Dynamics and description after relaxation of disordered quantum systems after a sudden quench

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in transition to

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Dynamics and Thermodynamics in Isolated Quantum Systems
Kavli Institute for Theoretical Physics, UC Santa Barbara
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Supported by:

Quenches in disordered quantum systems

1. Introduction
   - Foundations of statistical mechanics
   - Experiments with ultracold gases in 1D
   - Unitary evolution and thermalization
   - Results for nonintegrable and integrable systems
2. Non-equilibrium dynamics in the presence of disorder
   - Nonintegrable system
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3. Summary

Foundations of quantum statistical mechanics

Quantum ergodicity: John von Neumann ‘29
(Proof of the ergodic theorem and the H-theorem in quantum mechanics)

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Recent works:

Goldstein, Lebowitz, Tumulka, and Zanghi ‘06
(Canonical Typicality)

Popescu, Short, and A. Winter ‘06
(Entanglement and the foundation of statistical mechanics)

Goldstein, Lebowitz, Mastrodonato, Tumulka, and Zanghi ‘10
(Normal typicality and von Neumann’s quantum ergodic theorem)

MR and Srednicki ‘12
(Alternatives to Eigenstate Thermalization)

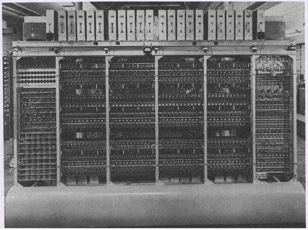
Foundations of classical statistical mechanics

One of the first numerical experiments:

Fermi, Pasta, Ulam, and Tsingou ’53

(Studies of nonlinear problems)

Vibrating chain of oscillators with non-linear couplings
Quasi-periodic behavior, lack of ergodicity



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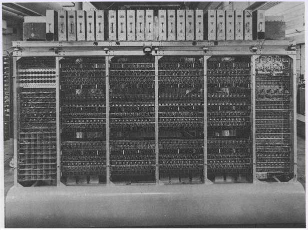

Chaos theory and modern approach to classical statistical mechanics

- Quasi-periodic behavior was not the result of Poincaré recurrences
- Korteweg-de Vries (KdV) equation and solitons in nonlinear systems
- Dynamical chaos and ergodicity
- Kolmogorov-Arnold-Moser (KAM) theorem

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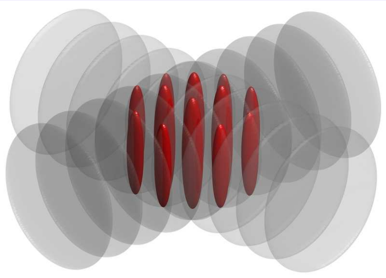

Effective one-dimensional δ potential
M. Olshanii, PRL **81**, 938 (1998).

$$U_{1D}(x) = g_{1D}\delta(x)$$

where

$$g_{1D} = \frac{2\hbar a_s \omega_\perp}{1 - C a_s \sqrt{\frac{m\omega_\perp}{2\hbar}}}$$

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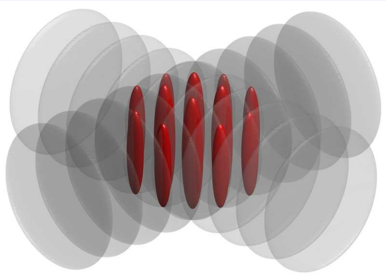

Girardeau '60

T. Kinoshita, T. Wenger, and D. S. Weiss, Science **305**, 1125 (2004).

T. Kinoshita, T. Wenger, and D. S. Weiss, Phys. Rev. Lett. **95**, 190406 (2005).

$$2 \gtrsim \gamma_{\text{eff}} = \frac{m g_{1D}}{\hbar^2 \rho} \gtrsim 20$$

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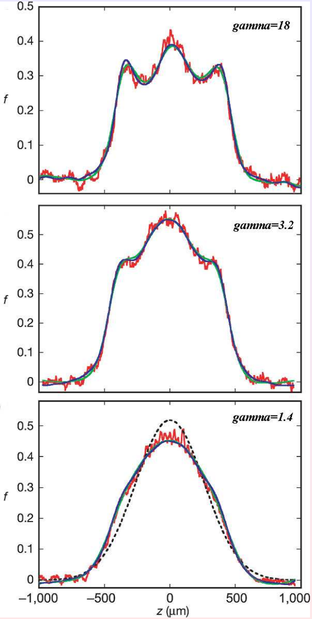
Lieb, Schulz, and Mattis '61

B. Paredes *et al.*,

Nature **429**, 277 (2004).

$\gamma_{\text{eff}} = \frac{U}{J} \approx$ 5–200

Absence of thermalization in 1D?



T. Kinoshita, T. Wenger, and D. S. Weiss, Nature **440**, 900 (2006).

$$\gamma = \frac{m g_{1D}}{\hbar^2 \rho}$$

g_{1D}: Interaction strength
ρ: One-dimensional density

If $\gamma \gg 1$ the system is in the strongly correlated Tonks-Girardeau regime

If $\gamma \ll 1$ the system is in the weakly interacting regime

Gring *et al.*, arXiv:1112.0013.

Quenches in one-dimensional superlattices

Quantum dynamics in a 1D superlattice

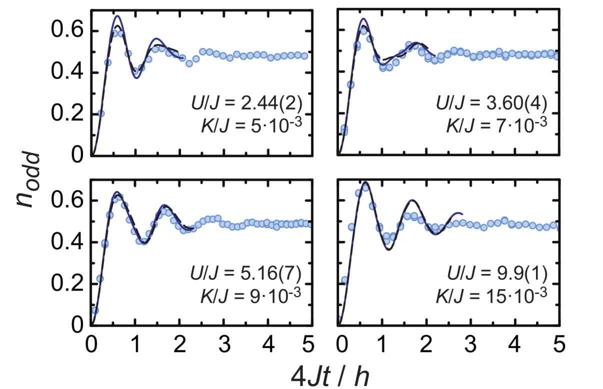
Trotzky *et al.*,

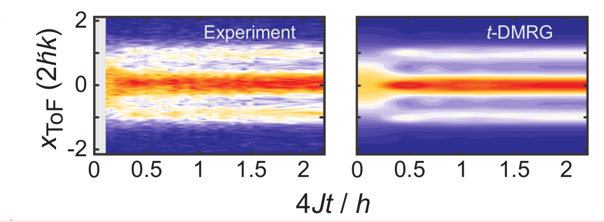
Nature Phys. **8**, 325 (2012).

Initial state $|01010\dots1010\rangle$

Unitary dynamics under the "Bose-Hubbard" Hamiltonian

Experimental results ($\circ$) vs exact t-DMRG calculations (lines) without free parameters





local observables (top)

vs

nonlocal observables (bottom)

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Exact results from quantum mechanics

If the initial state is not an eigenstate of $\widehat{H}$

$$|\psi_0\rangle \neq |\alpha\rangle \quad \text{where} \quad \widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle \quad \text{and} \quad E_0 = \langle\psi_0|\widehat{H}|\psi_0\rangle,$$

then a generic observable O will evolve in time following

$$O(\tau) \equiv \langle\psi(\tau)|\widehat{O}|\psi(\tau)\rangle \quad \text{where} \quad |\psi(\tau)\rangle = e^{-i\widehat{H}\tau}|\psi_0\rangle.$$

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What is it that we call thermalization?

$$\overline{O(\tau)} = O(E_0) = O(T) = O(T,\mu).$$

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One can rewrite

$$O(\tau) = \sum_{\alpha',\alpha} C^\star_{\alpha'} C_\alpha e^{i(E_{\alpha'}-E_\alpha)\tau} O_{\alpha'\alpha} \quad \text{where} \quad |\psi_0\rangle = \sum_\alpha C_\alpha|\alpha\rangle,$$

and taking the infinite time average (diagonal ensemble)

$$\overline{O(\tau)} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau d\tau' \langle\Psi(\tau')|\hat{O}|\Psi(\tau')\rangle = \sum_\alpha |C_\alpha|^2 O_{\alpha\alpha} \equiv \langle\hat{O}\rangle_{\text{diag}},$$

which depends on the initial conditions through $C_\alpha = \langle\alpha|\psi_0\rangle$.

Width of the energy density, sudden quench

Initial state $|\psi_0\rangle = \sum_\alpha C_\alpha |\alpha\rangle$ is an eigenstate of $\widehat{H}_0$. At $\tau = 0$

$$\widehat{H}_0 \to \widehat{H} = \widehat{H}_0 + \widehat{W} \qquad \text{with} \quad \widehat{W} = \sum_{j \in \sigma} \hat{w}(j) \quad \text{and} \quad \widehat{H}|\alpha\rangle = E_\alpha |\alpha\rangle.$$

MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

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The width of the energy density ΔE is then

$$\Delta E = \sqrt{\sum_\alpha E_\alpha^2 |C_\alpha|^2 - (\sum_\alpha E_\alpha |C_\alpha|^2)^2} = \sqrt{\langle\psi_0|\widehat{W}^2|\psi_0\rangle - \langle\psi_0|\widehat{W}|\psi_0\rangle^2},$$

or

$$\Delta E = \sqrt{\sum_{j_1, j_2 \in\sigma} [\langle\psi_0|\hat{w}(j_1)\hat{w}(j_2)|\psi_0\rangle - \langle\psi_0|\hat{w}(j_1)|\psi_0\rangle\langle\psi_0|\hat{w}(j_2)|\psi_0\rangle]} \stackrel{L\rightarrow\infty}{\propto} L^{d_\sigma/2}$$

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Since the width of the full spectrum diverges as L^{d_L}

$$\Delta\epsilon = \frac{\Delta E}{L^{d_L}} \overset{L\rightarrow\infty}{\propto} \frac{1}{L^{d_L - d_\sigma/2}},$$

$d_L(d_\sigma)$ is the dimensionality of the lattice (of the region affected by the quench).

since $d_L \geq d_\sigma$ then $\Delta\epsilon$ vanishes in the thermodynamic limit.

MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

Quenches in disordered quantum systems

Description after relaxation

Hard-core boson Hamiltonian

$$\hat{H} = \sum_{i=1}^{L} -t\left(\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.}\right) + V\hat{n}_i\hat{n}_{i+1} - t'\left(\hat{b}_i^\dagger \hat{b}_{i+2} + \text{H.c.}\right) + V'\hat{n}_i\hat{n}_{i+2} + \mu_i\hat{n}_i$$

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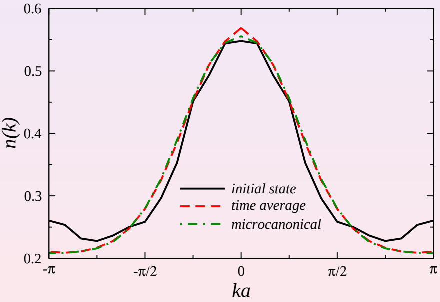
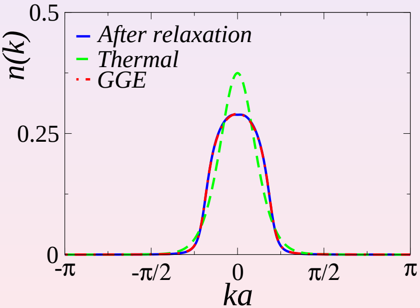
Dynamics vs statistical ensembles

Nonintegrable: $t' = V' \neq 0$, $\mu_i = 0$



MR, PRL **103**, 100403 (2009);
PRA **80**, 053607 (2009).

Integrable: $V = t' = V' = 0$, $\mu_i \neq 0$



MR, V. Dunjko, V. Yurovsky, and
M. Olshanii, PRL **98**, 050405 (2007).

Eigenstate thermalization

Eigenstate thermalization hypothesis

[Deutsch, PRA **43** 2046 (1991); Srednicki, PRE **50**, 888 (1994).]

- The expectation value $\langle\alpha|\widehat{O}|\alpha\rangle$ of a few-body observable $\widehat{O}$ in an eigenstate of the Hamiltonian $|\alpha\rangle$, with energy E_α, of a many-body system equals the thermal average of $\widehat{O}$ at the mean energy E_α:

$$\langle\alpha|\widehat{O}|\alpha\rangle = \langle\widehat{O}\rangle_{\text{microcan.}}(E_\alpha).$$

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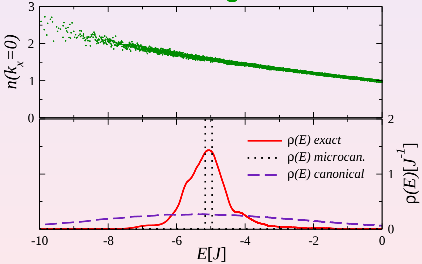
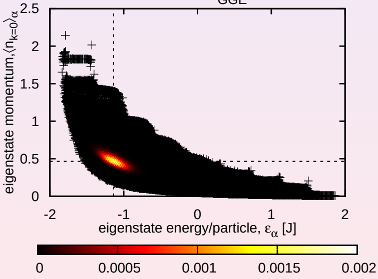


MR, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).

Integrable ($\hat{\rho}_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} e^{-\sum_m \lambda_m \hat{I}_m}$)



A. C. Cassidy, C. W. Clark, and MR, PRL **106**, 140405 (2011).

What changes in the presence of disorder?

Many-body localization

- D. M. Basko, I. L. Aleiner, and B. L. Altshuler, Ann. Phys. **321**, 1126 (2006).
- V. Oganesyan and D. A. Huse, Phys. Rev. B **75**, 155111 (2007).
- A. Pal and D. A. Huse, Phys. Rev. B **82**, 174411 (2010).
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Some questions we would like to address

- How is the relaxation dynamics?
- Will observables fail to equilibrate?

$$O(\tau) \neq \overline{O(\tau)}$$

- If an observable equilibrates, will it fail to thermalize?

$$\overline{O(\tau)} \neq O(E_0) = O(T) = O(T, \mu)$$

Quenches in disordered quantum systems

Model Hamiltonian and the localization transition

Spinless fermion Hamiltonian in 1D

$$\hat{H} = \sum_{ij} J_{ij} \left(\hat{f}_i^\dagger \hat{f}_j + \text{H.c.} \right) + V \sum_i \left(\hat{n}_i - \frac{1}{2} \right) \left(\hat{n}_{i+1} - \frac{1}{2} \right)$$

E. Khatami, MR, A. Relaño, and A. M. García-García, PRE **85**, 050102(R) (2012).

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Hopping amplitudes

Gaussian random distribution $\langle J_{ij} \rangle = 0$

$$\langle (J_{ij})^2 \rangle = \left[1 + \left(\frac{|i-j|}{\beta} \right)^{2\alpha} \right]^{-1}$$

Limit $V = 0$:

- Properties depend on α but not on $\beta > 0$
- $\alpha < 1$, eigenstates are delocalized
- $\alpha > 1$, eigenstates are localized
- $\alpha = 1$, eigenstates are multifractal

Mirlin *et al.*, PRE **54**, 3221 (1996).

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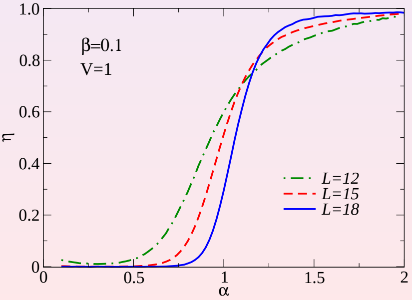
Metal-insulator transition

$$\eta = [\text{var} - \text{var}_{\text{WD}}]/[\text{var}_{\text{P}} - \text{var}_{\text{WD}}]$$

var: variance of level spacing distribution



Dynamics after a quench

Quench protocol

- Start from an eigenstate of $\hat{H}$ ($|\psi_0\rangle$) in a certain disorder realization.
- Evolve under another disorder realization with the same α.
- $E = \langle\psi_0|\hat{H}_{\text{fin}}|\psi_0\rangle$ is the energy of a thermal state with temperature $T = 10$.
- Everything is computed by means of full exact diagonalization.
- Normalized differences: $\Delta O = \frac{\sum_k |O_A(k) - O_B(k)|}{\sum_k O_B(k)}$, disorder averages: $\langle\Delta O\rangle_{\text{dis}}$

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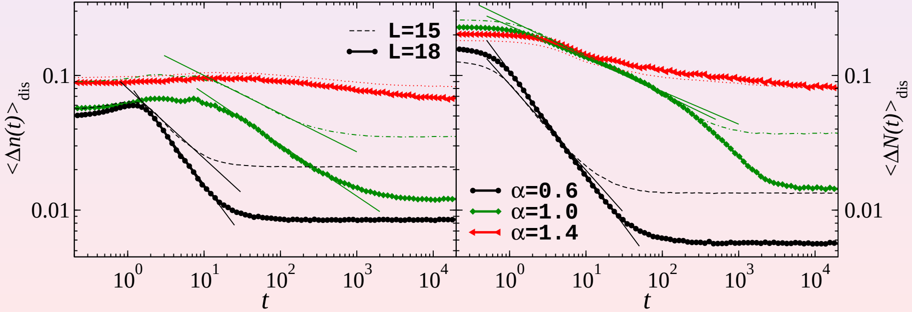
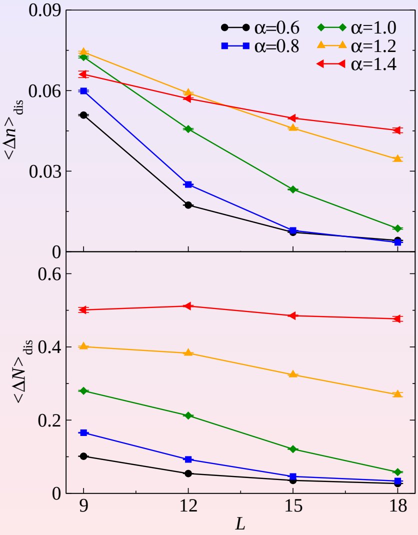
Time evolution



Model Hamiltonian and the localization transition

Microcanonical vs diagonal



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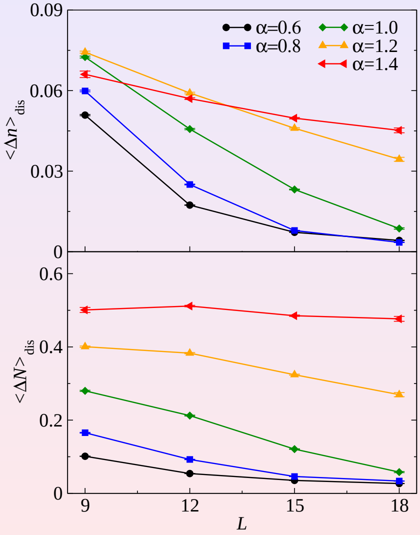
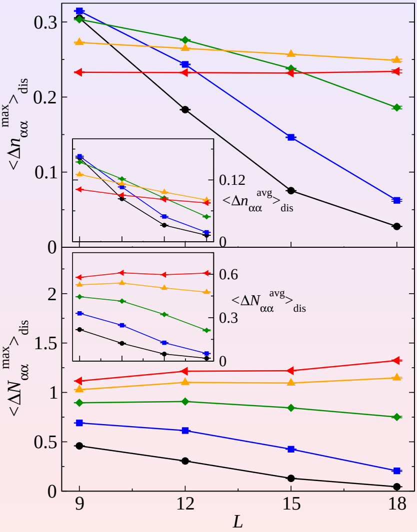


Eigenstate thermalization



Quenches in disordered quantum systems

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$$\hat{H} = -J \sum_{i=1}^{L-1} (\hat{b}_i^\dagger \hat{b}_{i+1} + \text{H.c.}) + \lambda \sum_i \cos(2\pi\sigma i + \delta)\, \hat{n}_i^b \quad \text{where} \quad \sigma = (\sqrt{5} - 1)/2$$

C. Gramsch and MR, arXiv:1206.3570.

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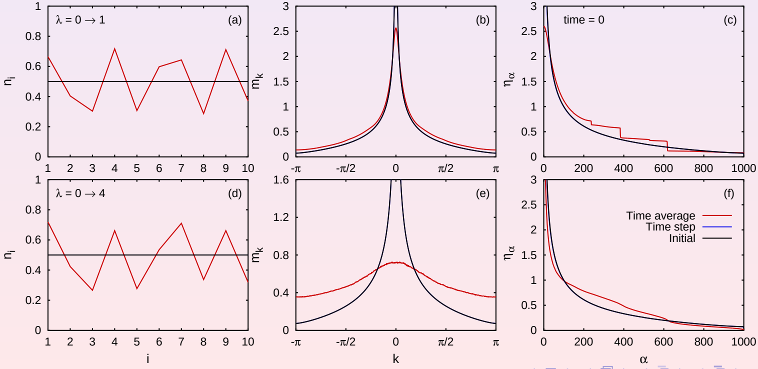
Dynamics after a quench from the ground state ($\lambda_I = 0 \to \lambda_F \neq 0$)



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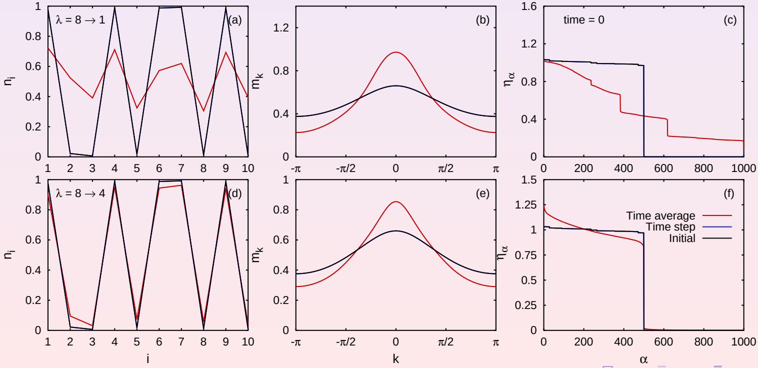
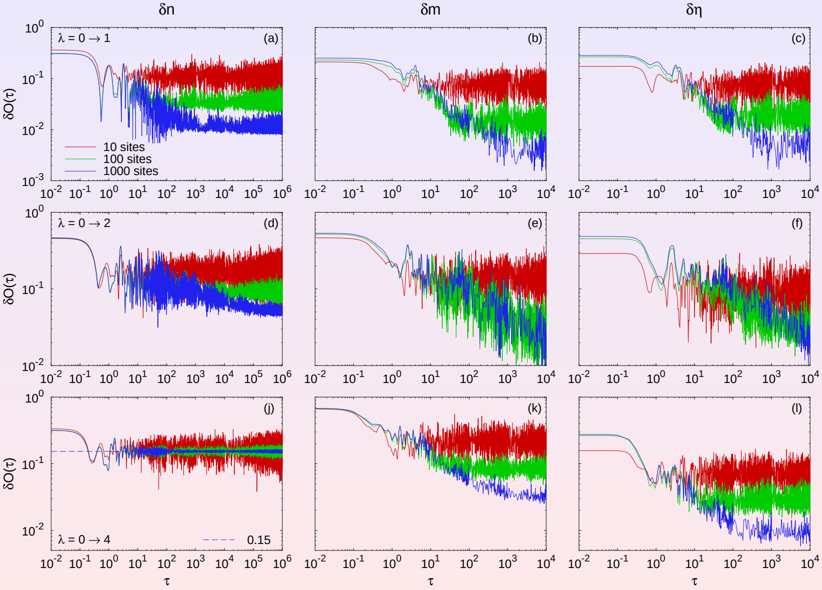
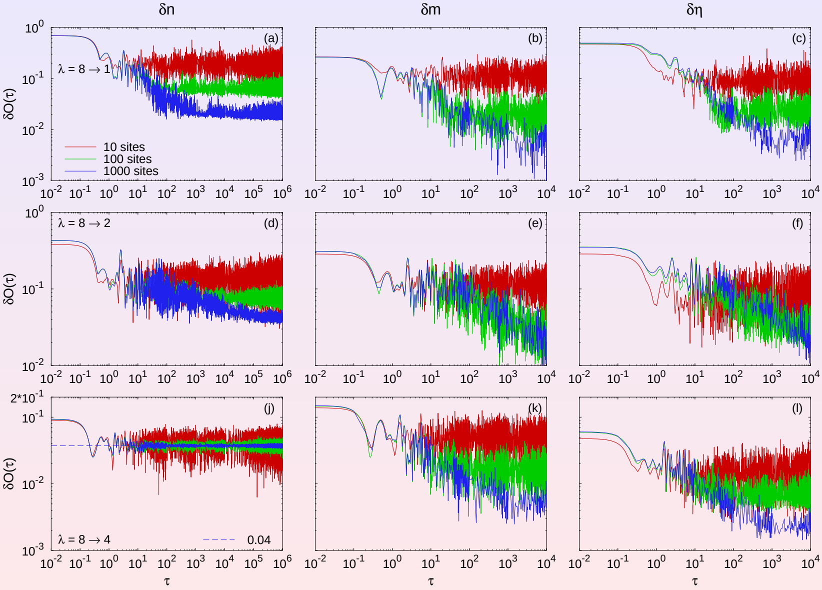
Dynamics after a quench from the ground state ($\lambda_I \neq 0 \rightarrow \lambda_F < \lambda_I$)



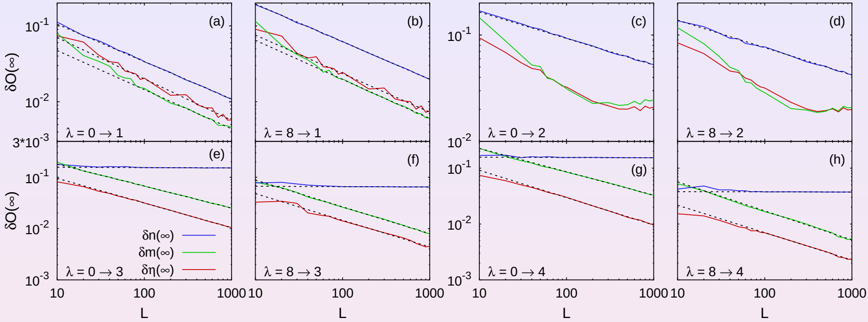
Dynamics after a quench from the ground state



Dynamics after a quench from the ground state

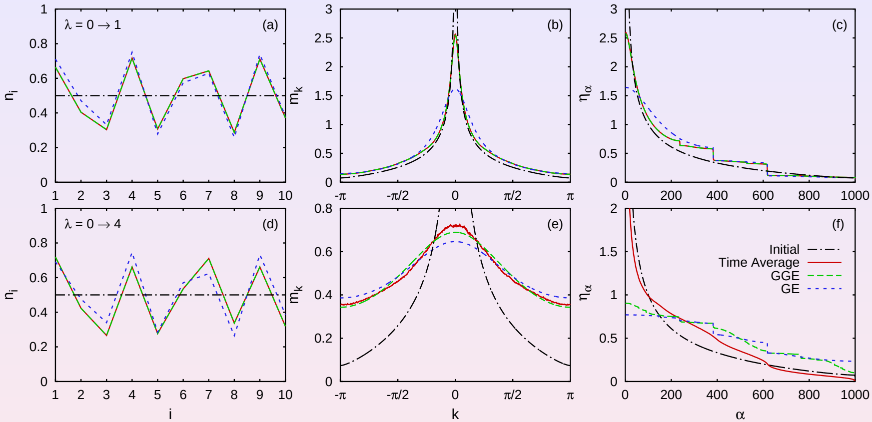


Scaling of $\delta O(\tau)$ after relaxation



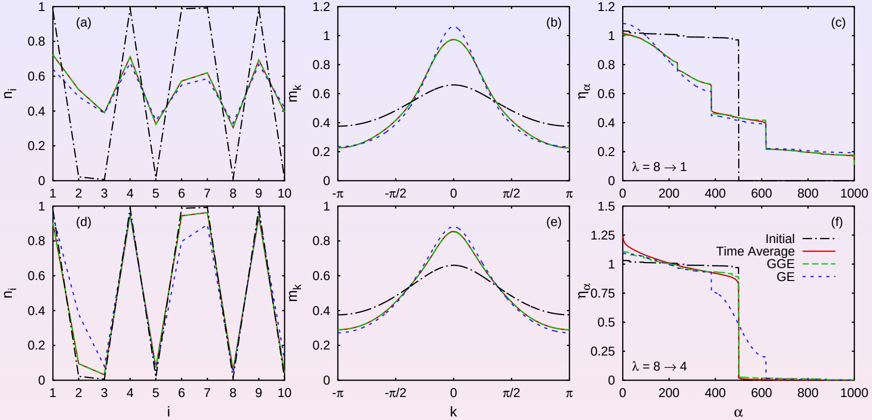
- Delocalized phase ($\lambda_F < 2$): $\delta n(\infty) \sim \delta m(\infty) \sim \delta\eta(\infty) \propto 1/\sqrt{L}$.
- Localized phase ($\lambda_F > 2$): $\delta m(\infty) \sim \delta\eta(\infty) \propto 1/\sqrt{L}$, $\delta n(\infty) =$ const.
- Critical point ($\lambda_F = 2$): $\delta n(\infty) \propto 1/L^{1/4}$

Results after relaxation vs statistical mechanics



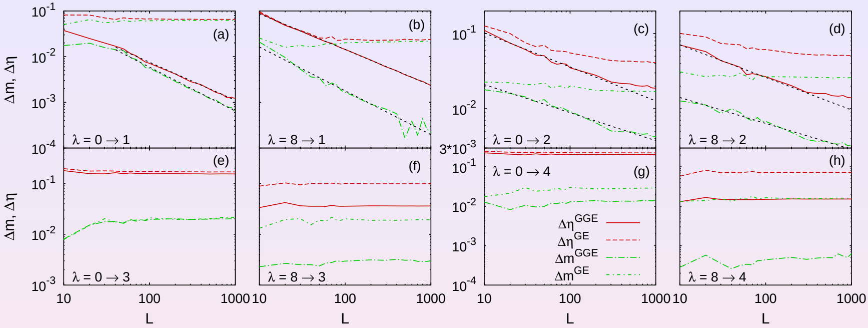
- Delocalized phase ($\lambda_F < 2$): GGE describes one-body observables, GE fails.
- Localized phase ($\lambda_F > 2$): GGE describes n_i but fails for m_k and η_α, GE fails.

Results after relaxation vs statistical mechanics



- Delocalized phase ($\lambda_F < 2$): GGE describes one-body observables, GE fails.
- Localized phase ($\lambda_F > 2$): GGE describes n_i and m_k but fails for η_α, GE fails.

Scaling of Δm and $\Delta\eta$ with L



- Delocalized phase ($\lambda_F < 2$): GGE describes one-body observables ($\Delta m^{\text{GGE}} \sim \Delta\eta^{\text{GGE}} \propto 1/L$), GE fails.
- Critical point ($\lambda_F = 2$): GGE describes one-body observables, GE fails.
- Localized phase ($\lambda_F > 2$): GGE describes n_i but fails for m_k and η_α, GE fails.

Summary

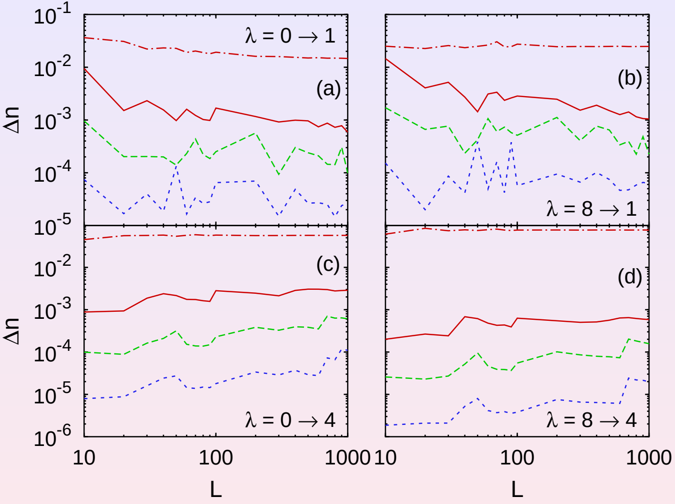
Nonintegrable case

- Localized regime: No eigenstate thermalization and the system does not thermalize
- Delocalized regime: Eigenstate thermalization and the system thermalizes. Power law relaxation?

Integrable case

- Localized regime: m_k and η_α equilibrate but GGE fails to describe them after relaxation
- Delocalized regime: n_i, m_k, and η_α equilibrate and they are described by GGE (despite the lack of translational invariance!). Power law relaxation?
- Critical point: Slower relaxation dynamics. GGE describes observables after relaxation

Scaling of Δn with L



- In all regimes: the differences go to zero as the accuracy in the calculation of the time average is increased.