Many-body localization as a Dynamical Renormalization Group Fixed Point

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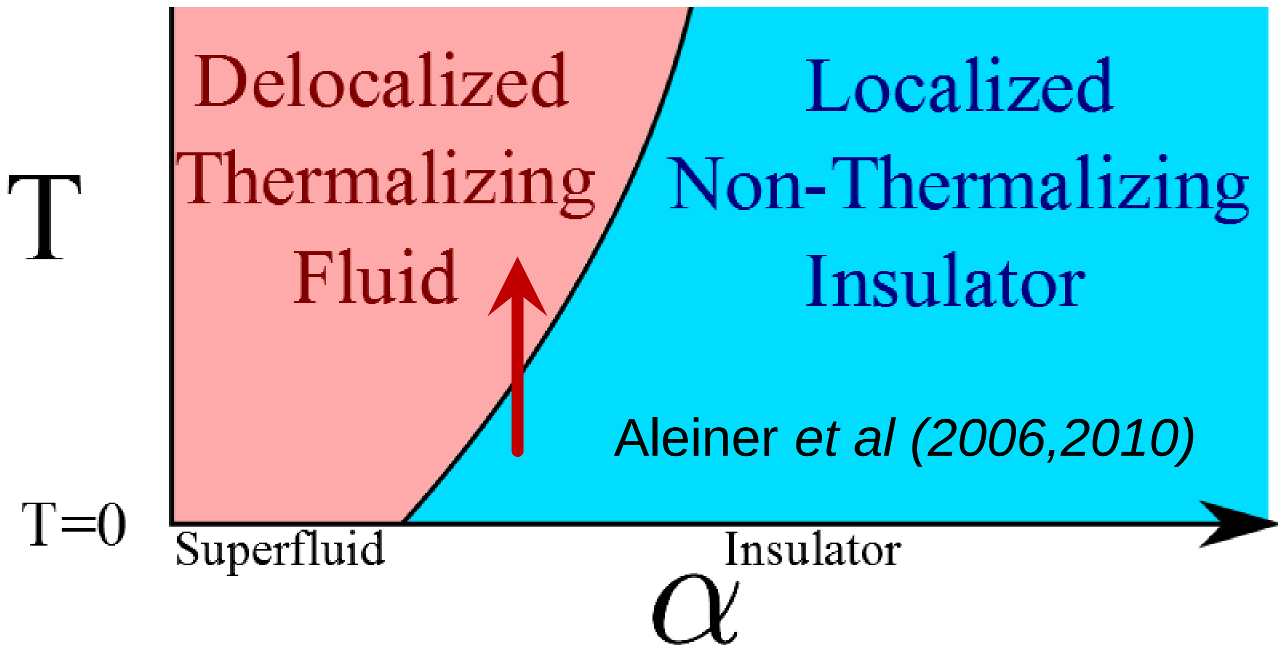
arXiv:1205.0026

Minerva-Weizmann program

Many-Body Localization



If the model has bounded spectrum, one can attempt to drive the transition at infinite temperature
Oganesyan and Huse *(2007), Pal and Huse (2010)*

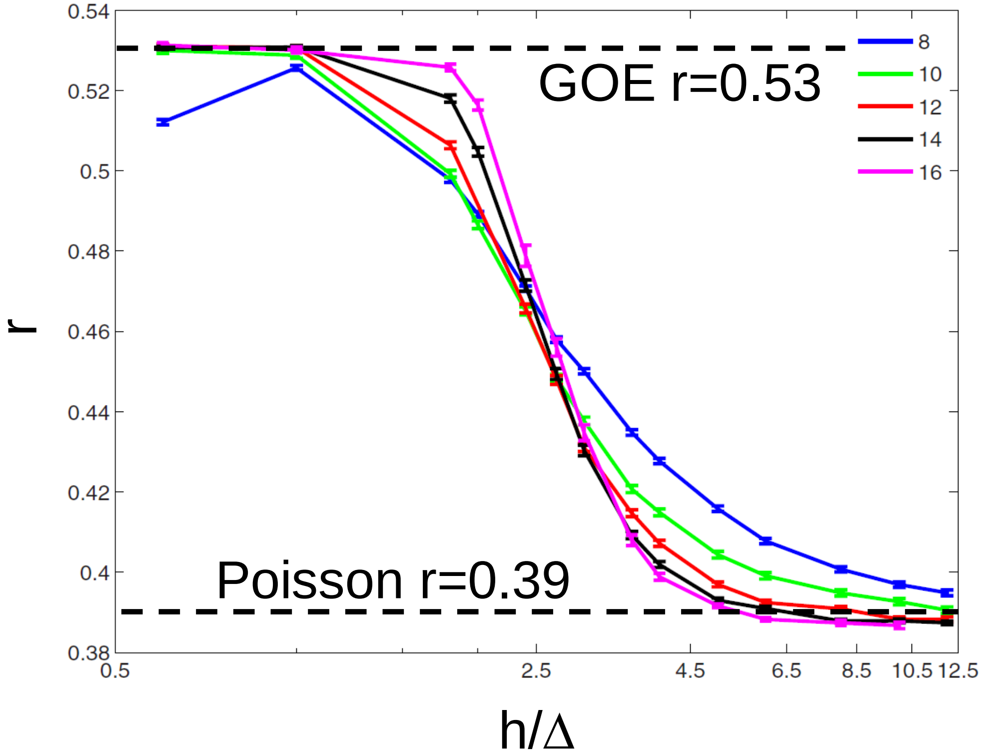
Disordered Spin Chains

A. Pal and D. Huse, Physical Review B **82**, 1 (2010)

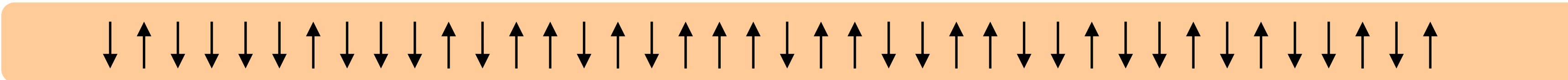
$$H = \frac{1}{2}\sum_{ij}(S_i^+ S_j^- + \text{H.c}) + \Delta \sum_{ij} S_i^z S_j^z + \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

= interacting fermions: $$H = \frac{1}{2}\sum_{ij}\left(a_i^\dagger a_j + \text{H.c.}\right) + \sum_i h_i n_i + \Delta \sum_{ij} n_i n_j$$

Ratio of adjacent energy gaps from exact diagonalization of 16 sites:



Thermalization and dynamics of entanglement entropy in disordered spin chains

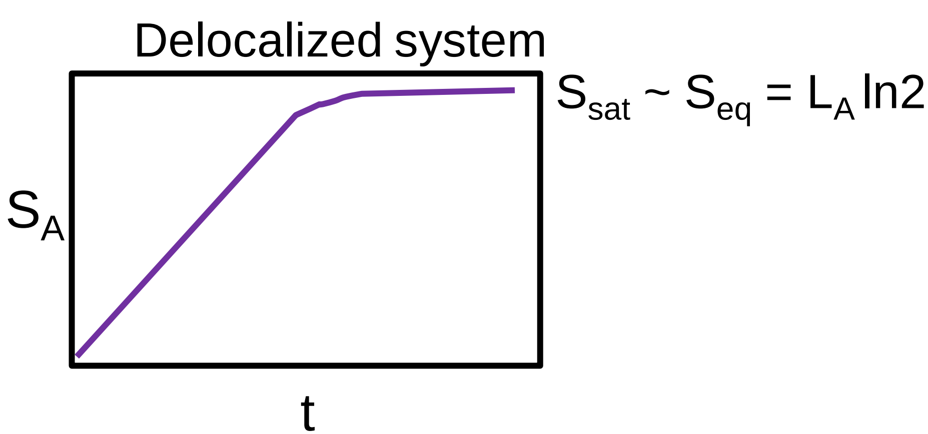


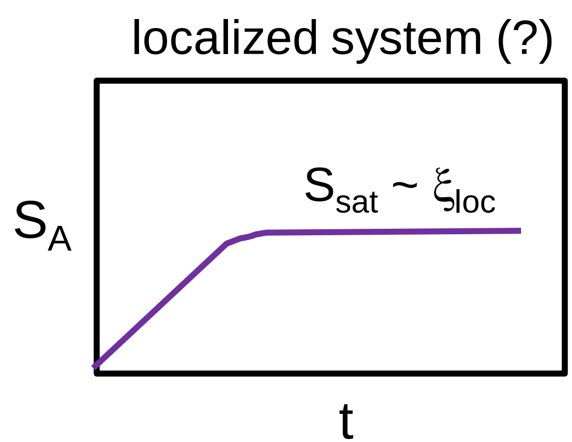
$$e^{-iHt}\,|\,\Psi_0\,\rangle \qquad H = \frac{J}{2}\sum_i \left(S_i^+ S_{i+1}^- + \mathrm{H.c.}\right) + J\Delta \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

= interacting fermions: $H = \frac{J}{2}\sum_i \left(a_i^\dagger a_{i+1} + \mathrm{H.c.}\right) + \Delta J \sum_i n_i n_{i+1} + \sum_i h_i n_i \qquad h_i \in [-h, h]$

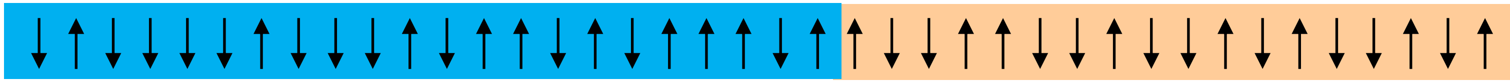
$\rho_A \qquad \rho_B$

Von-Neuman entropy generated in the dynamics: $S_A(t) = -Tr\left[\rho_A(t)\ln\rho_A(t)\right]$





Entanglement dynamics: numerics

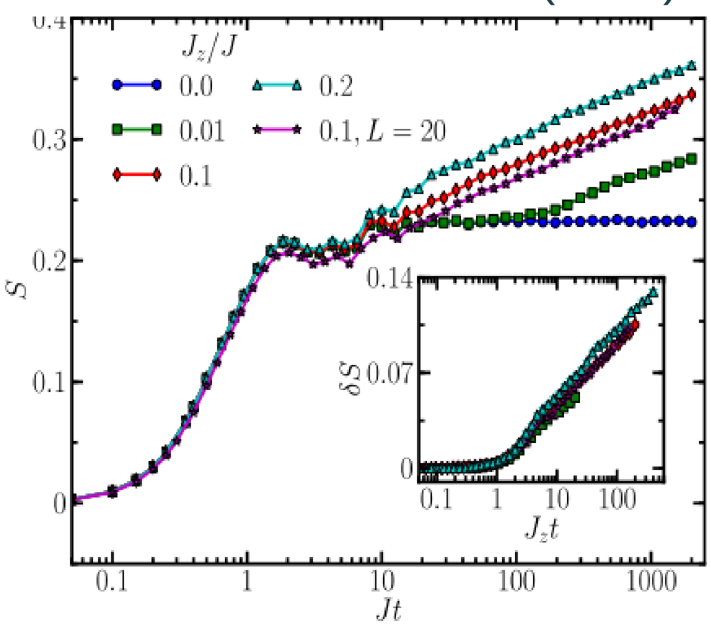


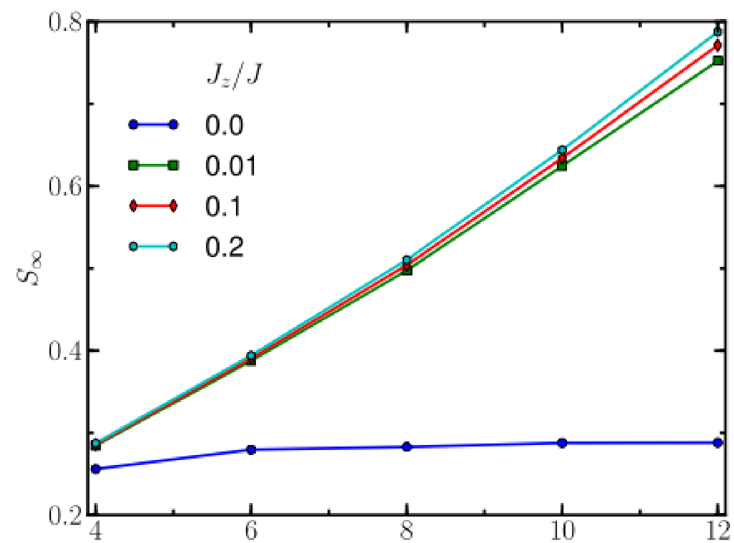
$$H = \frac{J}{2}\sum_i \left(S_i^+ S_{i+1}^- + \mathrm{H.c.}\right) + \underbrace{J\Delta}_{J_z}\sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z \qquad h_i \in [-h, h]$$

Entropy growth $S_A(t)$
- Non interacting: Saturation
- Interacting log(t) increase

Saturation in finite system
- Non interacting: S_{sat}=const
- Interacting: $S_{sat} = s_0 L$
  extensive, but much smaller
  then expected thermal entropy

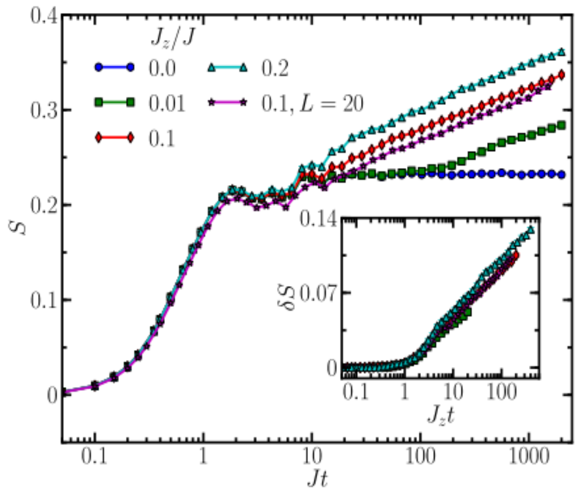
Bardarson et. al. PRL (2012)

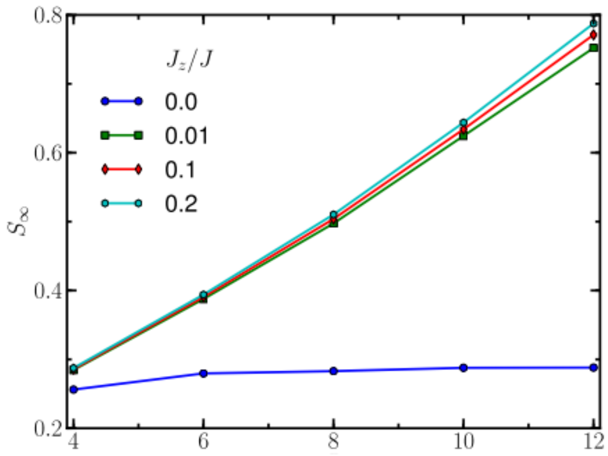




Earlier numerical studies: De Chiara et. al. (2006); Znidaric et. al. (2008)

Questions and goals for theory





- Explain the universal evolution of the entanglement entropy in this “localized” state as seen in numerics.
- Does the system thermalize? Description of the long time steady state?
- Nature of the transition to the delocalized state?

Outline

- Derivation of real space RG for quantum time evolution in strong disorder and application to model:

$$H = \frac{1}{2}\sum_i J_i \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2\Delta_i S_i^z S_{i+1}^z\right)$$

"Interaction"

- Main Results:

1. Flow to infinite randomness fixed point

2. Delayed logarithmic growth of entanglement entropy:

$$S(t) \approx \eta_1 \ln(t/t_{delay})\,\Theta(t - t_{delay}) + \left[\eta_2 \ln(t/t_{delay})\right]^{2/\phi}\Theta(t - t_*)$$

Compare to noninteracting case: $S(t) \sim \ln(\ln t)$

3. Emergent conserved quantities → no thermalization

Real space RG for dynamics

Working model:

$$H = \frac{1}{2}\sum_i J_i \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2\Delta_i S_i^z S_{i+1}^z\right)$$

$J_i \in [-\Omega, \Omega]$ $\quad$ $|\Delta_i| \ll 1$ $\quad$ drawn from uncorrelated broad distributions

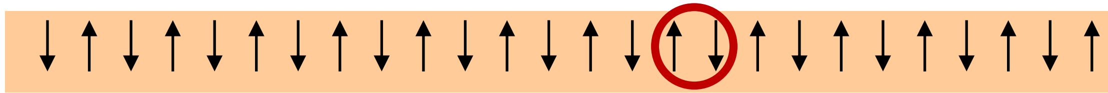
Neel initial state: $\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow$

We want to compute: $\quad \rho(t) = e^{i\,Ht}\rho(0)e^{-i\,Ht} = \,?$

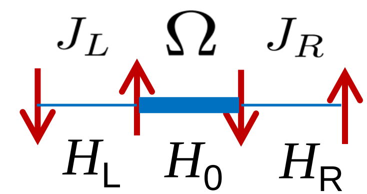
Use the basic idea of real-space RG for strong disorder
(Das gupta & Ma 79, D. S. Fisher 92)

But instead of targeting ground state target the long time dynamics.

Real space RG for the dynamics - application



1. Short times described by rapid oscillations (freq. Ω) performed by pairs of spins coupled by the strongest bonds.$J=\Omega$.



That is all we have at time scale $t \approx \Omega^{-1}$
all other spins are essentially frozen!

2. Compute effective dynamics at times $t \gg \Omega^{-1}$ (eliminating frequencies of order Ω)

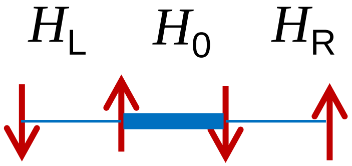
$$\rho(t) = \left[U_I^\dagger \rho_0 U_I\right]_{\Omega^{-1}} = e^{iH_{eff}t}\rho_0 e^{-iH_{eff}t}$$

2nd order expansion of U in the interaction picture w.r.t H_0

Average over rapid oscillations

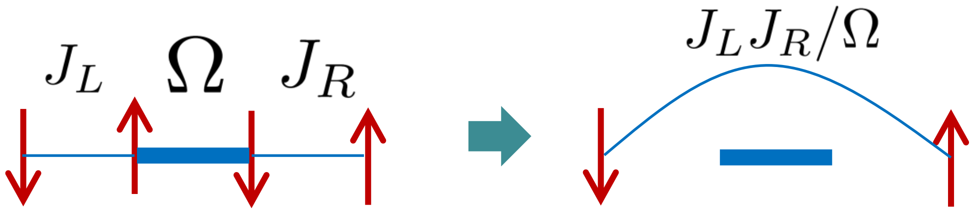
3. Iterate to obtain flow of the (distribution of) coupling constants

Perturbation expansion of the evolution operator:

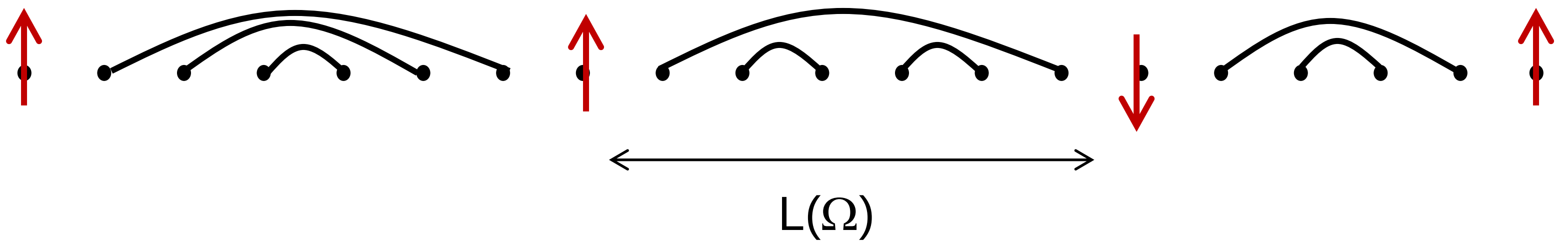


$$U_I = 1 - \frac{i}{\hbar}\int_0^t dt_1 e^{\frac{i}{\hbar}H_0 t_1}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_1}$$
$$-\frac{1}{\hbar^2}\int_0^t dt_1 \int_0^{t_1} dt_2 e^{\frac{i}{\hbar}H_0 t_1}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_1}e^{\frac{i}{\hbar}H_0 t_2}\left(H_R + H_L\right)e^{-\frac{i}{\hbar}H_0 t_2}$$

Real space RG – non interacting case (Δ=0)



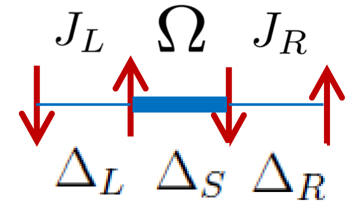
Effective spin-chain after many iterations:



Important outcome - a relation between length and time scales

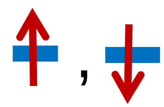
L(Ω) = mean separation between spins (length of clusters) at scale Ω=1/t

The RG decimation step for Δ>0



Need to keep track of a new spin on the strong bond

$$H_{eff} = \frac{J_L J_R}{2\Omega(1-\Delta_S^2)}\left[(1+\Delta_S S_n^z)(S_L^+ S_R^- + H.c.) - \Delta_L \Delta_R S_n^z (S_L^z S_R^z)\right]$$



$$= |\pm\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\Big)$$

The new spin initially points along x or –x therefore the evolution is a superposition of the dynamics given an up-spin on the bond and the dynamics with a down-spin:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{iH_{eff}^{(+)}t}|\psi_0^{LR}\rangle\,|\uparrow_n\rangle + e^{iH_{eff}^{(-)}t}|\psi_0^{LR}\rangle\,|\downarrow_n\rangle\right)$$

This leads to entanglement between decimated bond and the nearby spins after a time

$$t_{\text{ent}} = \frac{2\Omega}{J_L J_R \Delta_S}$$

But no effect on subsequent renormalization of coupling constants!

$$\tilde{J} \approx J_L J_R/\Omega \qquad |\tilde{\Delta}| \approx |\Delta_L||\Delta_R|/4$$

Flow of distributions

Scaling variables: $\Gamma = \ln(\Omega_0/\Omega) = \ln(\Omega_0 t)$

$\zeta = \ln(\Omega/J)$ $\beta = -\ln|\Delta|$

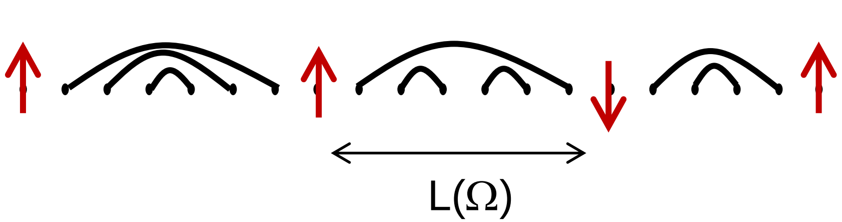
Flow equation for joint probability distribution:

$$\frac{\partial P}{\partial \Gamma} = \frac{\partial P}{\partial \zeta} + \rho(0;\Gamma)\int_0^\infty d\beta_L d\beta_R d\zeta_L d\zeta_R \delta(\zeta - \zeta_L - \zeta_R)\delta(\beta - \beta_L - \beta_R - \ln 4)P(\zeta_L, \beta_L;\Gamma)P(\zeta_R, \beta_R;\Gamma)$$

Exact solution for two crucial properties

$$\Gamma = \ln(\Omega_0/\Omega) = \ln(\Omega_0 t) \qquad \zeta = \ln(\Omega/J) \qquad \beta = -\ln|\Delta|$$

1. Individual distribution of J's. Flow to infinite randomness



$$\rho(\zeta;\Gamma) \equiv \int d\beta P(\beta,\zeta;\Gamma) = \alpha(\Gamma) e^{-\alpha(\Gamma)\,\zeta}$$

with: $\alpha(\Gamma) = \dfrac{1}{\Gamma + \alpha_0^{-1}}$

Analogous to the random singlet ground state
(Dasgupta&Ma 79, Fisher 94)

$$L(\Gamma) = (\alpha_0\Gamma + 1)^2 \rightarrow [\alpha_0 \ln(\Omega_0 t)]^2$$

Immediate consequence: decay of AF order as $m_{AF} \sim \left(\dfrac{1}{\ln(\Omega_0 t)}\right)^2$

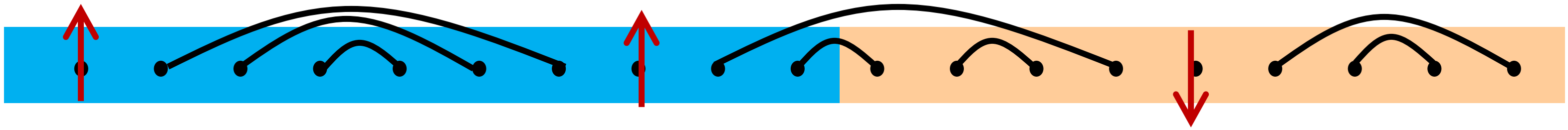
2. Conditional average value of *interaction*
(average β on a bond with given ζ)

$$\bar{\beta}(\zeta,\Gamma) \equiv \int_0^\infty d\beta\, \beta\, \frac{P(\zeta,\beta;\Gamma)}{\rho(\zeta;\Gamma)}$$

Entropy growth in the “non interacting” case (Δ=0)

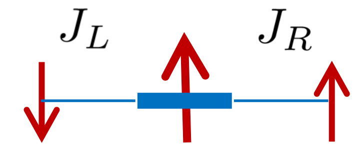
<u>In the case Δ=0 :</u> Only intra-pair entanglement

Compute entanglement entropy by counting the number of decimated bonds that cut the interface.
Each decimated bond crossing the interface contributes ~log2.
(As in the ground state of random singlet phase – Refael & Moore PRL 2004)



$$S_{ent} \sim \int_0^{\Gamma} \alpha(\Gamma')d\Gamma' = \ln(\Gamma + \alpha_0^{-1}) = \ln\left(\ln(\Omega_0 t) + \alpha_0^{-1}\right)$$

Entropy growth in the interacting case (Δ>0)



A bond eliminated at t_1 builds entanglement with neighbors only at a later time $t=t_1+t_{ent}$.

$$t_{\mathrm{ent}} = \frac{2\Omega}{J_L J_R \Delta_S}$$

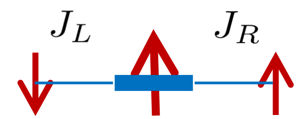
The interaction generates entanglement only after a delay time from the start of time evolution

$$t_{\mathrm{delay}} \approx \frac{2\Omega_0}{J_0^2 \Delta_0} = \left(\frac{2\Omega_0}{J_0}\right)\frac{1}{J_0^z}$$

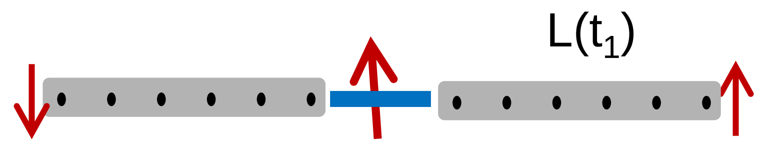
How much entanglement is generated?

Entropy growth in the interacting case (Δ>0)

Entanglement measured at time t originates from pairs eliminated at earlier time t_1



Remaining spins at t_1 are separated by decimated clusters of length $L(t_1)$



By the time $t=t_1+t_{ent}$ that these spins become entangled the decimated clusters between them must also be entangled

$$S(t) \approx L(t_1) = \Big(\alpha_0 \underbrace{\ln(\Omega t_1)}_{\Gamma_1} + 1\Big)^2$$

Relation between t_1 and t :

$$t = t_1 + t_{\text{ent}} = t_1\left(1 + \frac{2\Omega_1^2}{J_L J_R \Delta_S}\right) \approx t_1 \frac{2\Omega_1^2}{J_1^2 \Delta_1}$$

Taking log of both sides and using solutions for typical value of ζ and conditional average of β

$$\Gamma \ = \ 3\Gamma_1 + \tfrac{1}{b_0}(a_0\Gamma_1 + 1)^{\phi} + 2/a_0 + \ln 2$$

Evolution of the entanglement entropy (Δ>0)

$$S(t) \approx \eta_1 \ln(t/t_{delay})\,\Theta(t - t_{delay}) + [\eta_2 \ln(t/t_{delay})]^{2/\phi}\,\Theta(t - t_*)$$

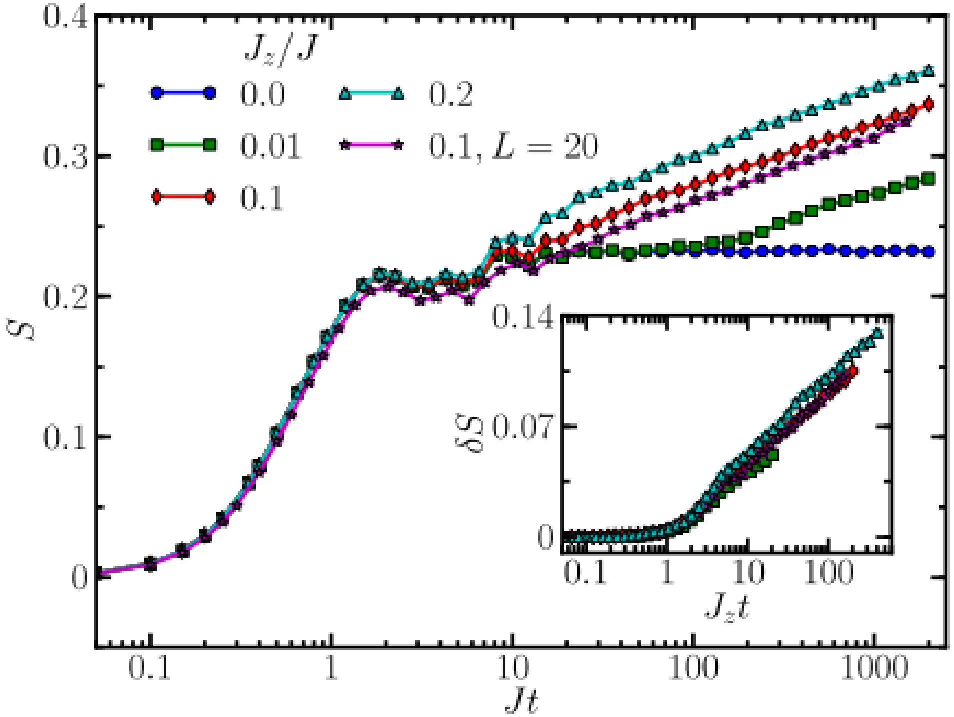
$$\eta_1 = \frac{1}{\ln(\Omega_0/J_0)}$$

$$\eta_2 = -1/\ln(\Delta_0)$$

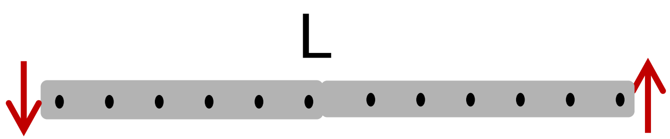
ϕ - Golden ratio

$$t_{\text{delay}} \approx \frac{2\Omega_0}{J_0^2\Delta_0} = \left(\frac{2\Omega_0}{J_0}\right)\frac{1}{J_0^z}$$

Compare with numerical results from Bardarson et. al. PRL (2012)



Saturation of entanglement entropy in a finite system

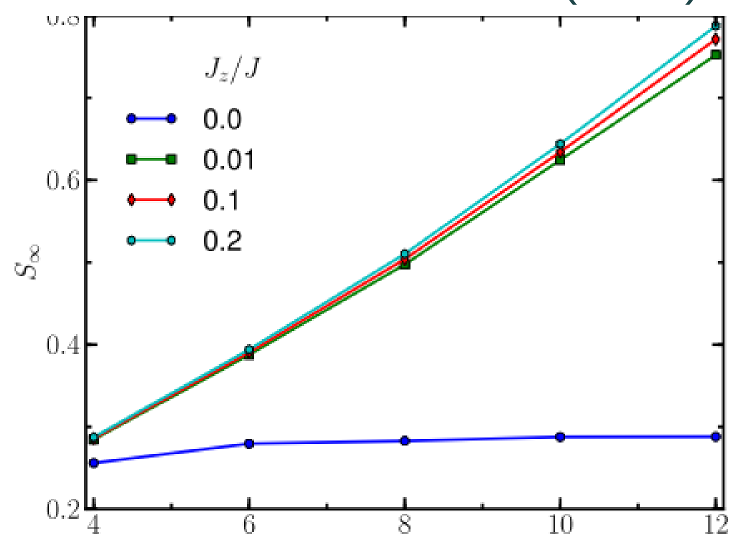


Saturation time: $t(L) = \Omega_0^{-1} e^{\Gamma(L)} \approx \Omega_0^{-1} e^{\sqrt{L}/\alpha_0}$

Entropy saturates to an extensive value: S(L) ~ L

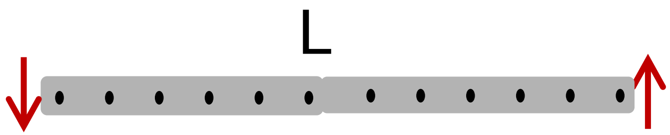
In agreement with the numerical results:

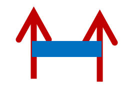
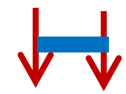
Bardarson et. al. PRL (2012)



Saturation value is not the expected thermalized value S(L)= L ln2. Why?

Emergent conservation laws

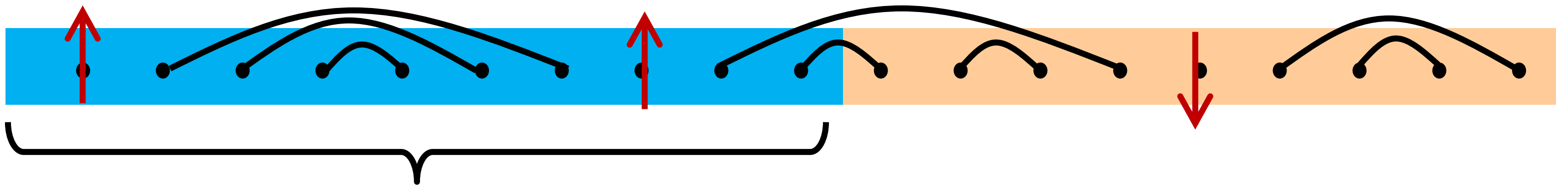


In every decimated pair of spins the states  and  are never populated therefore S(L)<(L/2)ln2

More generally $I_p = (S_1^z S_2^z)_p$ are approximate constants of motion (asymptotically exact for long distance pairs)

Many-body localization (non thermalization) $\stackrel{?}{=}$ emergent GGE

Evolution of particle number fluctuations



$$\langle \delta N^2(t) \rangle = ?$$

Since the $\uparrow\uparrow$ and $\downarrow\downarrow$ states of decimated pairs are not populated, only pairs that intersect the interface contribute to $\langle \delta N^2(t) \rangle$

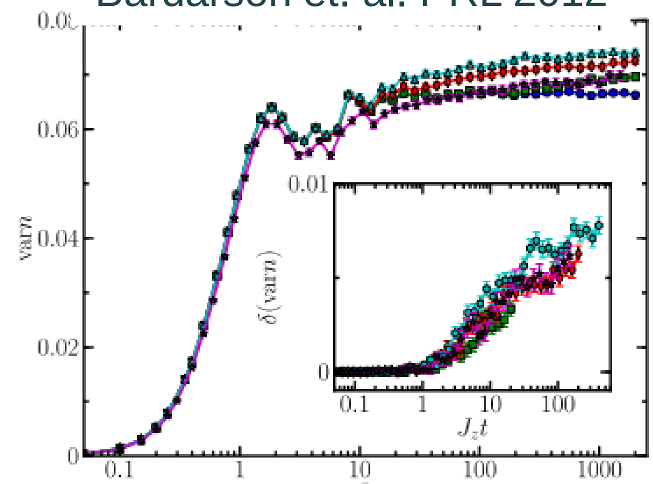
$$\langle \delta N^2(t) \rangle = \ln\left(\ln(\Omega_0 t) + \alpha_0^{-1}\right)$$

Much slower than entanglement growth and independent of interaction!

Saturates to a non-extensive value in a finite system: $\langle \delta N^2(\infty) \rangle \sim \ln L$

Localized

Bardarson et. al. PRL 2012



Range of validity of the RG scheme

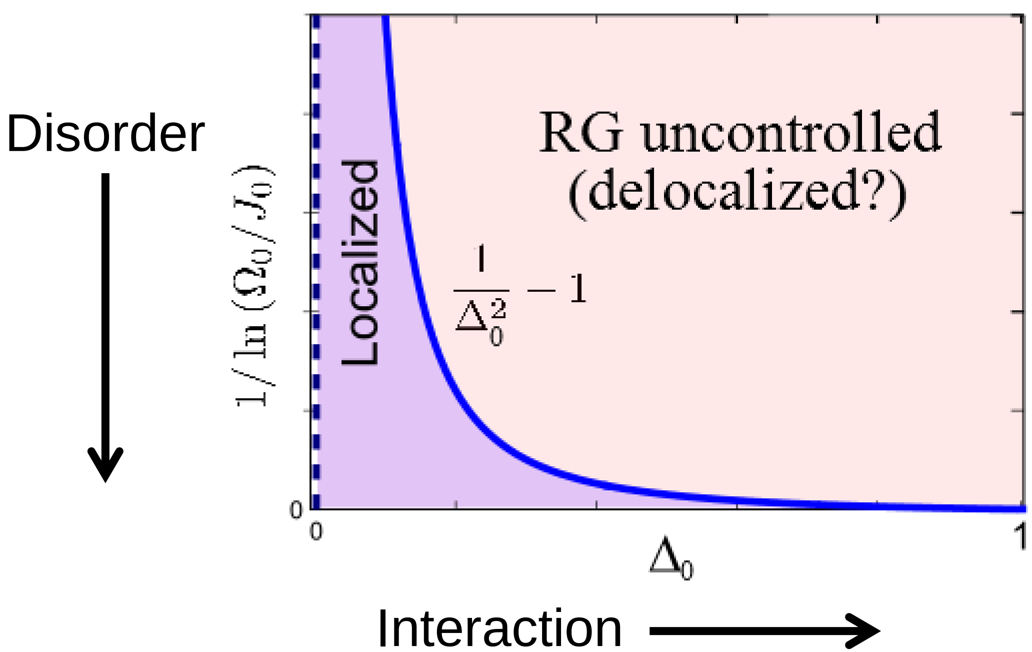
?= Extent of the localized state

A criterion for initial conditions that lead to the localized fixed point can be found from the RG rule:

$$\tilde{J} = \frac{J_L J_R}{\Omega(1-\Delta_S^2)}$$

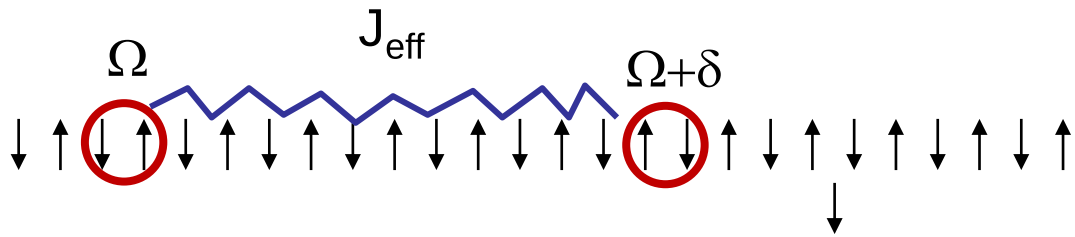
In order to flow to increasing randomness the typical J must decrease in the process. Therefore demand:

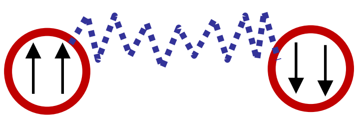
$$\frac{J_{typ}^2}{\Omega(1-\bar{\Delta}^2)} < J_{typ}$$



Rare distant resonances

The RG scheme dos not take such events into account



If $J_{eff} > \delta$ then the two pairs can switch to 

Increasingly rare with increasing disorder therefore expected to be irrelevant at the infinite randomness fixed point !

Moreover: Anderson localization arguments in Fock space (Basko et. al. 2006) imply they are irrelevant at some finite disorder).

Summary

- Formulated RG for dynamics of random spin chains
- Many-body localized state found for XXZ chain with initial Neel state. Identified as infinite randomness fixed point
- Entanglement growth:

$$S(t) \approx \eta_1 \ln\left(t/t_{\mathrm{delay}}\right) \qquad t_{\mathrm{delay}} < t \ll t_*$$

$$S(t) \approx \left[\eta_2 \ln\left(t/t_{\mathrm{delay}}\right)\right]^{2/\phi} \qquad t \gg t_*$$

- Particle number fluctuations: $\langle \delta N^2(t) \rangle \sim \ln\ln(\Omega_0 t)$
- Non thermal steady state can be understood as Generalized Gibbs ensemble with the asymptotic conserved quantities:

$$\left(S_1^z S_2^z\right)_{\mathrm{pair}}$$

Outlook / questions

- Nature of the steady state for generic initial conditions and generic disorder (allow local Zeeman fields)

- Critical point controlling the many-body localization transition?