Noncommutative field theory of the *Tkachenko mode*: symmetry and decay rate

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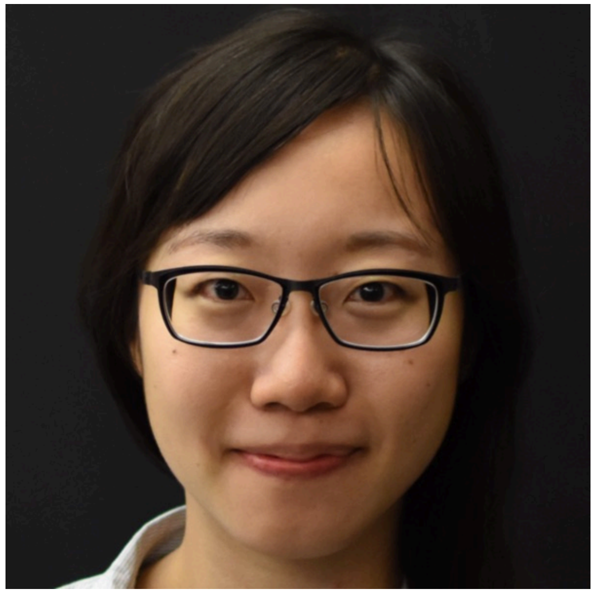

Yi-Hsien Du

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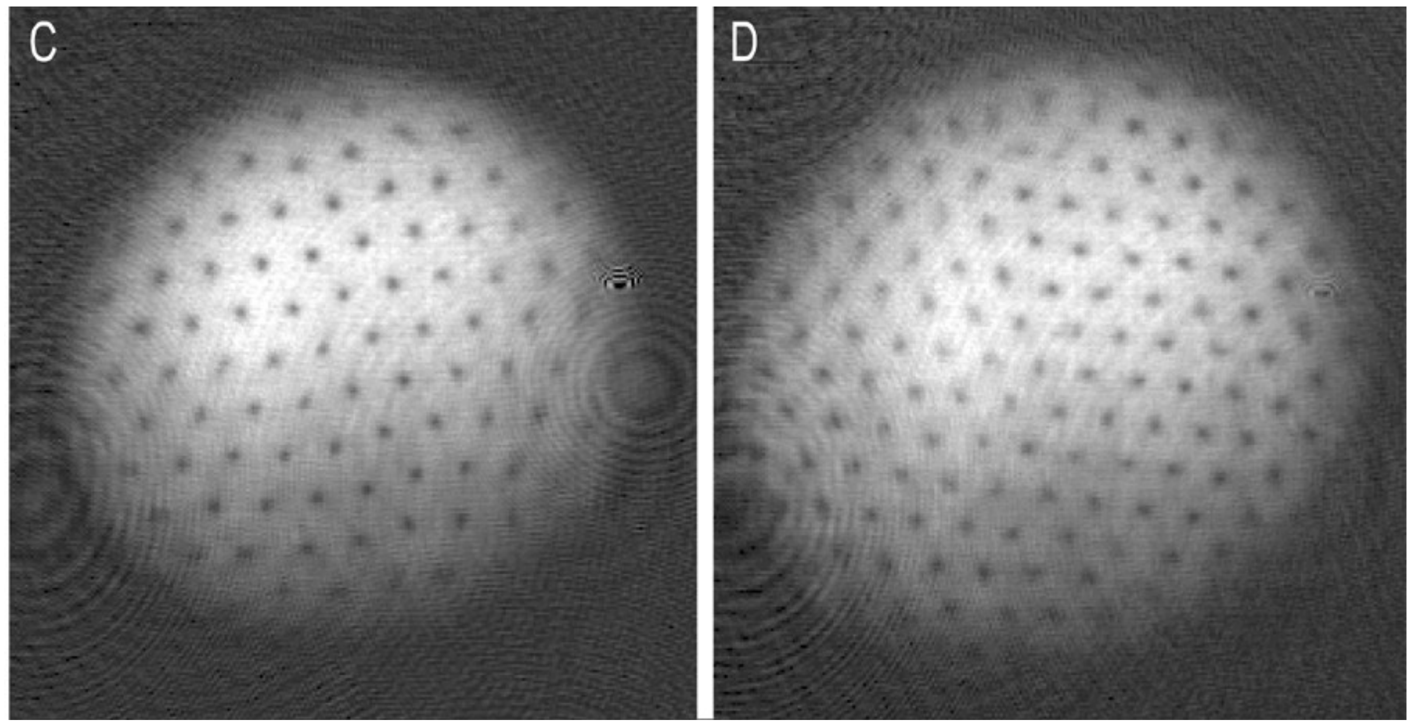
arXiv:2212.08671

Plan

- What is the Tkachenko mode?
- Tkachenko mode from the symmetry point of view
- Linear theory
- Nonlinear theory: use of non commutative field theory
- Prediction: decay rate of the Tkachenko mode

Rotating superfluid (2D)



- quantized circulation around each vortex

$$\int \mathrm{d}\mathbf{l} \cdot \mathbf{v} = \frac{\hbar}{m}\Delta\varphi = \frac{2\pi\hbar}{m}$$

Rotation as B field

- To counteract the centrifugal force, a quadratic trapping potential is imposed
- When the frequency of the potential is fine-tuned: only Coriolis force remains
- Equivalent to magnetic field $B = \frac{2\Omega}{m}$

What is Tkachenko mode

- Elastic waves on the vortex lattice
- first considered by Vladimir Tkachenko ca. 1965-1969
- observed in 2003 in BEC of ultracold atoms
- may play a role in compact stars ?
- (review by Sonin arXiv:1311.1781)

Strange features

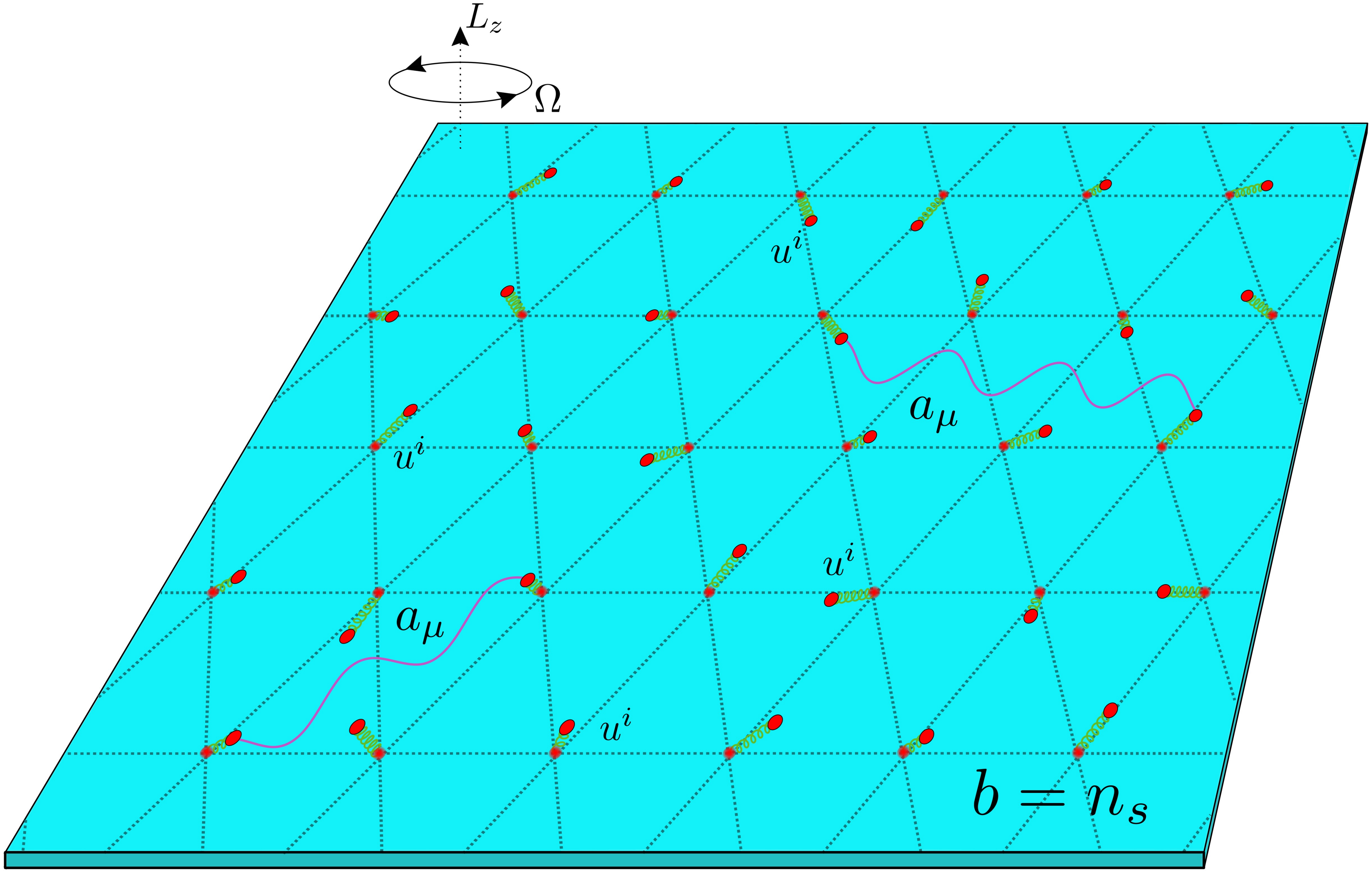
- quadratic dispersion $\omega = Ck^2$ at small k
- Only one polarization: transverse
- One Nambu-Goldstone boson for breaking of
  - 2 translations
  - U(1) phase rotation (Watanabe-Murayama 2013)

LLL regime

- At high rotation frequency: all bosons are on the lowest Landau level (LLL)
    - many questions in FQHE can be asked: role of GMP algebra, non-commutativity of spatial coordinates
- Filling factor ν can be >1 for bosons
- $\nu \gg 1$: “LLL mean-field regime”
- Transition to quantum Hall regime at $\nu \sim 1$

Dual description

- In 2+1D superfluid NGB = photon
- Vortex lattice = lattice of charge particles interacting with a dynamical gauge field
- Tkachenko mode is a mixture of superfluid photon and transverse acoustic phonon



L_z
Ω
u^i
u^i
a_μ
u^i
a_μ
u^i
$b = n_s$

Linear theory

- $\mathscr{L} = \varepsilon^{ij} u_i \dot{u}_j + \dot{\mathbf{u}}^2 - G(\partial_i u_j + \partial_j u_i)^2 + \mathbf{u} \cdot \mathbf{e} + \frac{1}{2}(\mathbf{e}^2 - b^2)$
- Regime of interest: $\omega \sim q^2$

Linear theory

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Linear theory

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Linear theory

- $\mathscr{L} = \varepsilon^{ij} u_i \dot{u}_j + \dot{\mathbf{u}}^2 - G(\partial_i u_j + \partial_j u_i)^2 + \mathbf{u} \cdot \mathbf{e} + \frac{1}{2}(\mathbf{e}^2 - b^2)$
- Regime of interest: $\omega \sim q^2 \ll q$
- $\dfrac{\delta S}{\delta a_0} = \nabla \cdot \mathbf{u} = 0 \qquad u^i = \varepsilon^{ij} \partial_j \phi$
    - only transverse wave survives in the IR (compression mode too costly in energy)
    - integrating out a_i : Lifshitz scalar $\mathscr{L} = \dot{\phi}^2 - (\nabla^2 \phi)^2$

Dipole symmetry

- Relationship between displacement u_i and ϕ:

$$u^i = \ell^2 \varepsilon^{ij} \partial_j \phi$$

Watanabe, Murayama 2013

- Spatial translation becomes dipole symmetry:

$$\phi \to \phi + \mathbf{a} \cdot \mathbf{x}$$
$$u^i \to u^i + c^i \qquad c^i = \ell^2 \epsilon^{ij} a_j$$

Problem with dipole symmetry

- In a magnetic field translations do not commute, rather

$$[P_x, P_y] = i\ell^2 Q$$

- This is not realized in the current version of dipole symmetry:

$$P_x : \phi \to \phi + \alpha y, \quad P_y : \phi \to \phi - \alpha x$$

- Need to treat the symmetry nonlinearly

Nonlinear treatment of the lattice

- A state of a solid is a map between external spatial coordinates and the frozen spatial coordinates

$$x^i \leftrightarrow X^a = \delta^a_i x^i - u^a$$

- Vortex number current:

$$j^\mu = \frac{n_0}{2}\varepsilon^{\mu\nu\lambda}\varepsilon^{ab}\partial_\nu X^a \partial_\lambda X^b \quad \rho = n_0\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}\right)$$

Coupling lattice to NGB

- $\mathscr{L} = \varepsilon^{\mu\nu\lambda}\varepsilon^{ab}a_\mu\partial_\nu X^a\partial_\lambda X^b$ − elastic energy $- b^2 - a_0\rho_0$

- Constraint:

$$\frac{\delta S}{\delta a_0} = 0 \to \frac{1}{2}\varepsilon^{ij}\varepsilon^{ab}\partial_i X^a\partial_j X^b = 1$$

- Mapping from x^i to X^a is area preserving

Noncommutative field

- On the lowest Landau level: $[\hat{x}, \hat{y}] = i\theta = -i\ell^2$
- $\frac{1}{2}\varepsilon^{ij}\varepsilon^{ab}\partial_i X^a \partial_j X^b = 1$ $\qquad \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x} = 1$

$$\stackrel{\sim}{\to} [\hat{X}, \hat{Y}] = i\theta$$

- x^i to X^a is a unitary transformation: $\hat{X}^a = e^{i\hat{\phi}}\hat{x}^a e^{-i\hat{\phi}}$
- Tkachenko mode = noncommutative field $\hat{\phi}(\hat{x})$

Noncommutative field theory

- Origin in high-energy physics
- Ideas of noncommutativity of spatial coordinates ~ 1940-1950
- Became popular in string theory ~ 1999, applications to FQHE
  Fradkin, Jejjala, Leigh; Goldman, Senthil…
- Coordinates do not commute $[\hat{x}, \hat{y}] = i\theta$,
  - $\hat{\Phi}_1(\hat{x}, \hat{y})\hat{\Phi}_2(\hat{x}, \hat{y}) \neq \hat{\Phi}_2(\hat{x}, \hat{y})\hat{\Phi}_1(\hat{x}, \hat{y})$

Weyl symbols

- Noncommutative field theory can be mapped to a familiar commutative field theory through a procedure:

- An operator is mapped to its Weyl symbol

$$\hat{x} \to x, \;\; \hat{y} \to \hat{y}, \quad \frac{1}{2}(\hat{x}\hat{y} + \hat{y}\hat{x}) \to xy$$
$$\frac{1}{3}(\hat{x}^2\hat{y} + \hat{x}\hat{y}\hat{x} + \hat{y}\hat{x}^2) \to x^2y$$

- Operator multiplication mapped to star (Moyal) product

$$\hat{A}\hat{B} \to A \star B = A \exp\!\left(\frac{i}{2}\theta\epsilon^{ij}\overleftarrow{\partial_i}\,\overrightarrow{\partial_j}\right)B = AB + \frac{i}{2}\theta\epsilon^{ij}\partial_i A \partial_j B + \cdots$$

Magnetic translations

- $\overrightarrow{X}(\vec{x}) \to \overrightarrow{X}'(\vec{x}) = \overrightarrow{X}(\vec{x} - \vec{c})$ translates to

$$e^{i\phi} \to e^{i\phi'} = \exp\left(\frac{i}{\theta}\vec{c} \times \vec{x}\right) \star e^{i\phi}$$

- $\phi \to \phi' = \phi + \frac{1}{\theta}\vec{c} \times \vec{x} + \frac{1}{2}\vec{c} \cdot \overrightarrow{\nabla}\phi + \cdots$

- 2 translations do not commute: $[P_{\vec{c}}, P_{\vec{c}'}] = i(\vec{c} \times \vec{c}')Q$

  - $Q : \phi \to \phi + \alpha$

  - ϕ is the superfluid phase

Constructing interacting Lagrangian

- $D_\mu \phi \equiv -ie^{-i\phi} \star \partial_\mu e^{i\phi}$

- Under magnetic translation:

$$D_0\phi \to e^{-i\vec{\alpha}\cdot\vec{x}} \star D_0\phi \star e^{i\vec{\alpha}\cdot\vec{x}}$$
$$D_i\phi \to e^{-i\vec{\alpha}\cdot\vec{x}} \star D_i\phi \star e^{i\vec{\alpha}\cdot\vec{x}} + \alpha_i$$

- Invariant Lagrangian: $L = L(D_t\phi, \partial_i D_j\phi)$

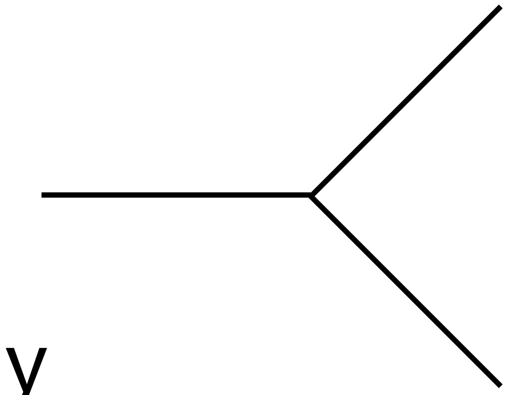
- Magnetic rotation symmetry $e^{i\phi} \to e^{i\omega\vec{x}^2} \star e^{i\phi}$: further constraints

Decay of Tkachenko mode

- Expanding the Lagrangian to cubic order:

$$S = \int d^3x \left[\dot{\phi}^2 - (\nabla^2 \phi)^2 + g_1 \dot{\phi}^3 + g_2 \dot{\phi} (\nabla^2 \phi)^2 + g_3 (\nabla^2 \phi)^3 \right]$$

- $\Gamma_{\phi \to 2\phi} \sim g^2 E^3$



sharply defined quasiparticle at low energy

(previous result $\Gamma \sim E$ Matveenko and Shlyapnikov 2011)

Further comments

- Dipole symmetry on a torus:

  magnetic field breaks translation symmetry to discrete

$$x \to x + \frac{\ell^2}{L_y} 2\pi n_x \,, \quad n_x \in \mathbb{Z}$$

- corresponds to dipole transformation

$$\phi \to \phi + \frac{2\pi}{L_y} n_x y$$

Girvin-Macdonald-Platzman

- Gauging the U(1) particle number: nonabelian symmetry

  instead of $\phi(x) \to \phi(x) + \alpha(x)$

  $$e^{i\phi} \to e^{i\alpha} \star e^{i\phi}$$

- Gauge transformations are generated by charge density

  $$\int d^2x \alpha(x) \rho(x)$$

- Noncommutativity of gauge transformations: GMP algebra for $\rho(x)$

Algebraic order

- The correlator of ϕ diverges in the IR

$$\langle \phi(0,\vec{x})\phi(0,\vec{y})\rangle \sim \frac{\#}{\nu} \ln|\vec{x}-\vec{y}|$$

- which implies that there is strictly no U(1) SSB

$$\langle e^{i\phi(\vec{x})} e^{-i\phi(\vec{y})}\rangle \sim \frac{1}{|\vec{x}-\vec{y}|^{\#/\nu}}$$

Conclusions

- Tkachenko mode: sound waves but $\omega \sim q^2$
- common NGB of SSB of U(1) particle number and magnetic translations
- A NGB of a noncommutative field theory
- Decay rate $\Gamma(E) \sim E^3$