Dynamical Quantum Phase Transitions in the Transverse Field Ising Model

Stefan Kehrein (Universität Göttingen)
Markus Heyl (Technische Universität Dresden)
Anatoli Polkovnikov (Boston University)

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GEORG-AUGUST-UNIVERSITÄT
GÖTTINGEN

Dynamical Phase Transitions

Quantum Quench:

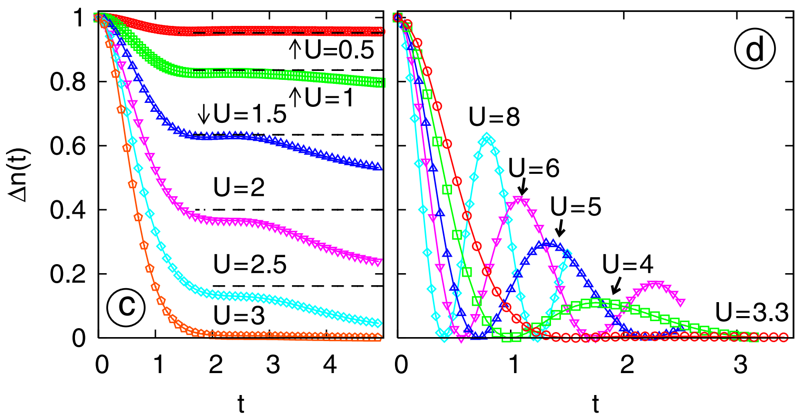
Initial Hamiltonian H_i: defines state $|\Psi_i\rangle$

Final Hamiltonian H_f: defines time evolution $|\Psi(t)\rangle = \exp(-iH_f t)\,|\Psi_i\rangle$

Dynamical phase transition = "Sudden" change in the dynamical behavior of observables as function of quench parameter and/or real time

Hubbard model: Eckstein *et al.*, PRL '09

Schiro and Fabrizio, PRL '10



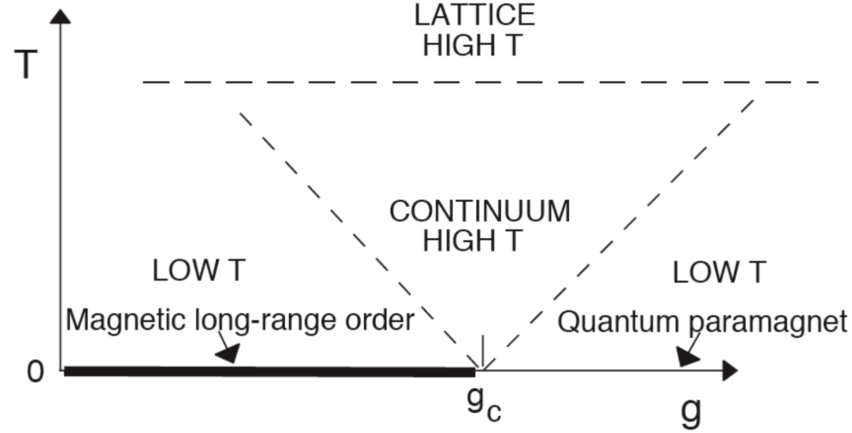
Infinite-dimensional Bose-Hubbard model Sciolla and Biroli, PRL '10

2D Bose gas: " Dynamical Kosterlitz-Thouless transition" Mathey *et al.*, arXiv:1112.1204

Sine-Gordon model A. Mitra, arXiv:1207.3777

Transverse field Ising model (d=1)

Hamiltonian: $$H(g) = -\sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z + g \sum_{i=1}^{N} \sigma_i^x$$



S. Sachdev, Quantum Phase Transitions

- Quantum phase transition at $g_c = 1$
- Exactly solvable via Jordan-Wigner mapping and Bogoliubov transformation

$$H(g) = \sum_{k>0} \epsilon_k(g) \left(\gamma_k^\dagger \gamma_k - \gamma_{-k} \gamma_{-k}^\dagger\right)$$

$$\epsilon_k(g) = \sqrt{(g-\cos k)^2 + \sin^2 k} \quad \Rightarrow \quad m = |g-1|$$

Interaction quench (instantaneous): Magnetic field $g_0 \to g_1$

- Quenches within same phase: $g_0, g_1 < 1$ or $g_0, g_1 > 1$

- Quenches across quantum critical point: $g_0 < 1, g_1 > 1$ or $g_0 > 1 , g_1 < 1$

Initial state: Ground state of $H(g_0)$

$$|\psi_0(g_0)\rangle = \frac{1}{\mathcal{N}} \exp\left(\sum_{k>0} B^*(k)\, \gamma_k^\dagger \gamma_{-k}^\dagger \right) |0\rangle \;= \otimes_{k>0} \;|u_k\rangle$$

Eigenoperators of $H(g_1)$ — Ground state

determined by quench (g_0, g_1)

$\rightarrow$ Mode occupation numbers $n_k = \gamma_k^\dagger \gamma_k$ are time invariant

First: Interpretation of quench dynamics in fermionic model

Lee-Yang theory and Fisher zeroes

Observation: Non-analyticities in real time evolution in thermodynamic limit

$$|\Psi(t)\rangle = \sum_j e^{-iE_j^f t} |E_j^f\rangle \langle E_j^f|\Psi_i\rangle$$

Rotating phases: How can this become non-analytic?

Analogous to Lee-Yang theory of thermodynamic phase transitions:

Grand canonical partition function

$$Z^{gc}(T,z) = \sum_{m=0}^{M} Z_m^{can}(T)\, z^m$$

canonical partition function of m particles

Fugacity $z = \exp(\mu/k_B T)$

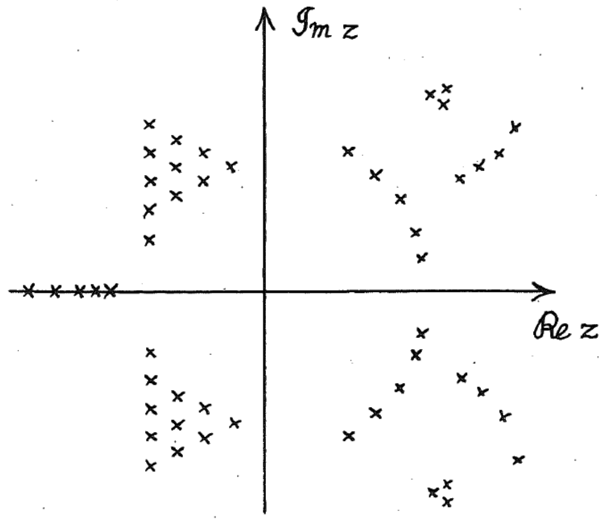
$\Rightarrow$ Polynomial of degree M in fugacity: Zeroes $z_i(T)$ away from positive real axis

$\Rightarrow$ Pressure is analytic

$$p(T,z) = \frac{k_B T}{V} \ln Z^{gc} = k_B T \frac{1}{V} \sum_{j=1}^{M} \ln\left(1 - \frac{z}{z_j(T)}\right)$$

Unless one takes the thermodynamic limit $V \rightarrow \infty$:

Zeroes can converge to a branch cut on positive real axis indicating a phase transition

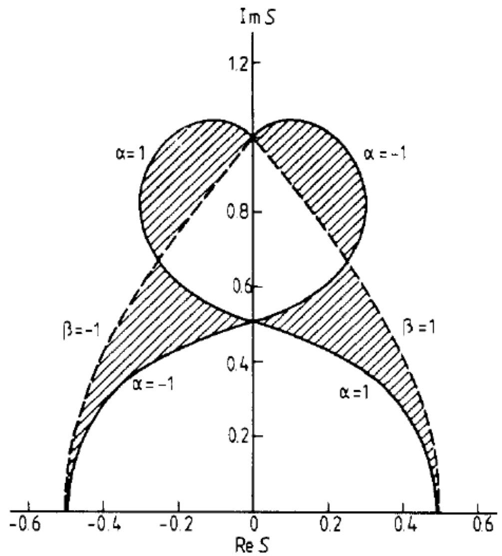


M. E. Fisher, The nature of critical points

Likewise for zeroes in the complex temperature plane:

Fisher zeroes

Thermodynamic phase transition
= free energy/particle is nonanalytic in thermodynamic limit



W. van Saarloos, D. A. Kurtze; J. Phys. A (1984)

Boundary partition function: $$Z(z) = \langle \Psi_0 | e^{-zH(g_1)} | \Psi_0 \rangle$$

- for real z: partition function of equilibrium system with boundaries separated by z [Le Clair et al., Nucl. Phys. B (1995)]

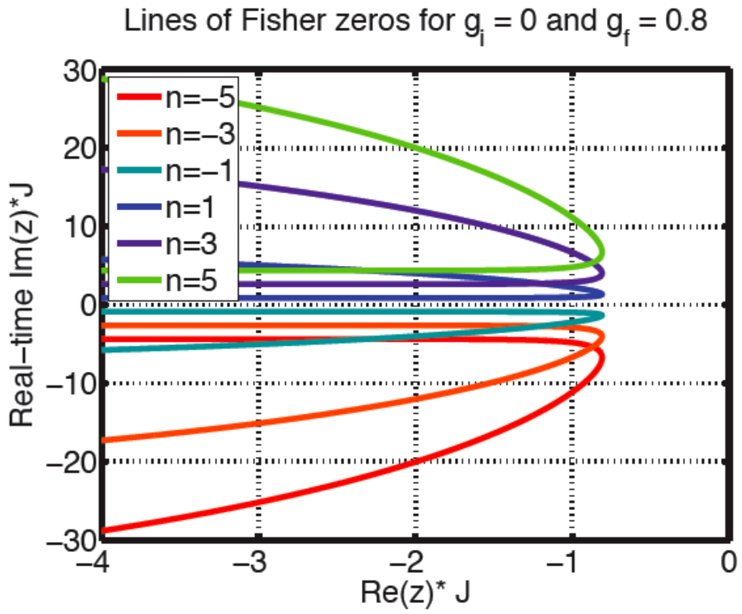
- for z = it and $|\Psi_0\rangle = |\Psi_{gs}(g_0)\rangle$

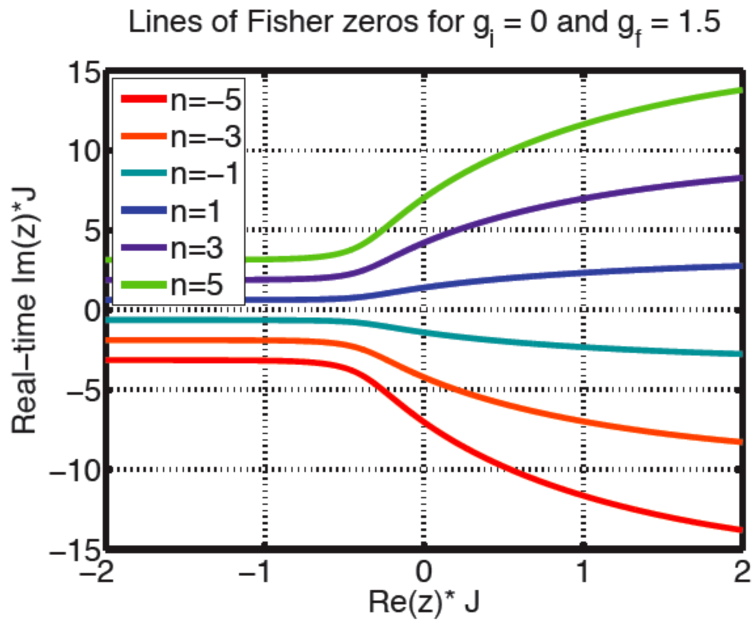
  Amplitude of quench dynamics $$G(t) = \langle \Psi_{gs}(g_0) | e^{-iH(g_1)t} | \Psi_{gs}(g_0) \rangle$$

Thermodynamic limit: $$f(z) = \lim_{N\to\infty} \frac{1}{N} \ln Z(z)$$

$$= \int_0^{\pi} \frac{dk}{2\pi} \ln \left(\frac{1 + |B(k)|^2 e^{-2\varepsilon_k z}}{1 + |B(k)|^2} \right)$$

Lines of Fisher zeroes in complex plane



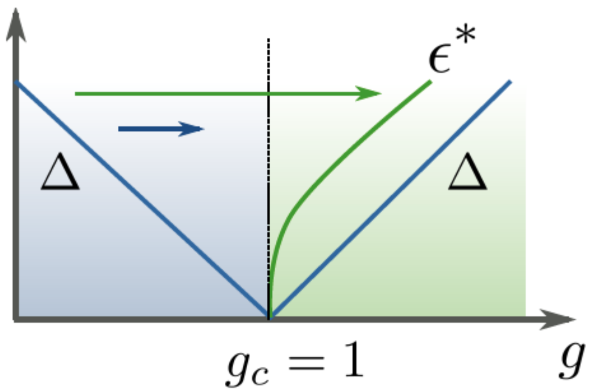


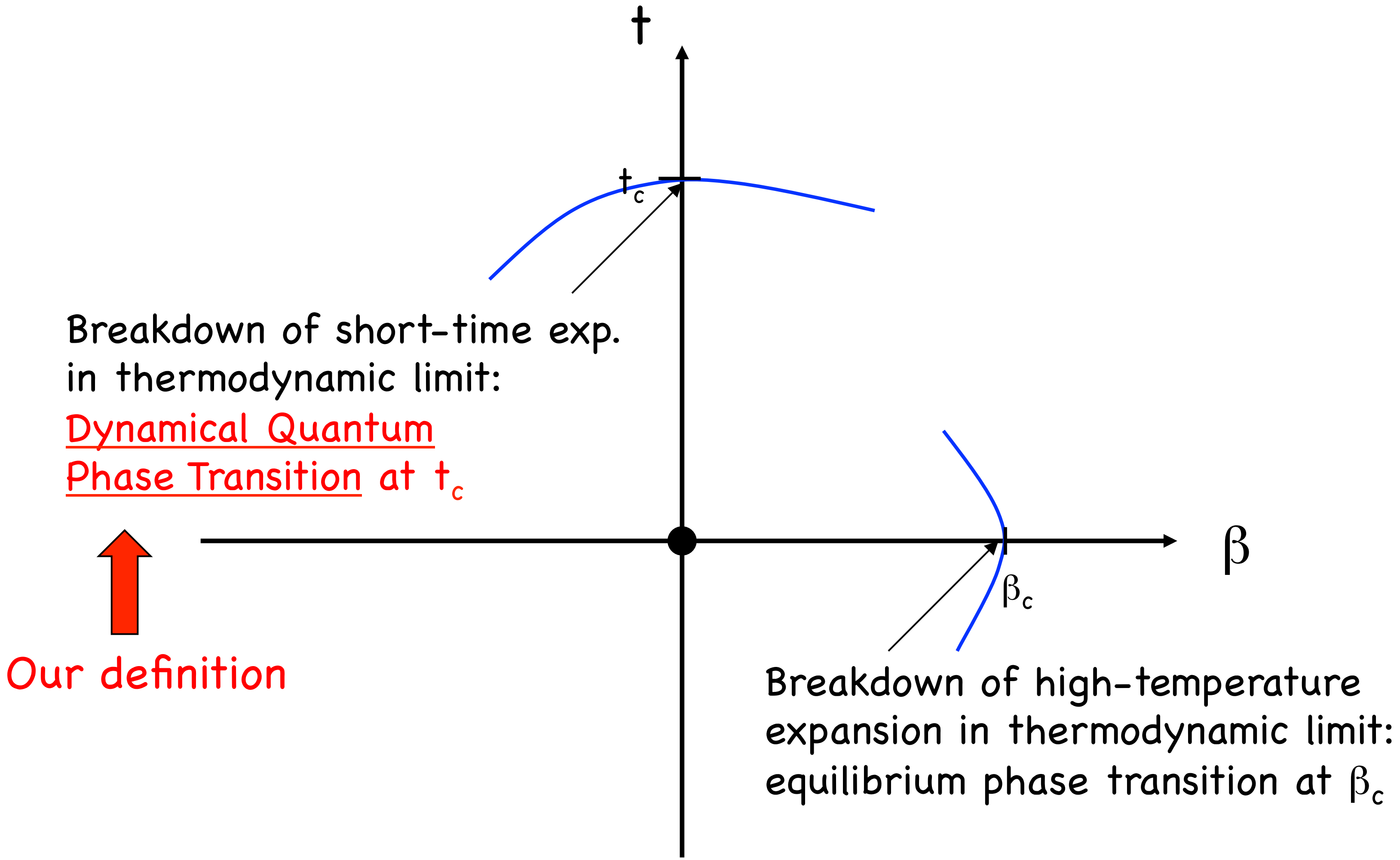
For quench across quantum critical point:

Fisher zeroes on time axis at $t_n = \frac{\pi}{2\varepsilon_{k*}} n \quad , \quad n = \pm 1, \pm 3, \ldots$

with new non-equilibrium energy scale

$$\cos k^* = \frac{1 + g_0 g_1}{g_0 + g_1}$$




t
t_c
β
$β_c$
Breakdown of short-time exp.
in thermodynamic limit:
Dynamical Quantum
Phase Transition at t_c
Our definition
Breakdown of high-temperature
expansion in thermodynamic limit:
equilibrium phase transition at $β_c$

Work distribution functions

Work performed during quench: Talkner et al., Phys. Rev. E (2007)

Two energy measurements

$$P(W) = \sum_j \delta(W - (E_j - E_0(g_0)) \, |\langle E_j | \psi_0(g_0) \rangle|^2$$

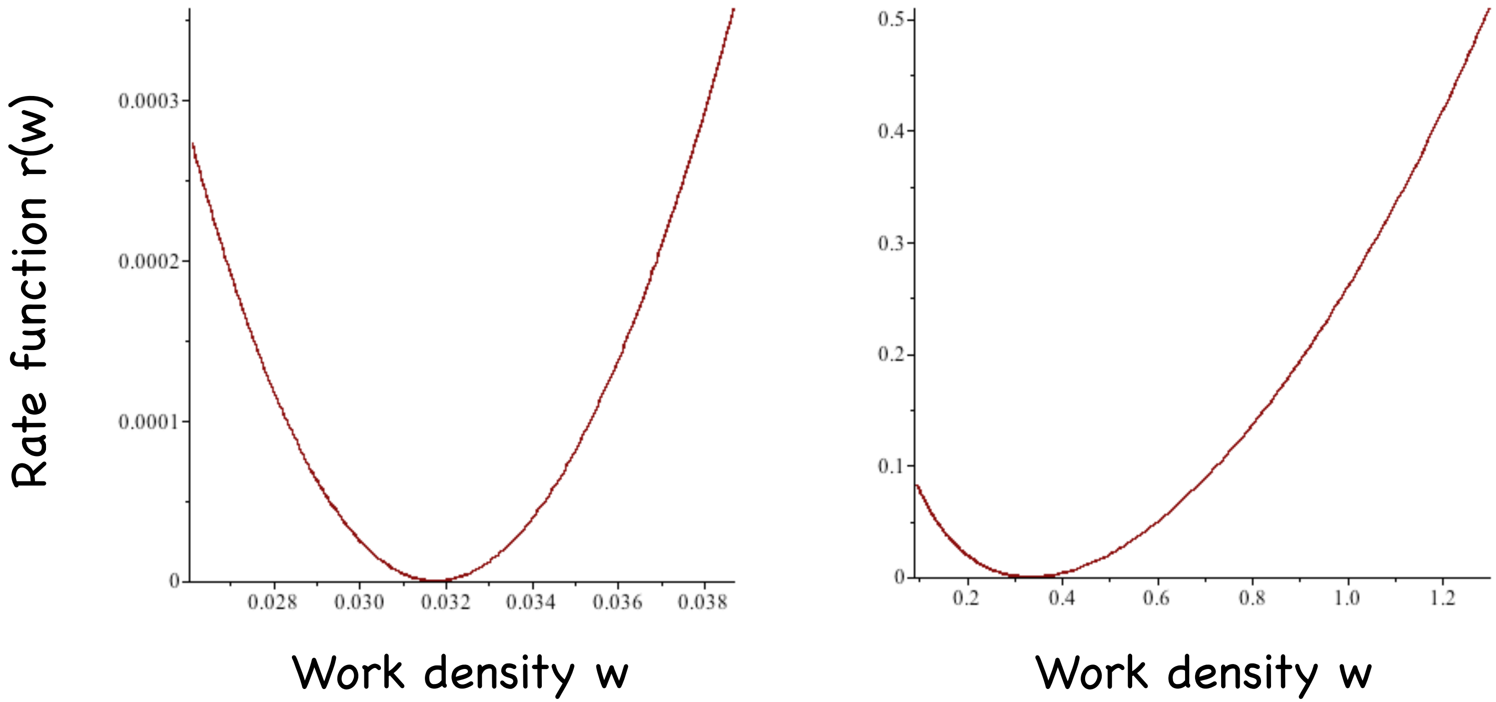
eigenstates of $H(g_1)$

- P(W) becomes δ-function in thermodynamic limit [Silva, PRL (2008)]
- Fluctuations?
- Large deviation form $P(W) = e^{-N\, r(w)}$

  with rate function r(w) ≥ 0 and work density w=W/N

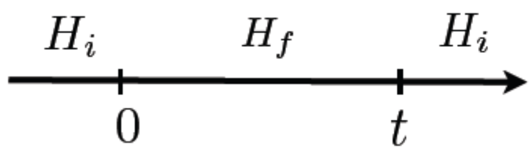
  [Expectation value w_{ex} = <w> corresponds to zero of rate function: $r(w_{ex})=0$]

$g_0 = 0,\ g_1 = 0.5$ $\qquad$ $g_0 = 0,\ g_1 = 1.5$

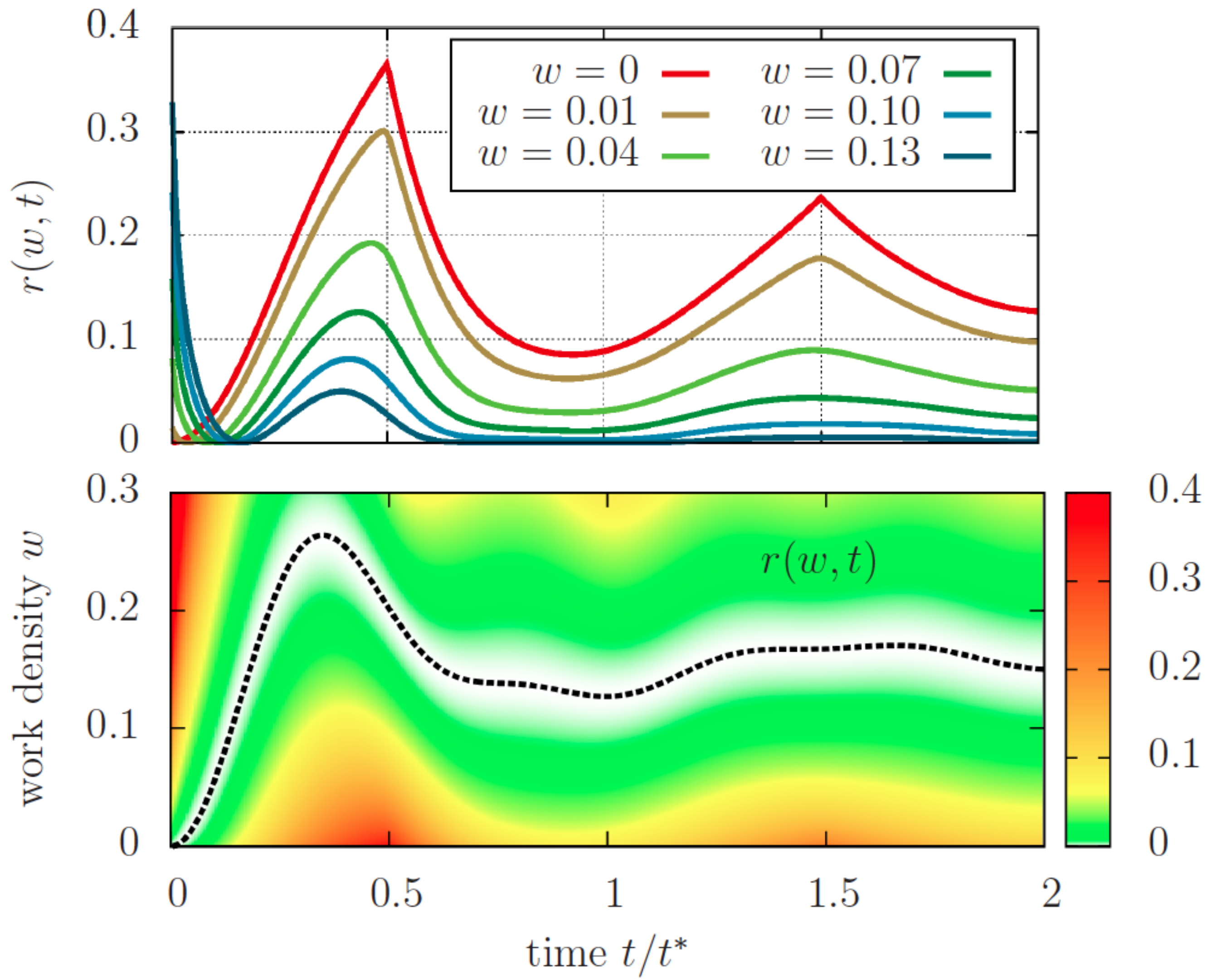


- Rate functions asymmetric: Gaussian only around $w = w_{ex}$
- For $w \to 0$: universal behavior $\propto w \ln w$

Rate function r(w;t) for work distribution of double quench



is non-analytic at Fisher zeroes: $r(w=0,t) = |f(it)|^2$



$g_0 = 0.5$

$g_1 = 2.0$

Influence of non-analytic behavior extends to w>0 (dynamical quantum phase transition)

“Critical” mode k^*

- Non-analytic behavior of $$f(it) = \frac{1}{N} \ln \prod_k \langle u_k(t=0) \,|\, u_k(t) \rangle$$

➔ related to mode k^* with (see also Pollmann et al., PRE (2010))

$$0 = \langle u_{k^*}(t=0) \,|\, u_{k^*}(t) \rangle = \frac{1 + 2n(k^*)\, e^{-2i\epsilon(k^*)t}}{1 + 2n(k^*)}$$

➔ $n(k^*) = 1/2$ (“infinite temperature”)

- Existence of k^* guaranteed for quench across QCP since

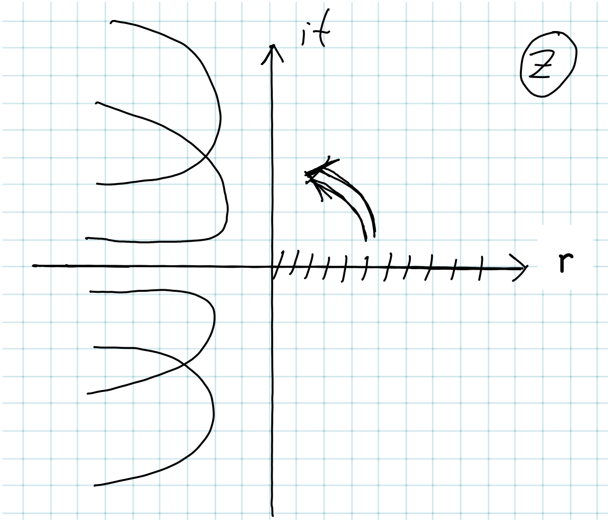
  n(k=0) = 1 ,   n(UV) = 0

  also for:

  - non-instantaneous ramping
  - other dispersion relations with same IR, UV-behavior

Equilibrium vs. Dynamics

Quench within same phase

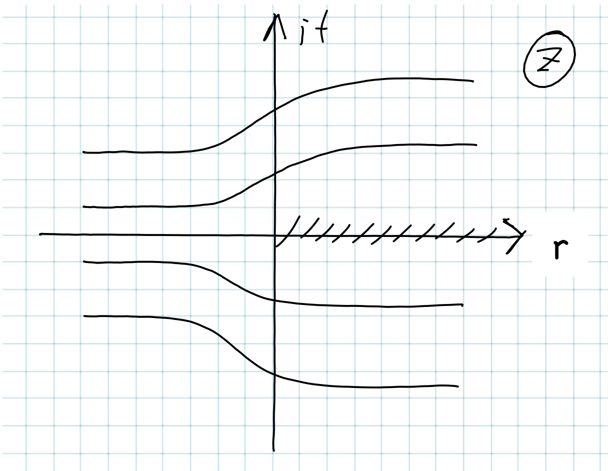


f(r) equilibrium

↓ Wick rotation ↓

f(it) dynamics

Quench across QCP



"Naive" Wick rotation not possible

Non-equilibrium time evolution is not described by equilibrium properties

Subtleties (?)

Ground state of fermionic model for $g_0 < 1$ is not thermodynamic pure ground state of the spin model.

Thermodynamic pure ground state (obeys cluster decomposition)

$$|\pm(g_0)> = \frac{1}{\sqrt{2}}\left(\,|\mathrm{GS}(g_0)>_{\mathrm{NS}} \pm |\mathrm{GS}(g_0)>_{\mathrm{R}}\,\right)$$

↑ Neveu-Schwarz sector (even) ↑ Ramond sector (odd)

Well-known (see e.g. Calabrese et al., arXiv:1204.3911):

- Even operators O_e $\quad \left(\prod_{j=1}^{N}\sigma_j^x\right) O_e \left(\prod_{j=1}^{N}\sigma_j^x\right) = O_e$

  can be evaluated only in NS-sector
  ➔ fermionic model from before

- Odd operators O_o involve R-sector ➔ difficult

Example: Calculate longitudinal magnetization m = < σ_z > via

$$\lim_{r\to\infty} \langle \sigma_i^z \, \sigma_{i+r}^z \rangle = m^2$$

↑

even operator

in fermionic model

BUT: (not fully appreciated in previous literature on quench dynamics)

Hamiltonian H = H_e + H_o

➔ Return amplitude (Loschmidt echo)

$$\langle \mathrm{GS}(g_0) \, | \, e^{-iH_f(g_1)t} \, | \, \mathrm{GS}(g_0) \rangle \overset{\text{in general}}{\neq} \begin{cases} \langle +(g_0) \, | \, e^{-iH(g_1)t} \, | \, +(g_0) \rangle \\ \langle -(g_0) \, | \, e^{-iH(g_1)t} \, | \, -(g_0) \rangle \end{cases}$$

fermionic model

spin model

Careful reinterpretation

Quench from paramagnetic phase:

Unique ground state in spin model $= |\mathrm{GS}(g_0)\rangle_{\mathrm{NS}}$

$$ {}_{\mathrm{NS}}\langle \mathrm{GS}(g_0)\,|\,e^{-i(H_e+H_o)t}\,|\,\mathrm{GS}(g_0)\rangle_{\mathrm{NS}} = {}_{\mathrm{NS}}\langle \mathrm{GS}(g_0)\,|\,e^{-iH_e t}\,|\,\mathrm{GS}(g_0)\rangle_{\mathrm{NS}} $$

= previous result from fermionic model

➔ everything unchanged (incl. Fisher zeroes, nonanalytic behavior in time, etc.)

Quench from ferromagnetic phase:

Return amplitude/Loschmidt matrix

$$\lim_{N\to\infty} \frac{1}{N} \ln \langle \pm(g_0) \,|\, e^{-iH(g_1)t} \,|\, \pm(g_0) \rangle = \begin{pmatrix} f_{++}(it) & f_{+-}(it) \\ f_{-+}(it) & f_{--}(it) \end{pmatrix}$$

$$[\, f_{++}(z) = f_{--}(z) \quad , \quad f^*_{+-}(z) = f_{-+}(z^*) \,]$$

Based on numerical evaluation for finite systems (up to N=200):

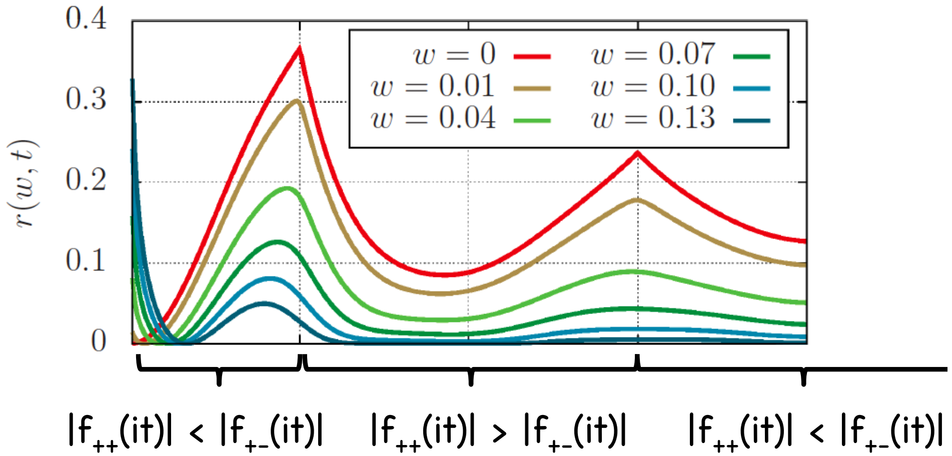
- $f_{++}(z)$ and $f_{+-}(z)$ are analytic
- t < first Fisher time: $|f_{++}(it)| < |f_{+-}(it)|$
  first Fisher time < t < second Fisher time: $|f_{++}(it)| > |f_{+-}(it)|$
  second Fisher time < t < third Fisher time: $|f_{++}(it)| < |f_{+-}(it)|$

  Larger overlap between same or other thermodynamic pure ground states alternates at Fisher times.

Work distribution function for double quench:

w=0 corresponds to return |+> -> |+> and |+> -> |->

➔ same result as in fermionic model, r(w=0,it) non-analytic at Fisher times



Entries in Loschmidt matrix are analytic,
but work distribution function shows non-analytic behavior.

Longitudinal Magnetization

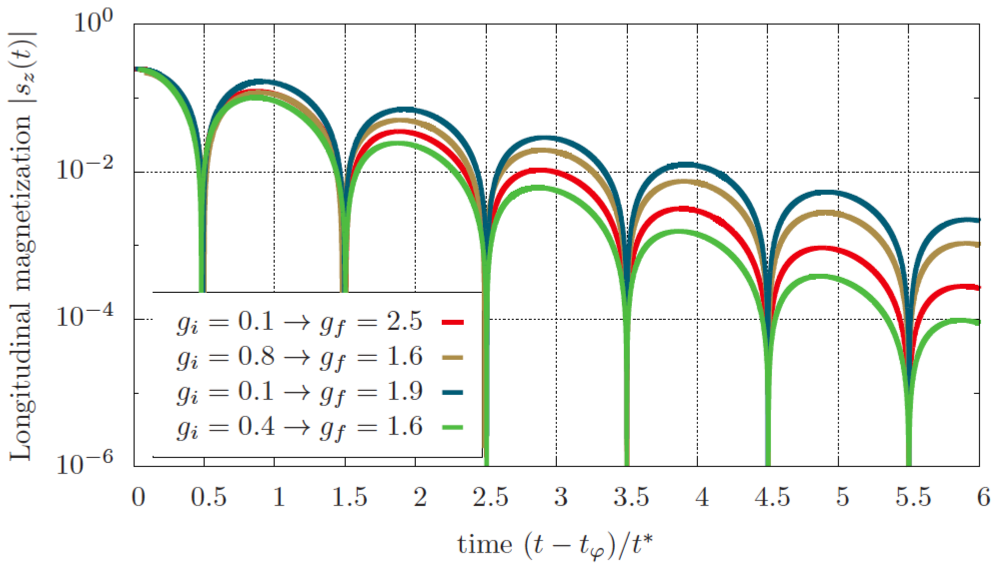
Quenches within ferromagnetic phase: Exponential decay $\rho_z(t) \overset{t\to\infty}{\sim} e^{-\Gamma t}$

[Calabrese et al., PRL (2011)]

Quenches across the QCP:

Longitudinal magnetization has

- analytical time evolution
- shows oscillatory decay with zeroes with period = Fisher time (see also Calabrese et al., arXiv:1204.3911)
- consistent with behavior of rate functions in Loschmidt matrix



Summary

I. Existence of dynamical quantum phase transitions in close analogy to equilibrium phase transitions

II. Breakdown of short-time expansion in thermodynamic limit
$\Rightarrow$ rate functions non-analytic in real time

III. Existence of new non-equilibrium energy scale $\varepsilon(k^*)$

IV. Dynamical quantum phase transition is protected

V. Signatures in other observables (longitudinal magnetization)

VI. Main question: Universality away from quadratic/integrable Hamiltonian