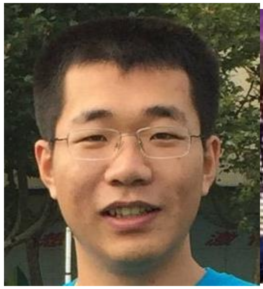
Xiuqi
Ma


Wilbur
Shirley


Meng
Cheng


Michael
Levin


John
McGreevy


Ho Tat
Lam

Phys. Rev. B 105, 195124 (2022)
arXiv:2211.10458

Fracton and Chern-Simons Theory

XIE CHEN, CALTECH

KITP, APR. 2023

NSF

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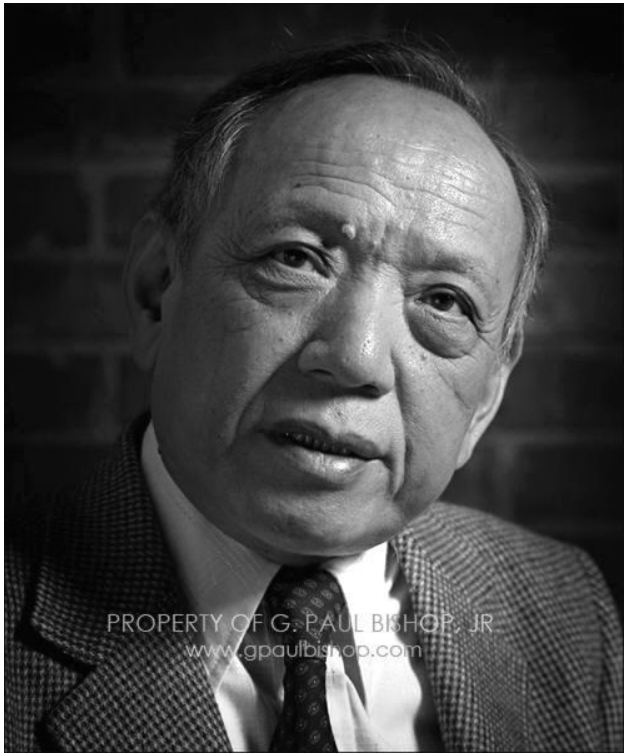
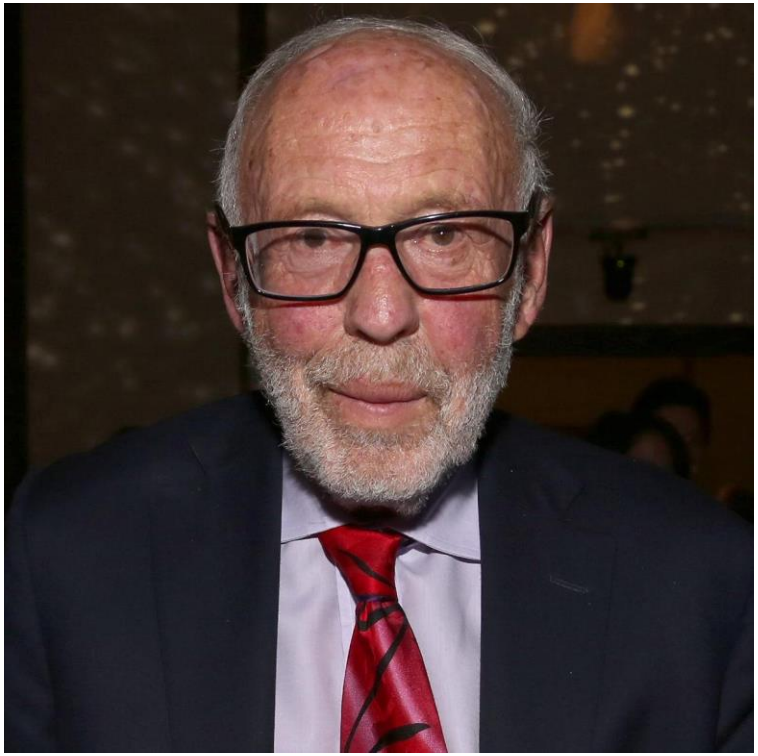
Simons Collaboration on
Ultra-Quantum Matter

Chern-Simons Theory



Shiing-Shen Chern



James Simons

Chern-Simons Theory

2+1D

$$\mathcal{L} = \frac{1}{4\pi} A \wedge dA \qquad \mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} A^I \wedge dA^J$$

Symmetric Integer Matrix

$K = 1$ Integer quantum Hall

$K = 3$ Fractional quantum Hall

$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ Z_2 gauge theory

Chern-Simons Theory

2+1D

$$\mathcal{L} = \frac{1}{4\pi} A \wedge dA \qquad \mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} A^I \wedge dA^J$$

Symmetric Integer Matrix

$\det(K)$	Ground state degeneracy on torus
K^{-1}	Fractional statistics, fractional charge

Chern-Simons Theory

2+1D

$$\mathcal{L} = \frac{1}{4\pi} A \wedge dA \qquad \mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} A^I \wedge dA^J$$

Symmetric Integer Matrix

$K = 1$ Integer quantum Hall $K^{-1} = 1$

$K = 3$ Fractional quantum Hall $K^{-1} = 1/3$

$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ Z_2 gauge theory $K^{-1} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$

Chern-Simons and Fracton

$$\mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} A^I \wedge dA^J$$

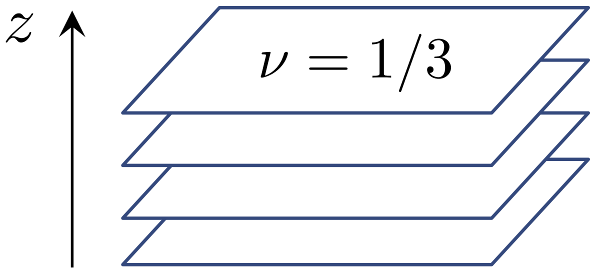
3+1D $$I, J = ... - 1, 0, 1, 2, ...$$

Chern-Simons and Fracton

$$\mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} A^I \wedge dA^J$$

3+1D $\quad I, J = \dots -1, 0, 1, 2, \dots$

$$K = \begin{pmatrix} \ddots & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$

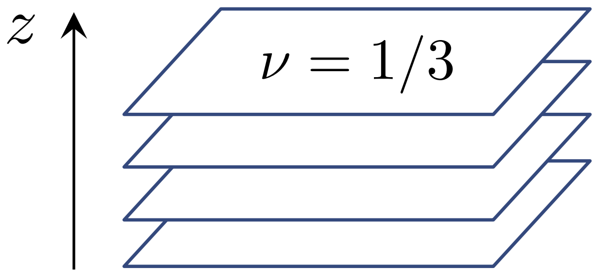


Chern-Simons and Fracton

$$\mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} A^I \wedge dA^J$$

$$I, J = ... -1, 0, 1, 2, ...$$

$$K = \begin{pmatrix} \ddots & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$



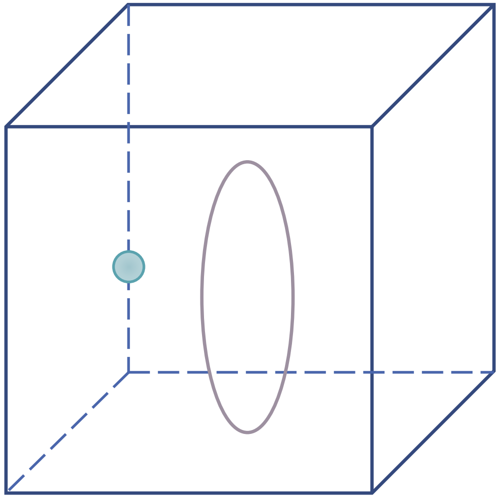
- Ground state degeneracy 3^N exponential in height in z
- Fractional point excitations move in xy planes only – planons

Compared to 3+1D topo Order

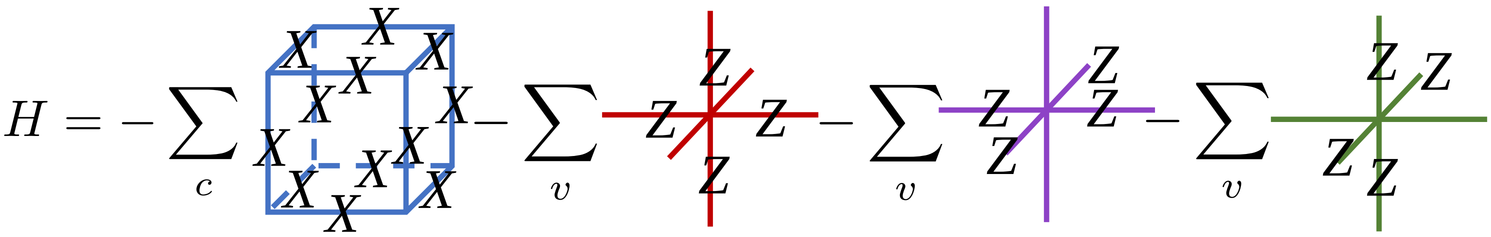
$$\mathcal{L} = \frac{2}{4\pi}\epsilon^{\mu\nu\rho\sigma}B_{\mu\nu}\partial_\rho A_\sigma$$

Z_2 gauge theory

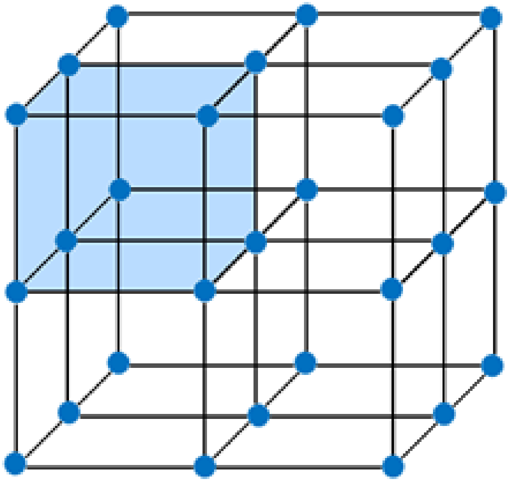
- Gapped
- Fractional point excitation
- Full motion in 3D space
- Fractional loop excitation
- Ground state degeneracy = 8



Compared to X-cube

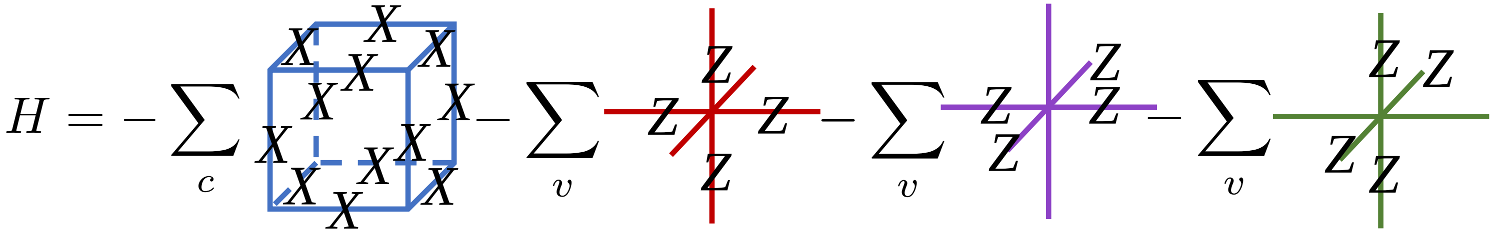


$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$



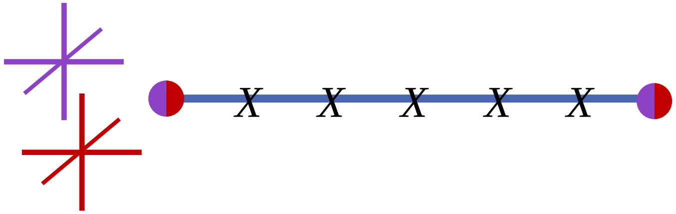
Vijay, Haah, Fu, 16

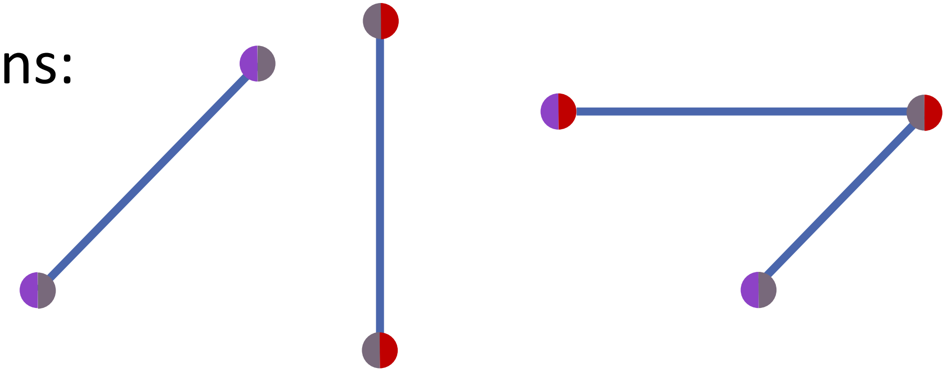
X-cube



Log(Ground state degeneracy) = 6L-3

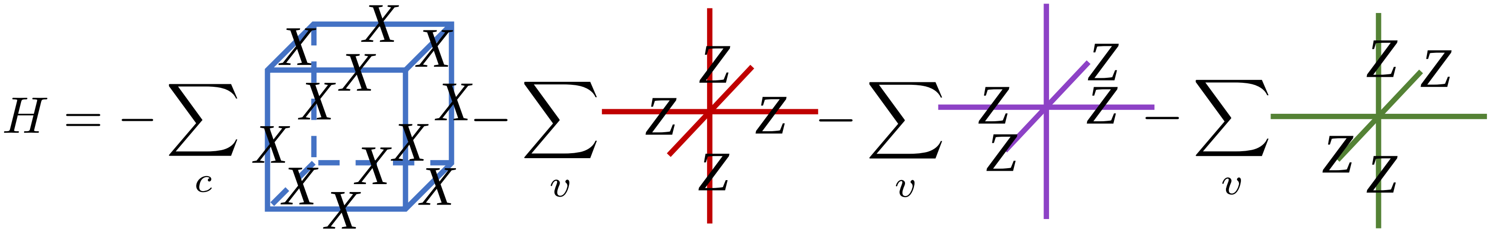
Dimension 1 excitations:



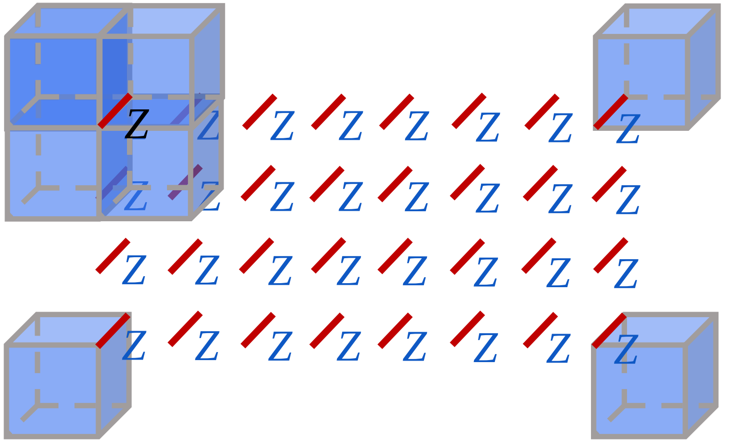


Vijay, Haah, Fu, 16

X-cube

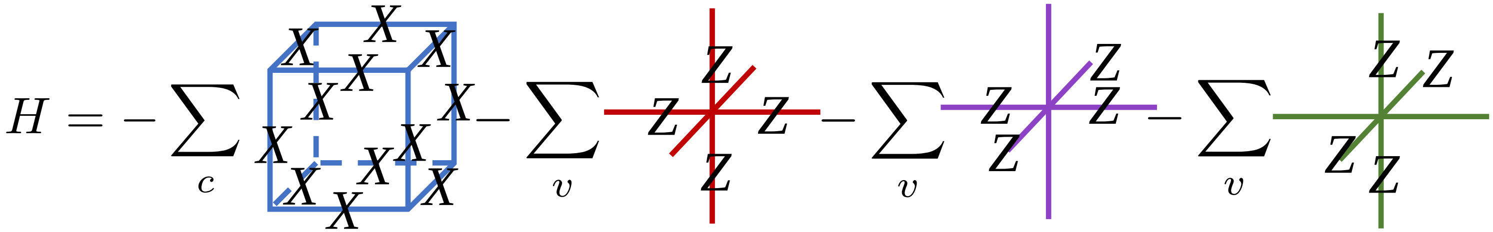


Fracton Excitations

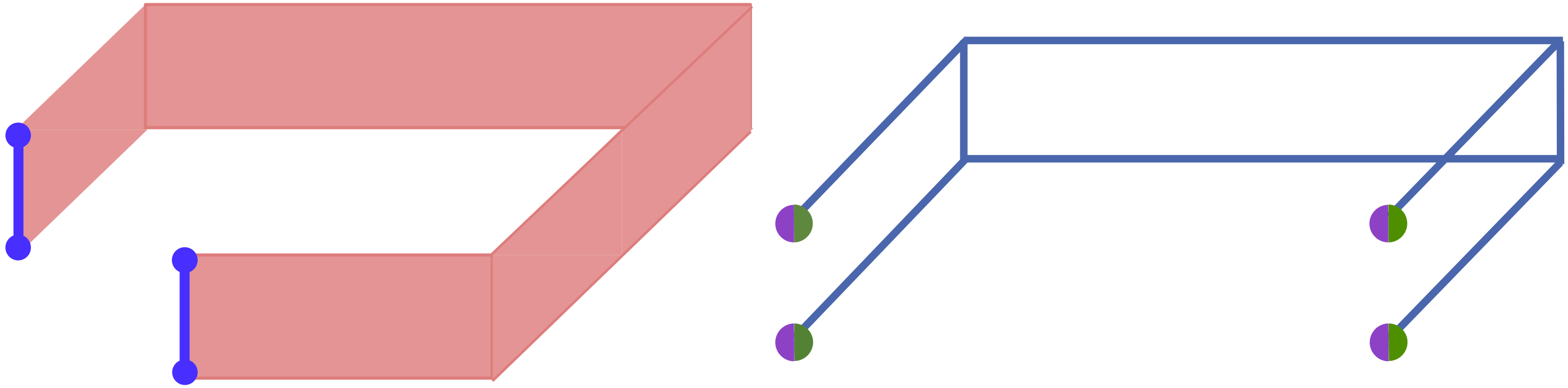


Vijay, Haah, Fu, 16

Compare to X-cube

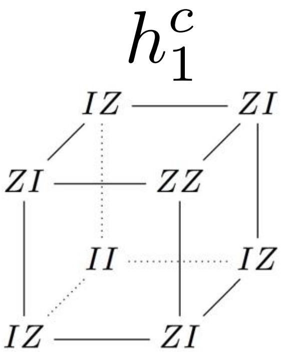


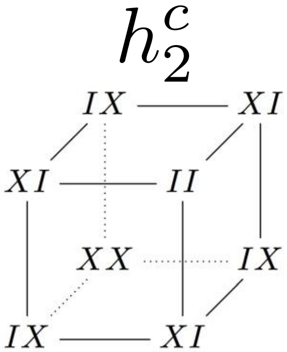
Planon excitations:



Vijay, Haah, Fu, 16

Compared to Haah's code

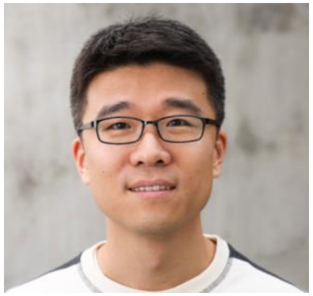




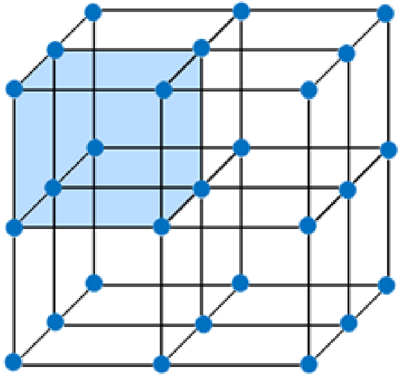
$$X = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$Z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$H = -\sum_c h_1^c - \sum_c h_2^c$$

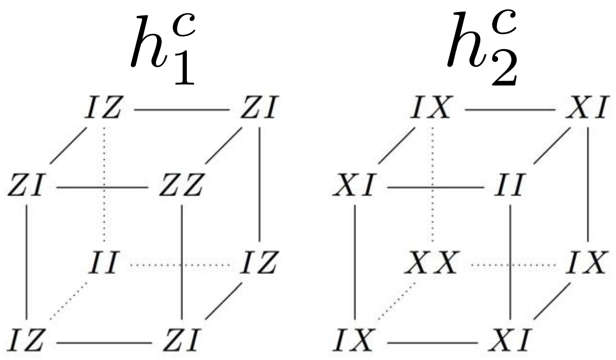


Jeongwan Haah
2011



- Cubic lattice model
- Two qubits per site
- All Hamiltonian terms commute

Compared to Haah’s code



$$H = -\sum_c h_1^c - \sum_c h_2^c$$

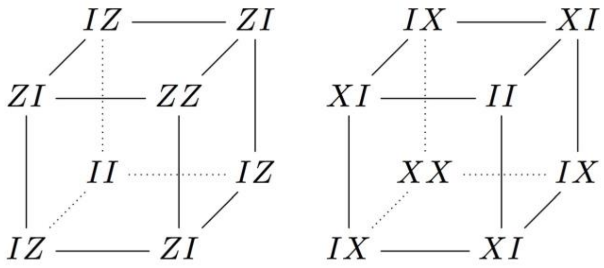
Stable ground state degeneracy

$$log\ \mathrm{GSD} =$$

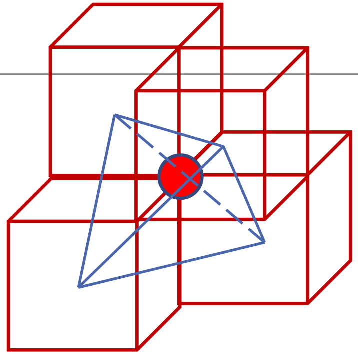
$$\frac{k+2}{4} = \deg_x\ \gcd\begin{bmatrix} 1+(1+x)^L, \\ 1+(1+\omega x)^L, \\ 1+(1+\omega^2 x)^L \end{bmatrix}_{\mathbb{F}_4}$$

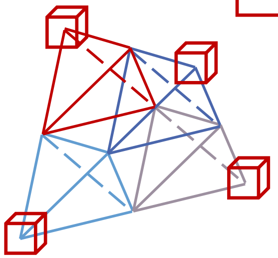
$$= \begin{cases} 1 & \text{if } L = 2^p+1\ (p \geq 1), \\ L & \text{if } L = 2^p\ (p \geq 1) \end{cases}$$

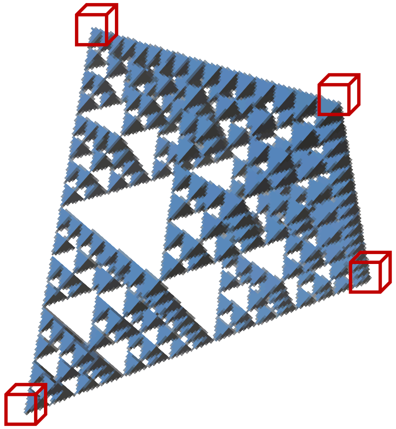
Haah’s code



- Cube excitations created four at a time
- Individual cube excitation cannot move
- Cube excitations separated through a fractal process

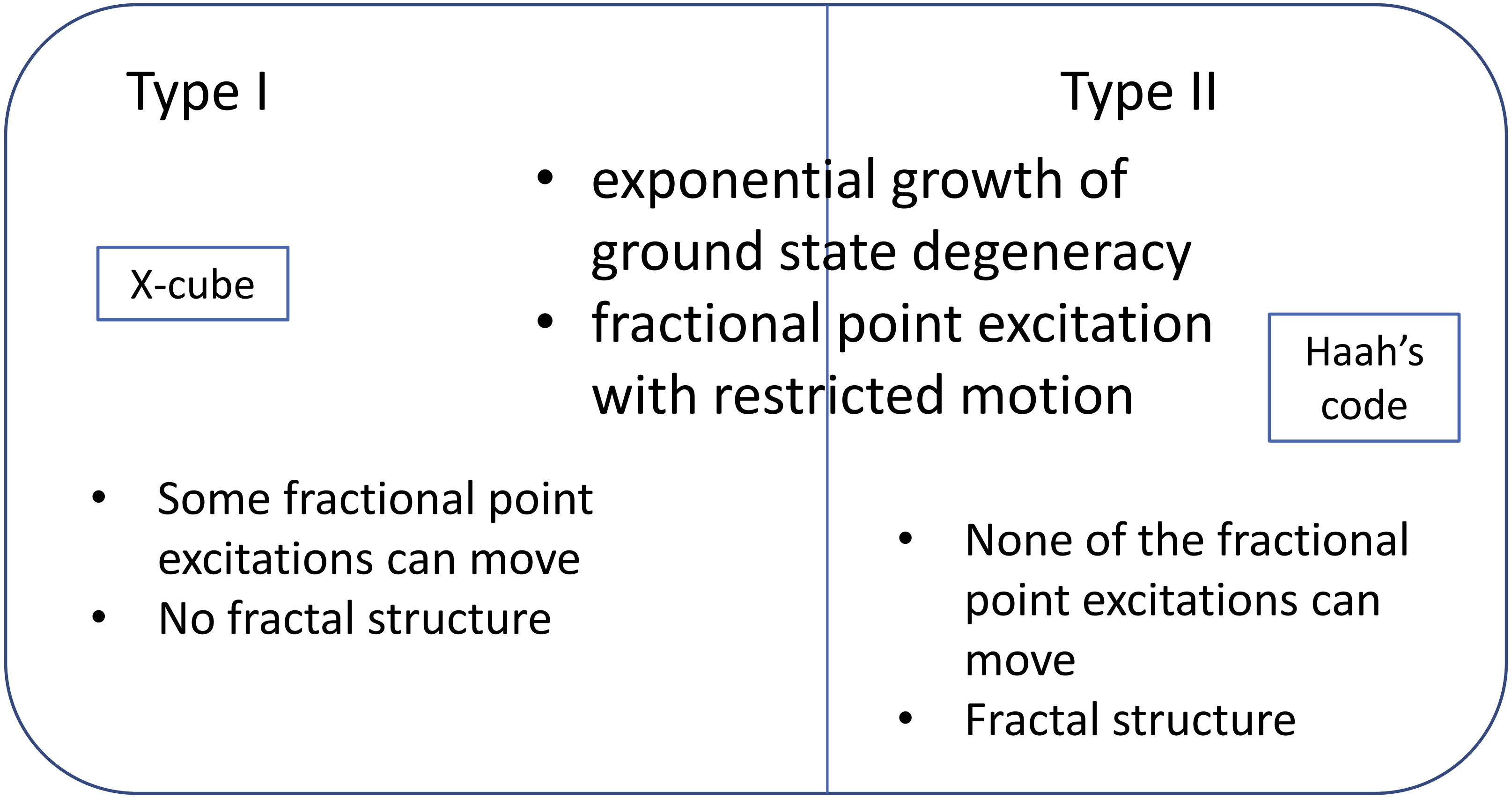






Fracton excitation

Fracton



Type I

Type II

- exponential growth of ground state degeneracy
- fractional point excitation with restricted motion

X-cube

Haah's code

- Some fractional point excitations can move
- No fractal structure

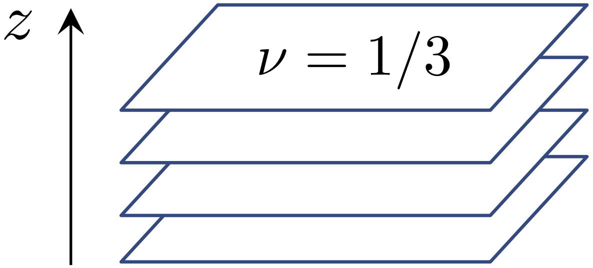
- None of the fractional point excitations can move
- Fractal structure

Chern-Simons and Fracton

$$\mathcal{L} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} A^I \wedge dA^J$$

$$I, J = \ldots -1, 0, 1, 2, \ldots$$

$$K = \begin{pmatrix} \ddots & & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$



- Ground state degeneracy 3^N exponential in height in z
- Fractional point excitations move in xy planes only – planons

Fracton

Type I

X-cube

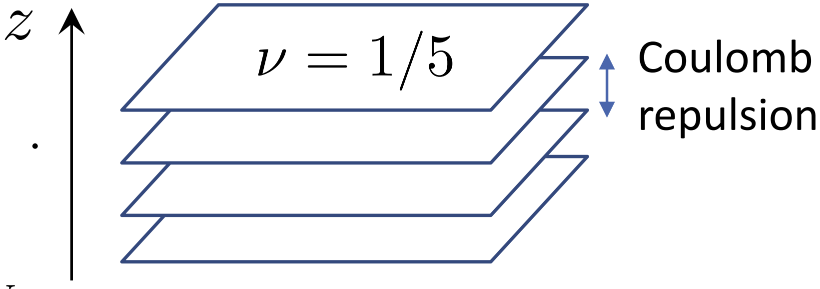
diagonal CS

Type II

Haah's code

Tri-diagonal K matrix

$$K(131) = \begin{pmatrix} 3 & 1 & & & 1 \\ 1 & 3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3 & 1 \\ 1 & & & 1 & 3 \end{pmatrix}_{N\times N} .$$



Qiu, Joynt, MacDonald, 1990; Naud, Pryadko, Sondhi, 2000

Tri-diagonal K matrix

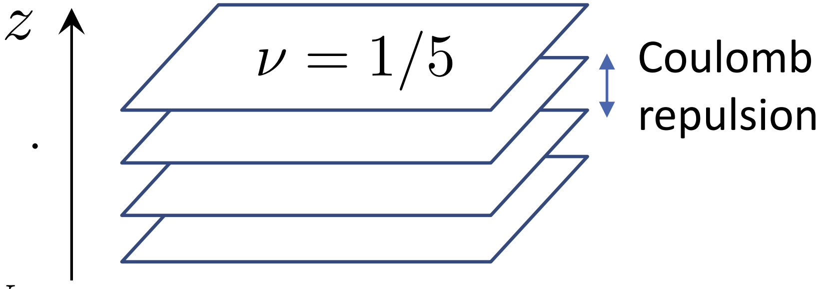
$$K(131) = \begin{pmatrix} 3 & 1 & & & & & 1 \\ 1 & 3 & 1 & & & & \\ & 1 & 3 & 1 & & & \\ & & 1 & 3 & 1 & & \\ & & & 1 & 3 & 1 & \\ & & & & 1 & 3 & 1 \\ 1 & & & & & 1 & 3 \end{pmatrix}$$

$$K(131)^{-1} = \frac{1}{65} \begin{pmatrix} 29 & -11 & 4 & -1 & -1 & 4 & -11 \\ -11 & 29 & -11 & 4 & -1 & -1 & 4 \\ 4 & -11 & 29 & -11 & 4 & -1 & -1 \\ -1 & 4 & -11 & 29 & -11 & 4 & -1 \\ -1 & -1 & 4 & -11 & 29 & -11 & 4 \\ 4 & -1 & -1 & 4 & -11 & 29 & -11 \\ -11 & 4 & -1 & -1 & 54 & -11 & 29 \end{pmatrix}$$

Shirley, Slagle, Chen, 2020

Tri-diagonal K matrix

$$K(131) = \begin{pmatrix} 3 & 1 & & & 1 \\ 1 & 3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3 & 1 \\ 1 & & & 1 & 3 \end{pmatrix}_{N\times N} .$$



$N \to \infty$ Irrational statistics

$$\theta_{IJ} = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}-3}{2} \right)^{|I-J|}$$

Ground state degeneracy

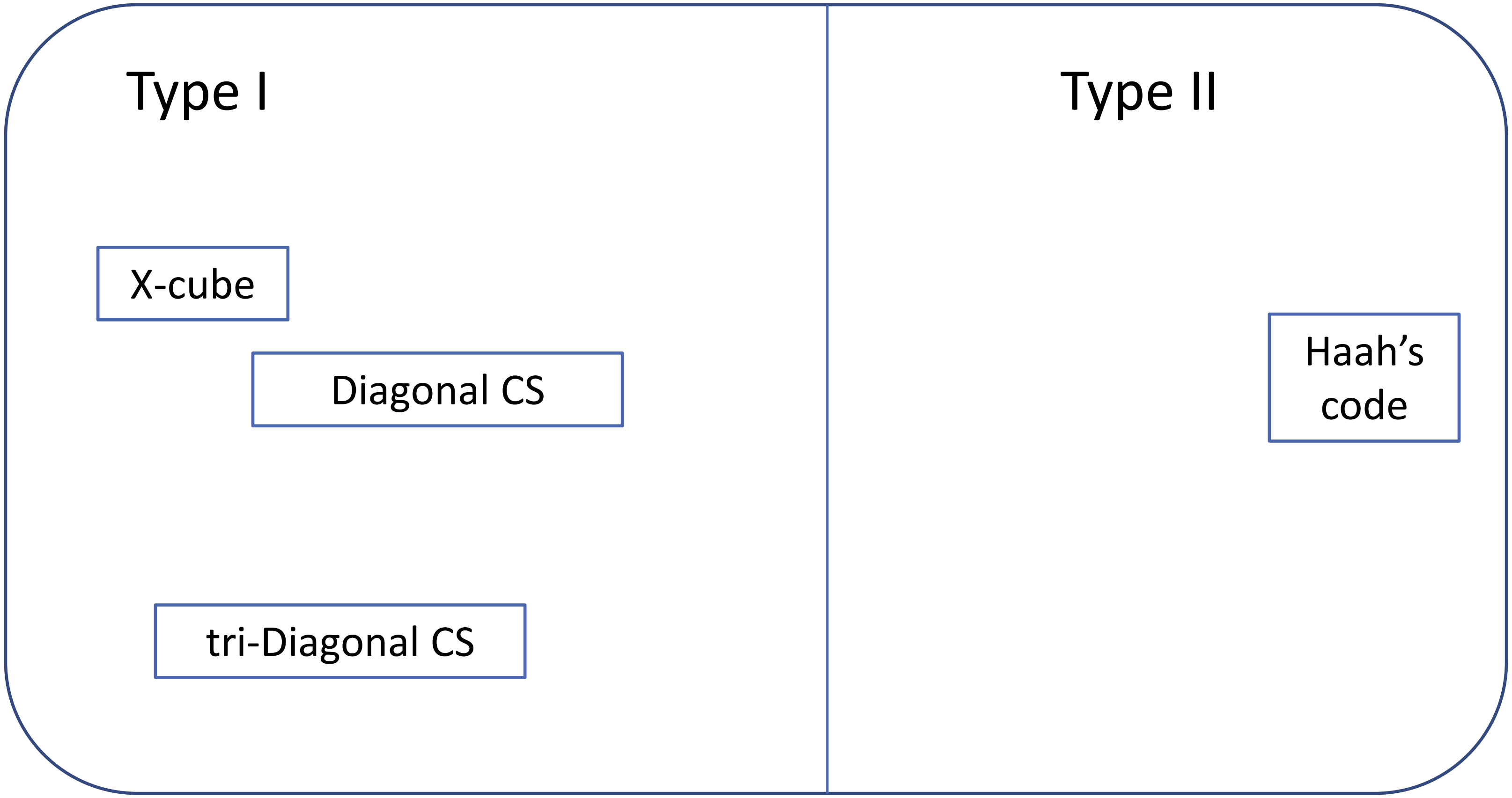
$$\left(\tfrac{3+\sqrt{5}}{2}\right)^N + \left(\tfrac{3-\sqrt{5}}{2}\right)^N - 2(-1)^N$$

$$\sim \left(\tfrac{3+\sqrt{5}}{2}\right)^N$$

Planon Fusion group $G_N = \mathbb{Z}_{F_N} \times \mathbb{Z}_{5F_N}$ F_N Nth number in the Fibonacci sequence

Qiu, Joynt, MacDonald, 1990; Naud, Pryadko, Sondhi, 2000

Fracton



Type I

X-cube

Diagonal CS

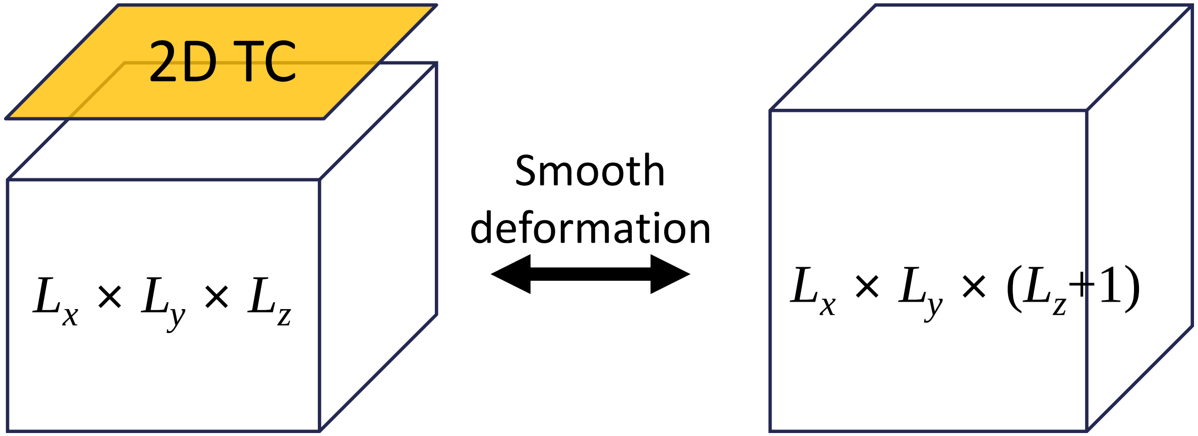
tri-Diagonal CS

Type II

Haah's code

X-cube and foliation

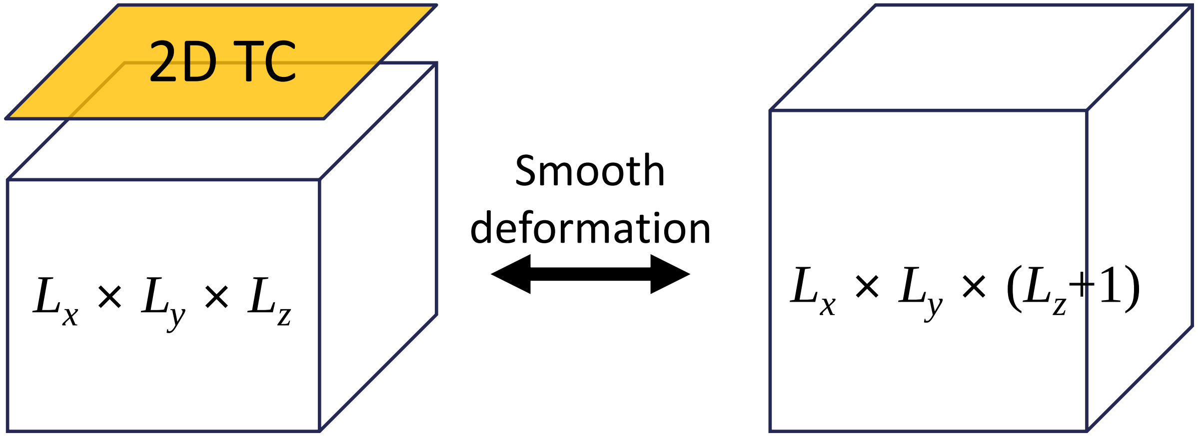
Foliation



Shirley, Slagle, Wang, Chen, 18

X-cube and foliation

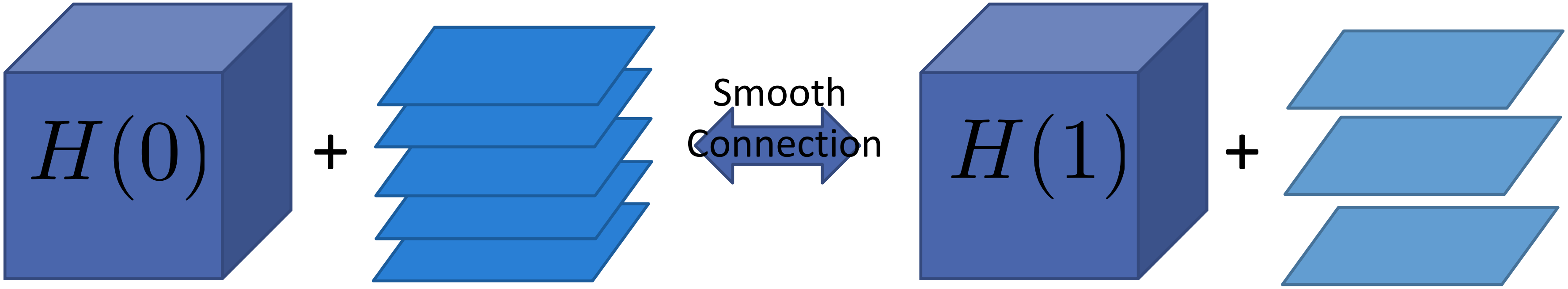
Foliation



- Exponential scaling of ground state degeneracy
- Planon excitation

Shirley, Slagle, Wang, Chen, 18

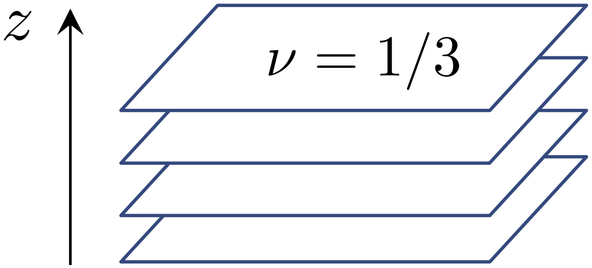
Foliated Fracton Phase



Comparing different models: mod contribution from 2D layers

Excitations, entanglement …

$$K = \begin{pmatrix} \ddots & & & \\ & 3 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}$$



A trivial foliated fracton model

Foliated K matrix

$$K = \begin{array}{c} \begin{array}{ccccccccc} & e_1 & m_1 & e_2 & m_2 & e_3 & m_3 & e_4 & \end{array} \\ \begin{pmatrix} \ddots & & & & & & & & \\ & 0 & 2 & \text{-1} & & & & & \\ & 2 & 0 & & & & & & \\ & \text{-1} & & 0 & 2 & \text{-1} & & & \\ & & & 2 & 0 & & & & \\ & & & \text{-1} & & 0 & 2 & \text{-1} & \\ & & & & & 2 & 0 & & \\ & & & & & \text{-1} & & 0 & \\ & & & & & & & & \ddots \end{pmatrix} \end{array}.$$

$$K^{-1} = \frac{1}{4} \begin{array}{c} \begin{array}{ccccccccc} & m_0 & e_1 & m_1 & e_2 & m_2 & e_3 & m_3 & \end{array} \\ \begin{pmatrix} \ddots & & & & & & & & \\ & 0 & & 1 & & & & & \\ & & 0 & 2 & & & & & \\ & 1 & 2 & 0 & & 1 & & & \\ & & & & 0 & 2 & & & \\ & & & 1 & 2 & 0 & & 1 & \\ & & & & & & 0 & 2 & \\ & & & & & 1 & 2 & 0 & \\ & & & & & & & & \ddots \end{pmatrix} \end{array}.$$

Shirley, Slagle, Chen, 2020

Foliated K matrix

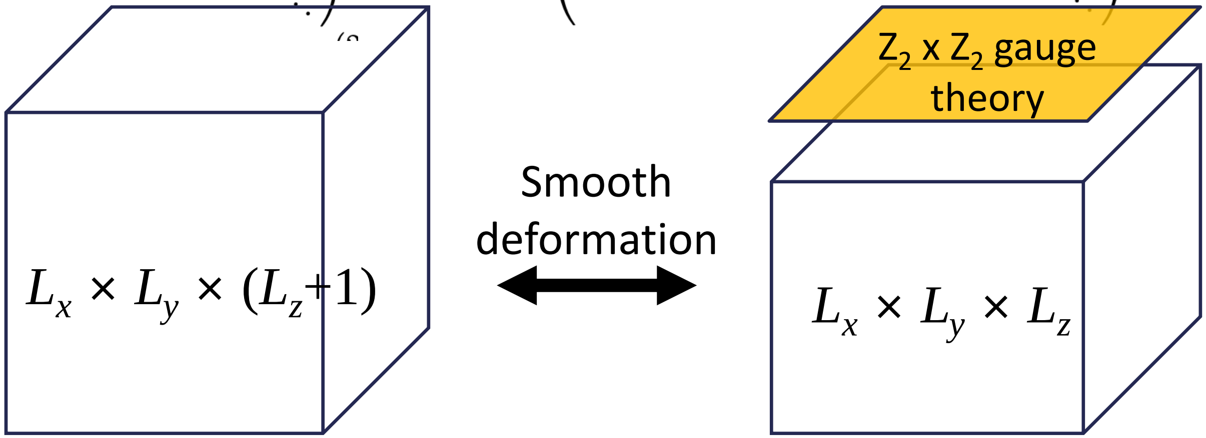
$$W = \begin{array}{c|ccccccccccc} & & \tilde{e}_1 & \tilde{m}_1 & \tilde{e}^A & \tilde{m}^A & \tilde{e}^B & \tilde{m}^B & \tilde{e}_2 & \tilde{m}_2 & \tilde{e}_3 & \\ \hline & \ddots & & & & & & & & & & \\ e_1 & & 1 & & & & & & & & & \\ m_1 & & & 1 & & & & 1 & & & & \\ e_2 & & & & 1 & & & & -1 & & & \\ m_2 & & & & & 1 & & & & & & \\ e_3 & & -1 & & & & 1 & & & & & \\ m_3 & & & & & & & 1 & & & & \\ e_4 & & & & & & & & 1 & & & \\ m_4 & & -1 & & & 1 & & & & 1 & & \\ e_5 & & & & & & & & & & 1 & \\ & & & & & & & & & & & \ddots \end{array} . \quad W^T K W = \begin{array}{ccccccccccc} & \tilde{e}_1 & \tilde{m}_1 & \tilde{e}^A & \tilde{m}^A & \tilde{e}^B & \tilde{m}^B & \tilde{e}_2 & \tilde{m}_2 & \tilde{e}_3 & \\ \ddots & & & & & & & & & & \\ & 0 & 2 & & & & & -1 & & & \\ & 2 & 0 & & & & & & & & \\ & & & 0 & 2 & -1 & 0 & & & & \\ & & & 2 & 0 & 0 & 0 & & & & \\ & & & -1 & 0 & 0 & 2 & & & & \\ & & & 0 & 0 & 2 & 0 & & & & \\ & -1 & & & & & & 0 & 2 & -1 & \\ & & & & & & & 2 & 0 & & \\ & & & & & & & -1 & & 0 & \\ & & & & & & & & & & \ddots \end{array} .$$

Shirley, Slagle, Chen, 2020

Foliated K matrix

$$W = \begin{array}{c}
 \\ e_1 \\ m_1 \\ e_2 \\ m_2 \\ e_3 \\ m_3 \\ e_4 \\ m_4 \\ e_5 \\ \\
\end{array}
\left(\begin{array}{ccccccccc}
\tilde{e}_1 & \tilde{m}_1 & \tilde{e}^A & \tilde{m}^A & \tilde{e}^B & \tilde{m}^B & \tilde{e}_2 & \tilde{m}_2 & \tilde{e}_3 \\
\ddots & & & & & & & & \\
1 & & & & & & & & \\
 & 1 & & & & 1 & & & \\
 & & 1 & & & & -1 & & \\
 & & & 1 & & & & & \\
-1 & & & & 1 & & & & \\
 & & & & & 1 & & & \\
 & & & & & & 1 & & \\
-1 & & & 1 & & & & 1 & \\
 & & & & & & & & 1 \\
 & & & & & & & & \ddots
\end{array}\right). \quad
W^T K W = \left(\begin{array}{ccccccccc}
\tilde{e}_1 & \tilde{m}_1 & \tilde{e}^A & \tilde{m}^A & \tilde{e}^B & \tilde{m}^B & \tilde{e}_2 & \tilde{m}_2 & \tilde{e}_3 \\
\ddots & & & & & & & & \\
0 & 2 & & & & & -1 & & \\
2 & 0 & & & & & & & \\
 & & 0 & 2 & -1 & 0 & & & \\
 & & 2 & 0 & 0 & 0 & & & \\
 & & -1 & 0 & 0 & 2 & & & \\
 & & 0 & 0 & 2 & 0 & & & \\
-1 & & & & & & 0 & 2 & -1 \\
 & & & & & & 2 & 0 & \\
 & & & & & & -1 & & 0 \\
 & & & & & & & & \ddots
\end{array}\right).$$

Nontrivial
Foliated
Fracton Model

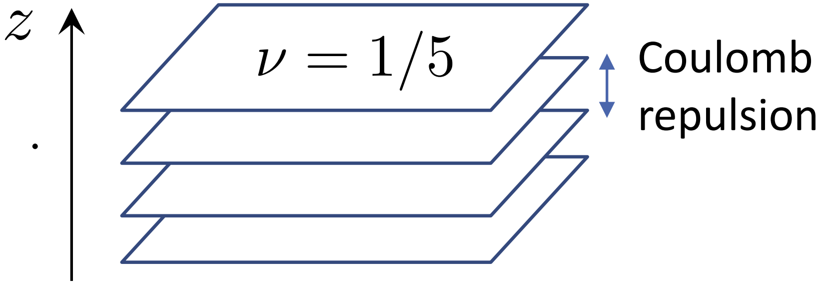


Smooth
deformation

Shirley, Slagle, Chen, 2020

Tri-diagonal K matrix

$$K(131) = \begin{pmatrix} 3 & 1 & & & 1 \\ 1 & 3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3 & 1 \\ 1 & & & 1 & 3 \end{pmatrix}_{N\times N} .$$



$N \to \infty$ Irrational statistics

$$\theta_{IJ} = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}-3}{2} \right)^{|I-J|}$$

Ground state degeneracy

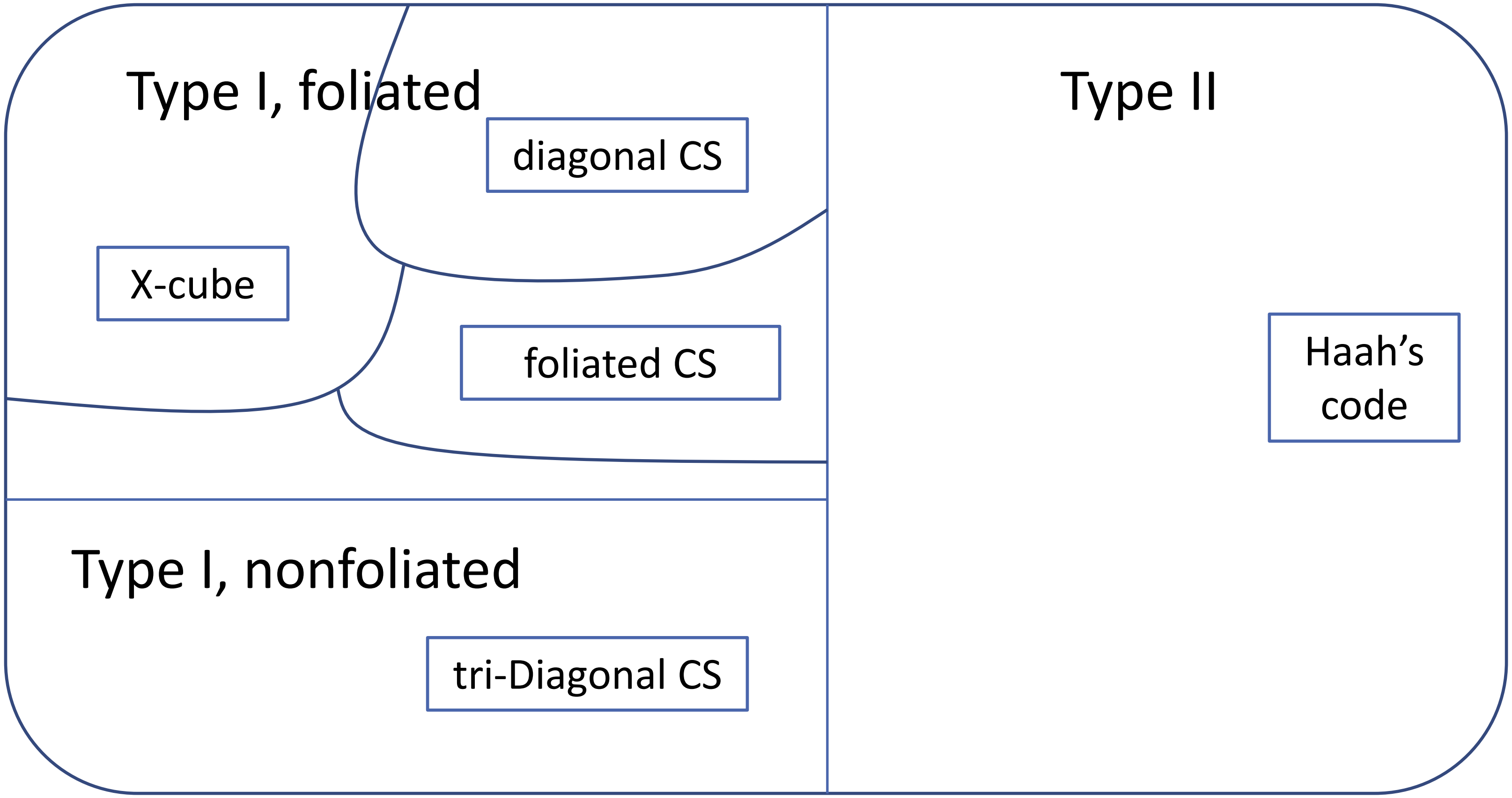
$$\left(\tfrac{3+\sqrt{5}}{2}\right)^N + \left(\tfrac{3-\sqrt{5}}{2}\right)^N - 2(-1)^N$$
$$\sim \left(\tfrac{3+\sqrt{5}}{2}\right)^N$$

Planon Fusion group $G_N = \mathbb{Z}_{F_N} \times \mathbb{Z}_{5F_N}$

F_N Nth number in the Fibonacci sequence

Qiu, Joynt, MacDonald, 1990; Naud, Pryadko, Sondhi, 2000

Fracton



Type I, foliated

diagonal CS

X-cube

foliated CS

Type I, nonfoliated

tri-Diagonal CS

Type II

Haah's code

Spectrum

$$\mathcal{L} = -\frac{1}{4e^2}\sum_I F^I_{\mu\nu}F^{I,\mu\nu} + \frac{1}{4\pi}\sum_{IJ} K_{IJ}\epsilon^{\mu\nu\lambda}A^I_\mu\partial_\nu A^J_\lambda$$

Gap of each mode is proportional to |eigenvalues of K |^2

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Gapless: $K(101), K(111), K(121)$

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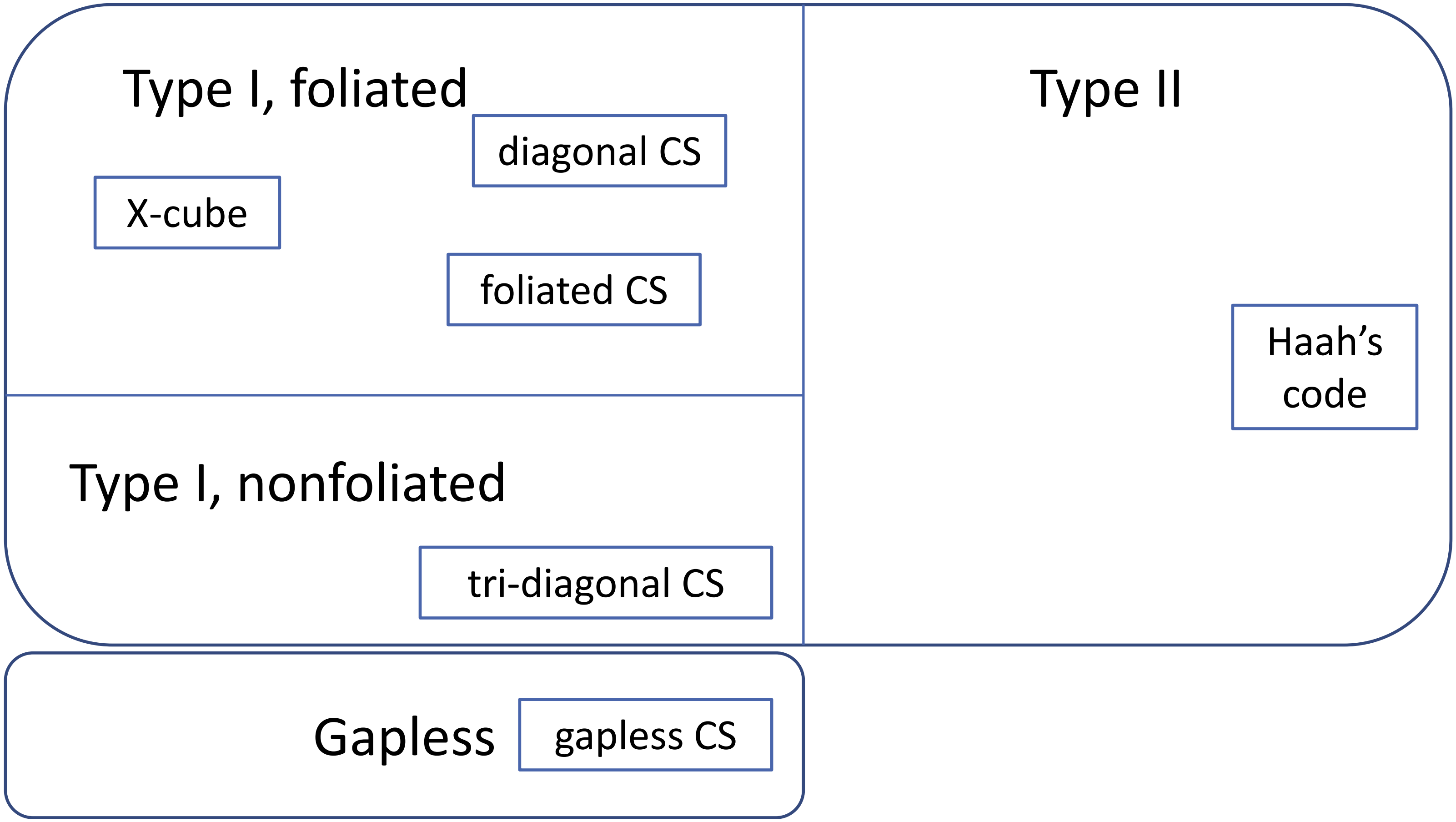
Sullivan, Dua, Cheng, arXiv: 2109.13267
Lam, Ma, Chen, arXiv:2211.10458

Fracton



Type I, foliated

diagonal CS

X-cube

foliated CS

Type I, nonfoliated

tri-diagonal CS

Type II

Haah’s code

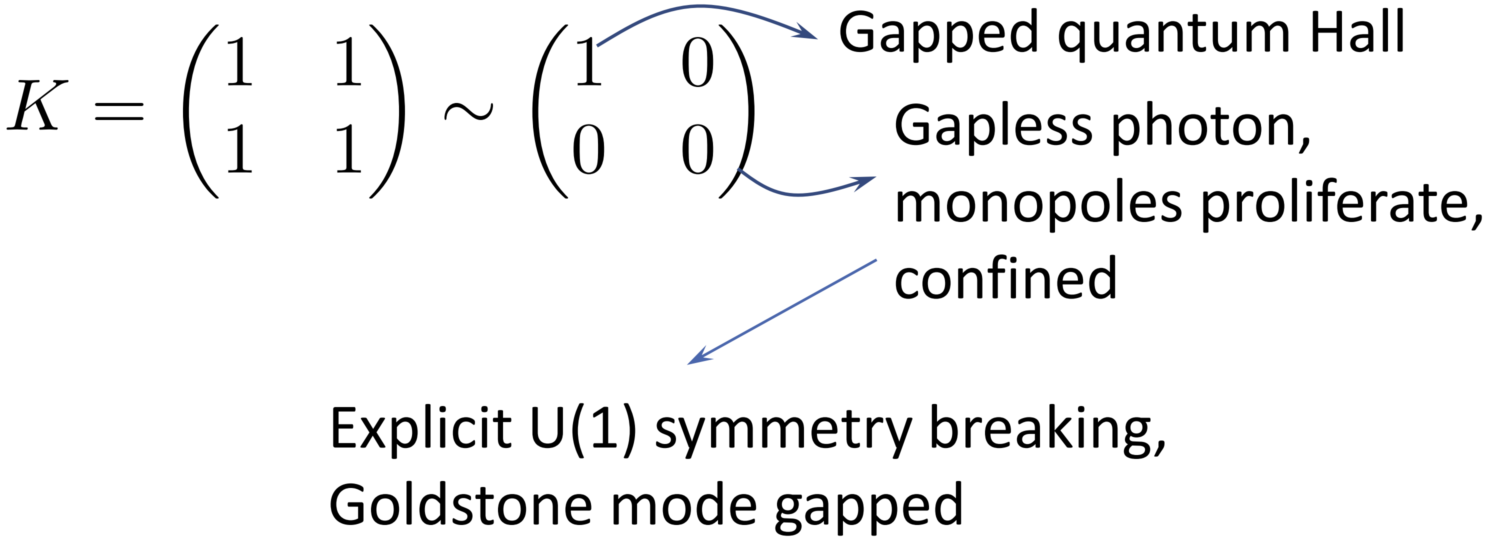
Gapless

gapless CS

Gapless 2+1D CS

$$\mathcal{L} = -\frac{1}{4e^2}\sum_I F^I_{\mu\nu}F^{I,\mu\nu} + \frac{1}{4\pi}\sum_{IJ} K_{IJ}\epsilon^{\mu\nu\lambda}A^I_\mu\partial_\nu A^J_\lambda$$

For example



$$K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Gapped quantum Hall

Gapless photon,
monopoles proliferate,
confined

Explicit U(1) symmetry breaking,
Goldstone mode gapped

Gapless infinite Chern Simons

K(121)	Contains mode with exactly zero energy $k_x = k_y = 0, k_z = \pi$	Quadratic dispersion
K(232)	no mode with exactly zero energy	Linear dispersion

- ☐ Stability?
- ☐ Topological feature?

Gapless infinite Chern Simons

- Stability
  - no gauge invariant local monopole operators
  - no relevant term to gap out the theory
  - robust gapless-ness
- Deconfined point excitation
  - polynominally decaying potential
- Topological feature (linear dispersion)
  - finite braiding phase between point excitations

Summary

- Generalizing 2+1D Chern Simons theory to 3+1D by including an infinite number of 2+1D gauge fields
- Type I fracton
- Foliated
- Non-foliated
- Stable gapless phases
- Continuous field theory description (linear dispersion)

Open questions

- Conditions for gapped foliated / gapped non-foliated
- Equivalence classes of foliated models (as equivalence class of K matrices)
- General properties of gapped non-foliated models (degeneracy, fusion group, statistics, RG, etc.)
- Building block for 3-foliated models (like x-cube)?
- Experiment?