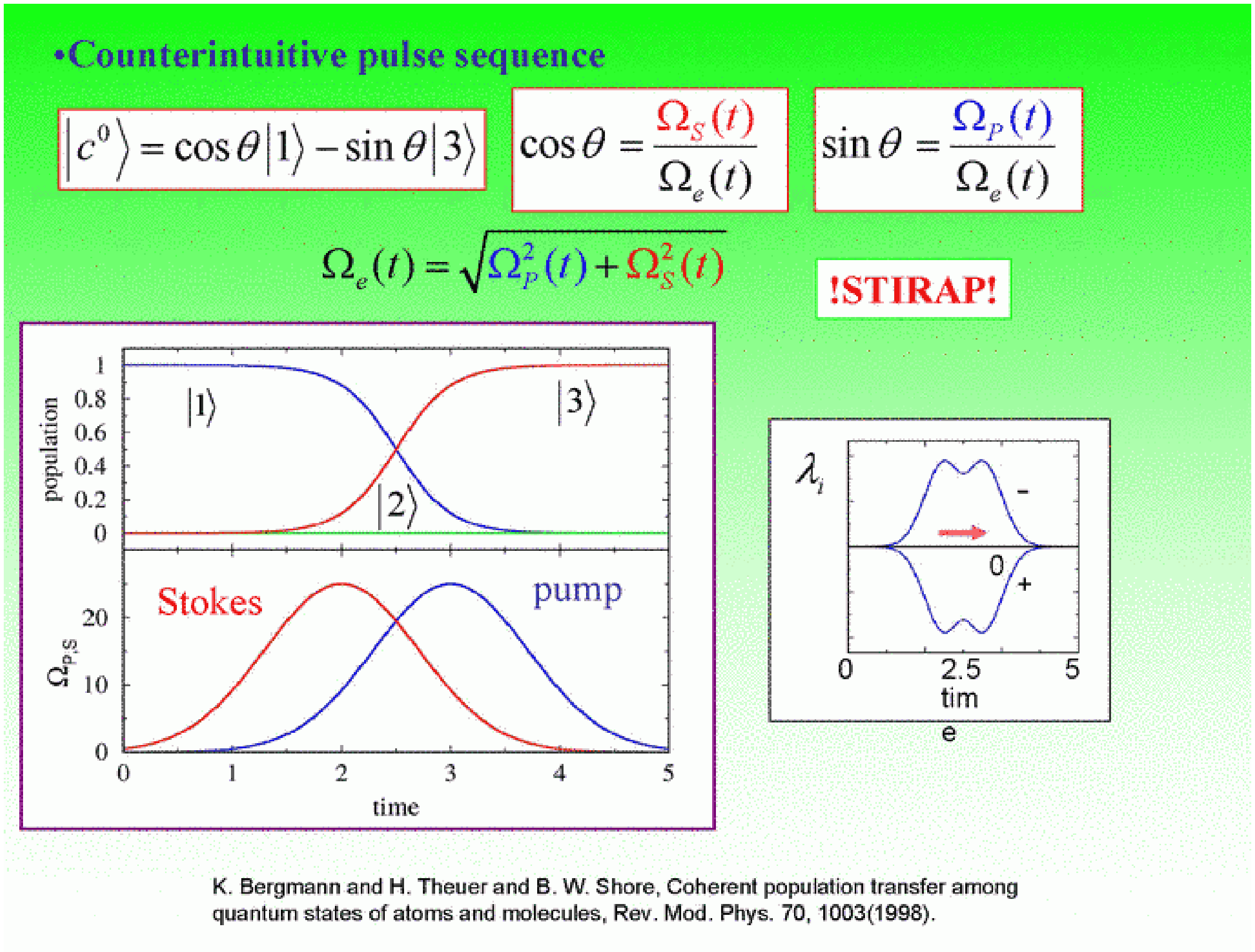
$$\phi_-(t)=\psi_1\sin\vartheta(t)\cos\varphi(t)-\psi_2\sin\varphi(t)+\psi_3\cos\varphi(t)\cos\vartheta(t)$$

$\phi_0(t)=\psi_1\cos\vartheta(t)-\psi_3\sin\vartheta(t)$ - dark state

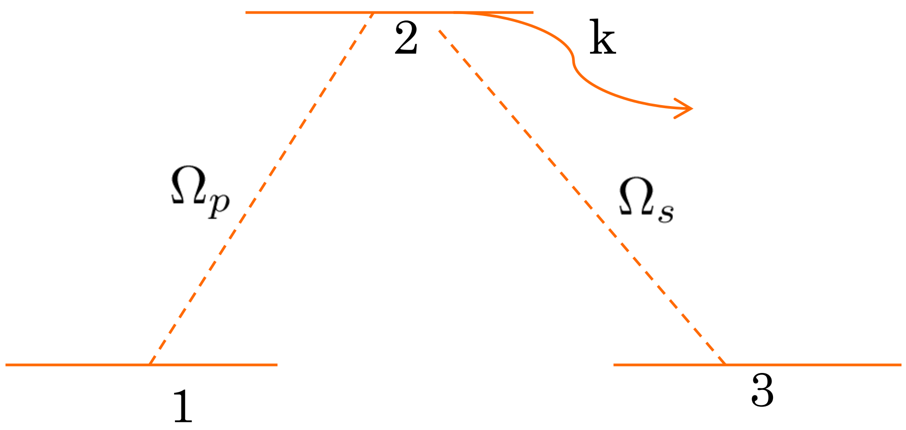
$$\phi_+(t)=\psi_1\sin\vartheta(t)\sin\varphi(t)+\psi_2\cos\varphi(t)+\psi_3\sin\varphi(t)\cos\vartheta(t)$$

$$\tan 2\varphi(t)=\frac{\sqrt{\Omega_p^2(t)+\Omega_s^2(t)}}{\Delta}\qquad \tan\vartheta(t)=\frac{\Omega_p(t)}{\Omega_s(t)}$$

STIRAP


•Counterintuitive pulse sequence
$|c^0\rangle = \cos\theta|1\rangle - \sin\theta|3\rangle$
$\cos\theta = \frac{\Omega_S(t)}{\Omega_e(t)}$
$\sin\theta = \frac{\Omega_P(t)}{\Omega_e(t)}$
$\Omega_e(t) = \sqrt{\Omega_P^2(t) + \Omega_S^2(t)}$
!STIRAP!
population
$|1\rangle$
$|2\rangle$
$|3\rangle$
Stokes
pump
$\Omega_{P,S}$
time
λ_i
-
0
+
0
2.5
5
tim
e
K. Bergmann and H. Theuer and B. W. Shore, Coherent population transfer among
quantum states of atoms and molecules, Rev. Mod. Phys. 70, 1003(1998).

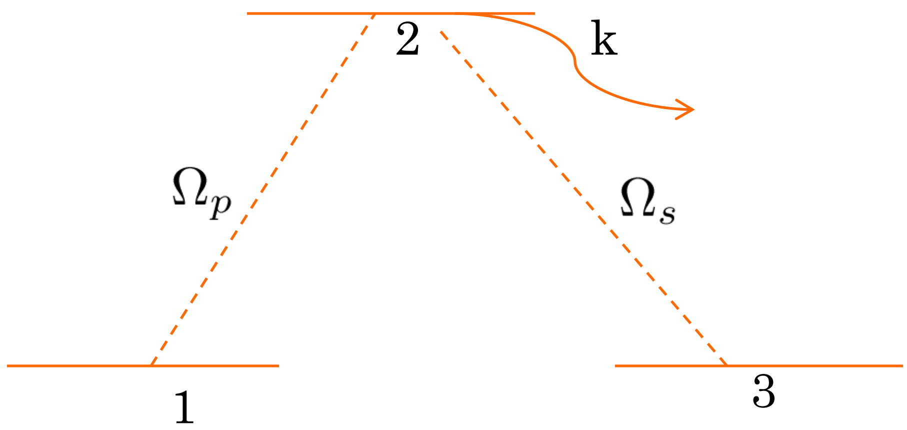
Population transfer in three level system



$$\frac{d}{dt}\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = -i\begin{pmatrix} 0 & \Omega_p & 0 \\ \Omega_p & -ik & \Omega_s \\ 0 & \Omega_s & 0 \end{pmatrix}\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}$$

$$|1\rangle \longrightarrow |3\rangle$$

Three level system



$$x_1 = x_1', \; x_2 = i x_2', \; x_3 = -x_3'$$

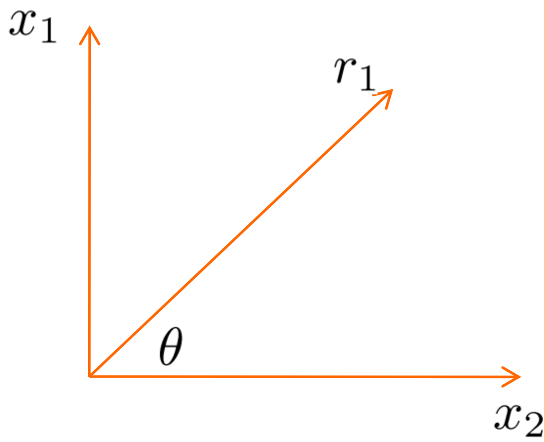
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & -\Omega_p & 0 \\ \Omega_p & -k & -\Omega_s \\ 0 & \Omega_s & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Initial $(1, 0, 0)$

Maximize $x_3(T)$

Optimal Control

$$r_1 = \sqrt{x_1^2 + x_2^2},\ r_2 = x_3$$



$$\frac{d}{dt}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -k\cos^2\theta & -\Omega_s\cos\theta \\ \Omega_s\cos\theta & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

$|\Omega_s| \leq A$ $\quad u = \cos\theta$ $\quad \theta \in [0, \frac{\pi}{2}]$

$$\frac{d}{dt}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -ku^2 & -Au \\ Au & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

Initial $(1, 0)$

Maximize $r_2(T)$

Optimal control

Denote the maximum achievable value of $r_2(T)$ by $V(r_1, r_2, t)$

$$V(r_1, r_2, t) = max_u V(r_1 + \delta r_1, r_2 + \delta r_2, t + \delta t)$$

$$\frac{\partial V}{\partial t} + \max_u H(u) = 0$$

$$H(u) = \frac{\partial V}{\partial r_1}\dot{r}_1 + \frac{\partial V}{\partial r_2}\dot{r}_2 = \begin{pmatrix} \frac{\partial V}{\partial r_1} & \frac{\partial V}{\partial r_2} \end{pmatrix} \begin{pmatrix} -ku^2 & -Au \\ Au & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

$$\lambda_1 = \frac{\partial V}{\partial r_1}, \lambda_2 = \frac{\partial V}{\partial r_2}$$

$$\frac{d}{dt}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} ku^2 & -Au \\ Au & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

Optimal control

$$H(u) = -k\lambda_1 r_1 \left\{ \left[u - \frac{A}{2k}\left(\frac{\lambda_2}{\lambda_1} - \frac{r_2}{r_1}\right)\right]^2 - \left[\frac{A}{2k}\left(\frac{\lambda_2}{\lambda_1} - \frac{r_2}{r_1}\right)\right]^2 \right\}$$

$$u = argmax_u H(u)$$

$$a = \frac{\lambda_2}{\lambda_1}, b = \frac{r_2}{r_1}$$

$$u = \begin{cases} 0 & \text{if } \frac{A}{2k}(a-b) < 0 \\ \frac{A}{2k}(a-b) & \text{if } 0 \leq \frac{A}{2k}(a-b) \leq 1 \\ 1 & \text{if } \frac{A}{2k}(a-b) > 1 \end{cases}$$

a-b ↗

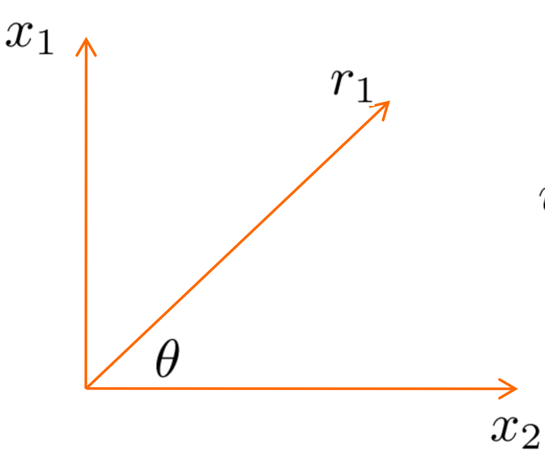
Optimal Control

$$u = \begin{cases} 0 & \text{if } \frac{A}{2k}(a-b) < 0 \\ \frac{A}{2k}(a-b) & \text{if } 0 \leq \frac{A}{2k}(a-b) \leq 1 \\ 1 & \text{if } \frac{A}{2k}(a-b) > 1 \end{cases}$$

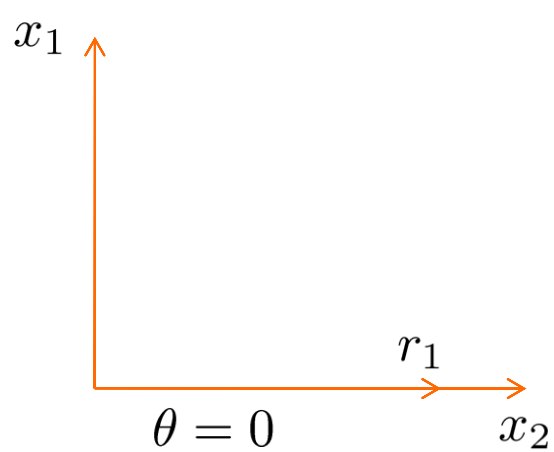
$$a = \frac{\lambda_2}{\lambda_1}, b = \frac{r_2}{r_1}$$

$$\frac{d}{dt}(a-b) = (a+b)u[A(a-b) - ku] \geq 0$$

a-b ↗



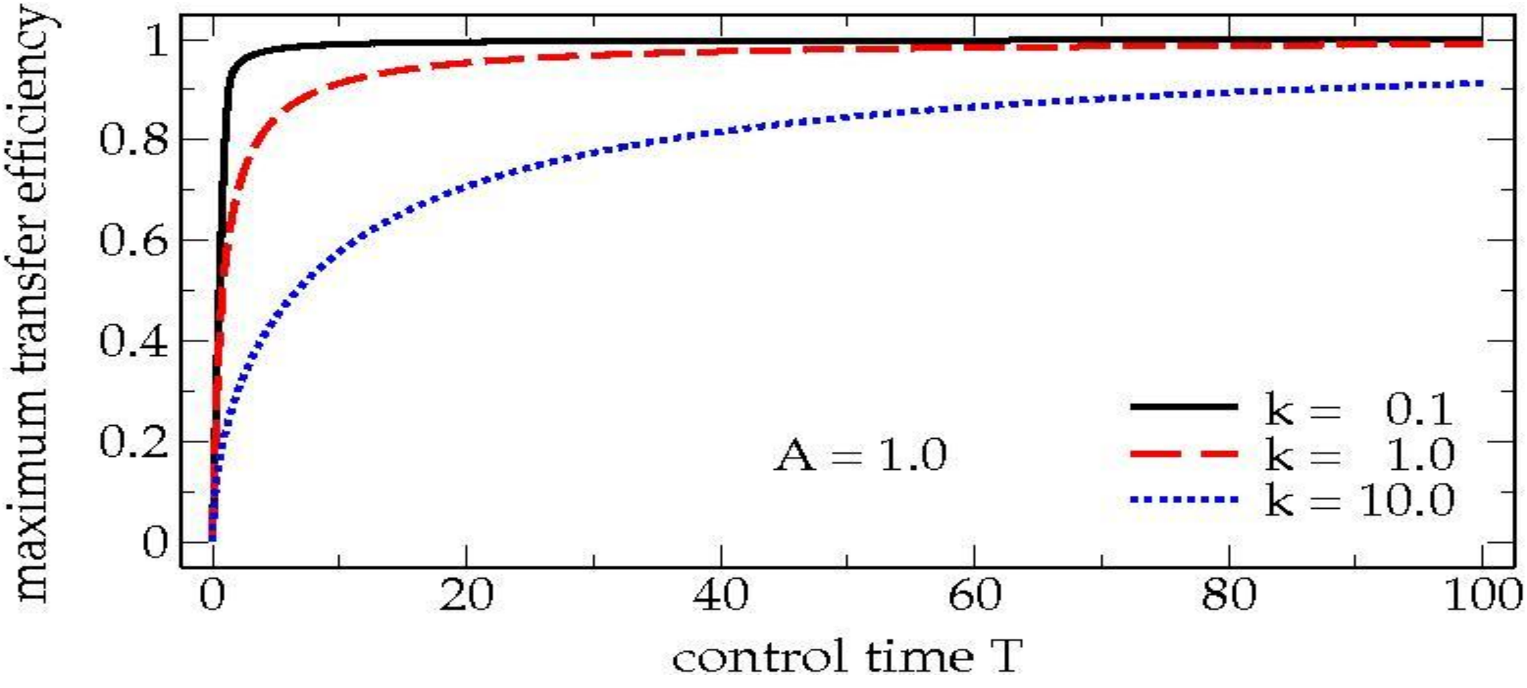
$$u = \cos\theta$$



Optimal control

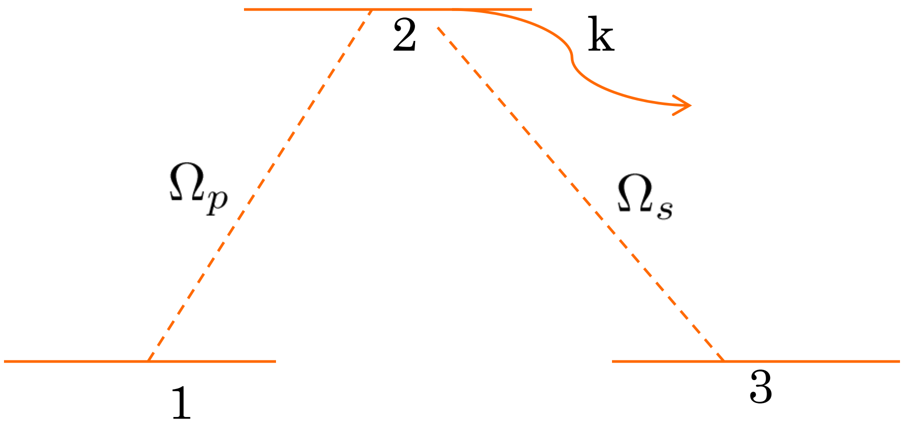
When $T \leq T_M$ $\qquad u^* = 1$

When $T > T_M$ $\qquad u^*(t) = \begin{cases} \frac{1}{\sqrt{A^2(\tau^2 - t^2) + 2k(\tau - t) + 1}} & \text{for } t \in [0, \tau] \\ 1 & \text{for } t \in [\tau, T] \end{cases},$



Connect to STIRAP

When $T \to \infty$ $\quad u^*(t) = \begin{cases} \frac{1}{\sqrt{A^2(\tau^2-t^2)+2k(\tau-t)+1}} & \text{for } t \in [0,\tau] \\ 1 & \text{for } t \in [\tau, T] \end{cases}$,



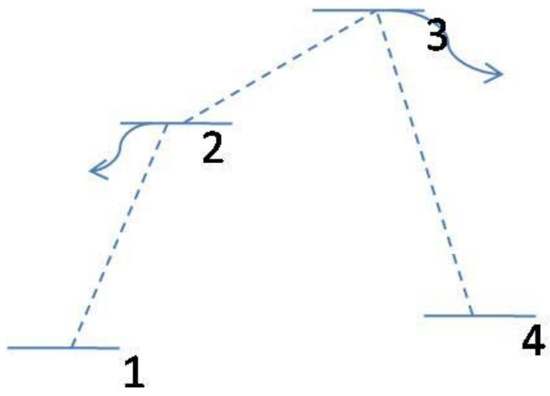
$$x_2 = r_1 u^* \sim 0$$

$$x_1|1\rangle + x_3|3\rangle$$

$$\frac{x_3}{x_1} = \frac{\Omega_p}{\Omega_s}$$

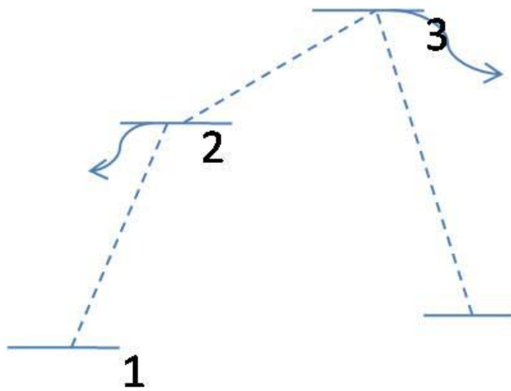
Dark state!

4-LEVEL SYSTEM



$$\frac{d}{dt}\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = -i \begin{pmatrix} 0 & \Omega_p & 0 & 0 \\ \Omega_p & -ik & \Omega_I & 0 \\ 0 & \Omega_I & -ik & \Omega_s \\ 0 & 0 & \Omega_s & 0 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix}$$

4-LEVEL SYSTEM



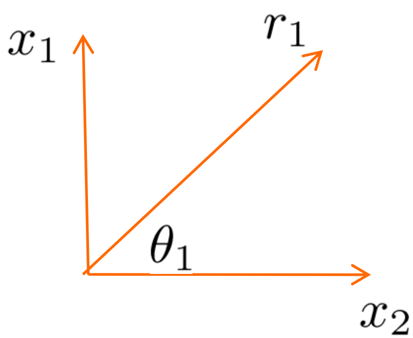
$$x_1 = x_1',\ x_2 = ix_2',\ x_3 = -x_3',\ x_4 = -ix_4'$$

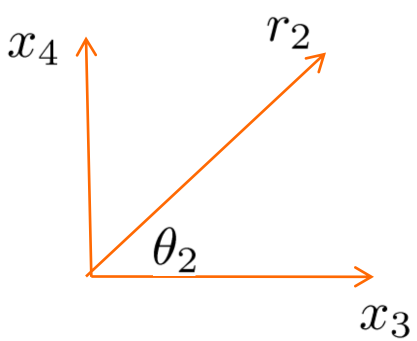
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & -\Omega_p & 0 & 0 \\ \Omega_p & -k & -\Omega_I & 0 \\ 0 & \Omega_I & -k & -\Omega_s \\ 0 & 0 & \Omega_s & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\text{Initial } (1, 0, 0, 0)$$

$$\text{Maximize } x_4(T)$$

4-LEVEL SYSTEM





$$r_1 = \sqrt{x_1^2 + x_2^2}$$

$$r_2 = \sqrt{x_3^2 + x_4^2}$$

$$\frac{d}{dt}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -k\cos^2\theta_1 & -\Omega_I \cos\theta_1 \cos\theta_2 \\ \Omega_I \cos\theta_1 \cos\theta_2 & -k\cos^2\theta_2 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

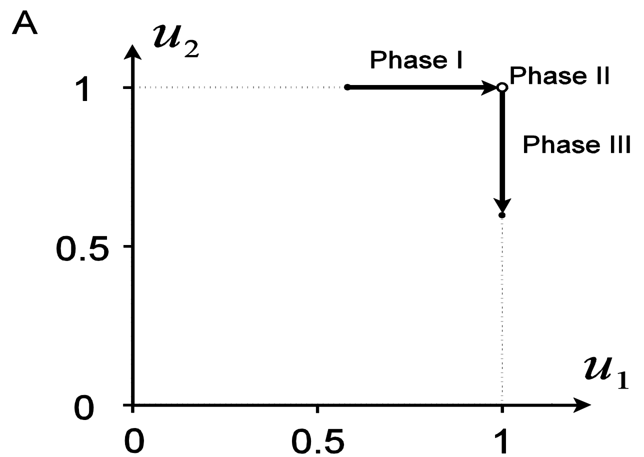
4-LEVEL SYSTEM

$$u_1 = \cos\theta_1 \qquad u_2 = \cos\theta_2 \qquad |\Omega_I| \leq A$$

$$\frac{d}{dt}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} -ku_1^2 & -Au_1u_2 \\ Au_1u_2 & -ku_2^2 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

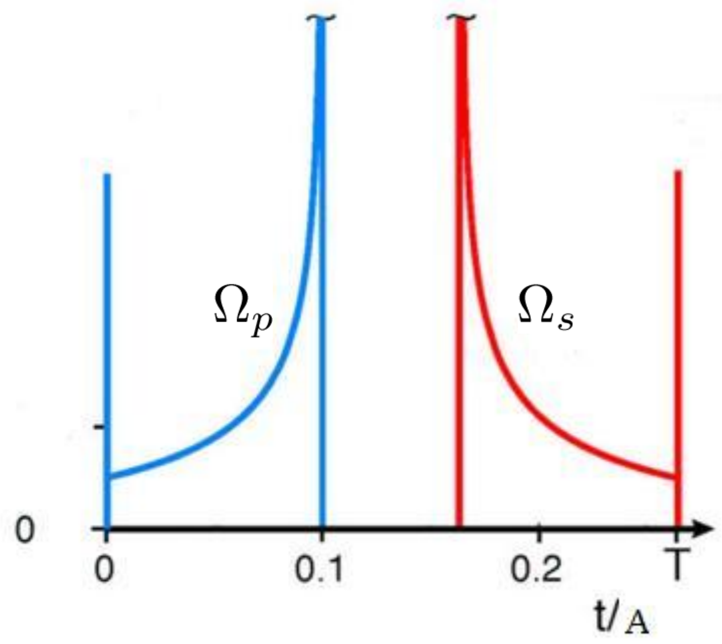
When $T \leq \frac{\cot^{-1}(2\xi)}{A}$ $\qquad u_1^*(t) = u_2^*(t) = 1$

When $T > \frac{\cot^{-1}(2\xi)}{A}$



$$\xi = \frac{k}{A}$$

Optimal control for 4-level system



$$T = 0.263/A$$

$$\eta_T = \frac{\exp(\xi(\theta_1 - \theta_2))(1 - \xi \sin 2\theta_2)}{\sin(\theta_1 + \theta_2)}$$

$$\theta_1 = \cot^{-1}\left(\frac{1-\kappa(\tau)}{2\xi\kappa(\tau)}\right)$$

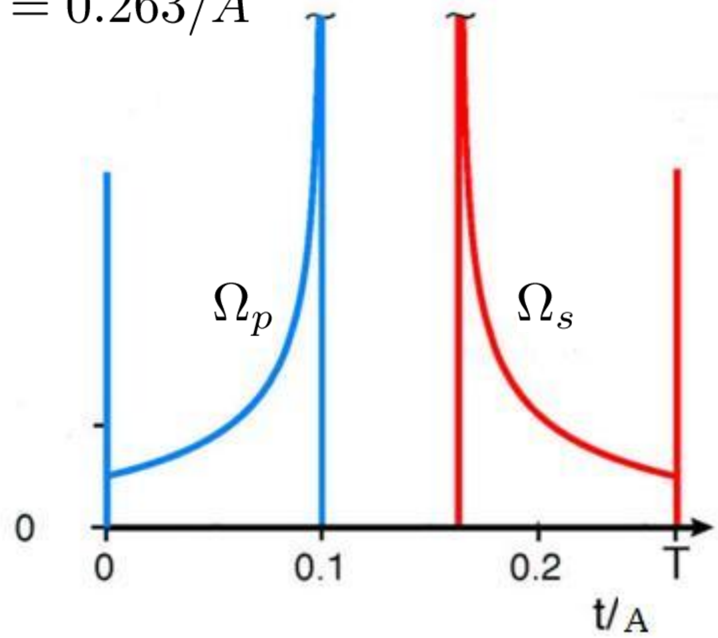
$$\theta_2 = \tan^{-1}\left(\frac{1-\kappa(\tau)}{2\xi}\right)$$

$$\kappa(\tau) = 1 + 2\xi^2 - 2\xi\sqrt{1+\xi^2}\coth(A\sqrt{1+\xi^2}\;\tau + 2\sinh^{-1}\xi)$$

$$T = 2\tau + \frac{\theta_2 - \theta_1}{A}$$

Optimal Control for 4-level system

$T = 0.263/A$



In the limit that T goes to infinity

$$\eta = \sqrt{1+\xi^2} - \xi$$

$$\xi = \frac{k}{A}$$

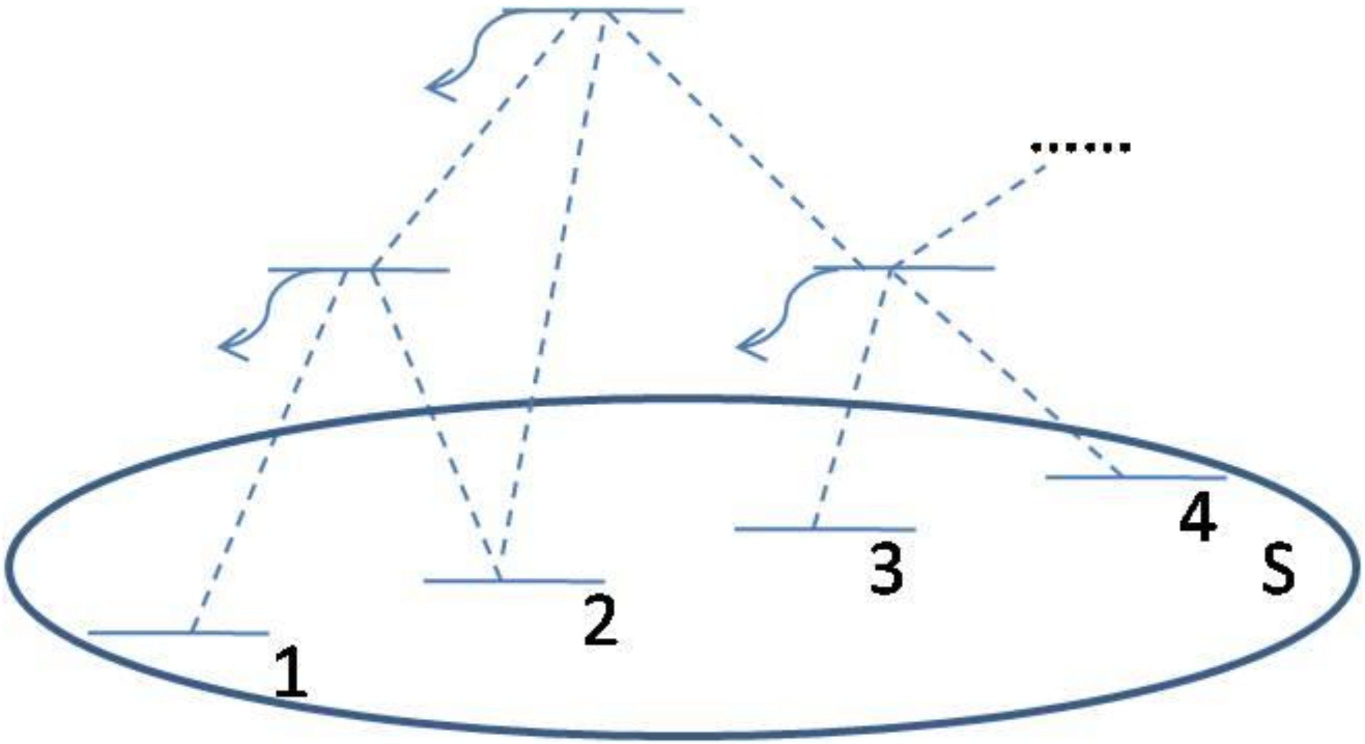
the transfer efficiency can reach unity only when $\xi = 0$, i.e, $\frac{k}{A} = 0$

WITH DETUNING

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = -i \begin{pmatrix} 0 & \Omega_p & 0 & 0 \\ \Omega_p & \Delta_2 - ik & \Omega_i & 0 \\ 0 & \Omega_i & \Delta_3 - ik & \Omega_s \\ 0 & 0 & \Omega_s & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

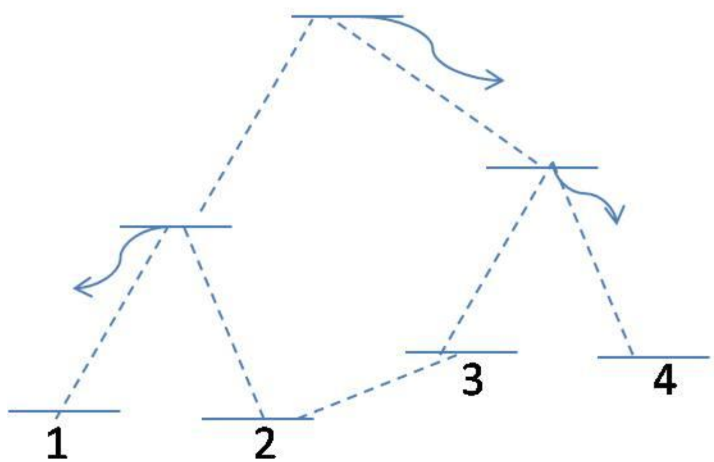
Suppose there is a trajectory achieving full population transfer with detuning, then on this trajectory x_2 and x_3 remain 0, which means the size of the detuning and relaxation rate has no effect, replacing the detuning with 0 will then lead to a contradiction with previous result.

Controllability on relaxation free subspace

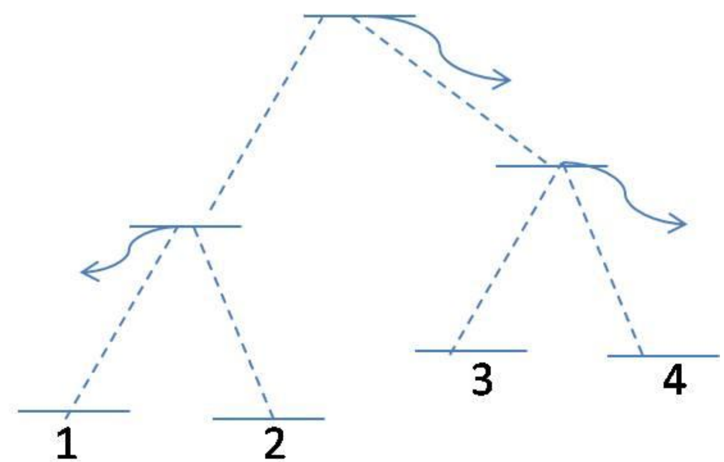


CONTROLLABILITY ON RELAXATION FREE SUBSPACE

If and only if any two eigenstates in the subspace can be connected by a path that never visits two consecutive states that both suffer relaxation.



a



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THANKS!