Prethermalization of weakly interacting quantum systems

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Overview

Relaxation of isolated quantum many-body systems:

- Integrable systems
  - Usually do not thermalize
  - Nonthermal state may be described by GGE
- Nearly integrable systems
  - Prethermalization may occur on intermediate time scale
  - Thermalization can occur later
- Nonintegrable systems
  - Thermalization can occur directly

Details can depend on

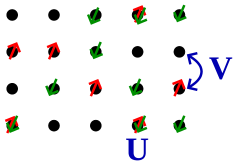
parameters, phases, initial states, observables, . . .

Hubbard model

Single-band Hubbard model: Gutzwiller '63; Kanamori '63; Hubbard '63

$$H = \underbrace{\sum_{ij\sigma} V_{ij}\, c^\dagger_{i\sigma} c_{j\sigma}}_{} \;+\; U \sum_i n_{i\uparrow} n_{i\downarrow}$$

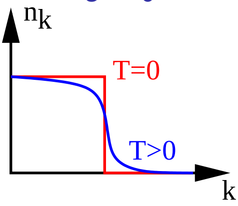
$$= \sum_{\boldsymbol{k}\sigma} \epsilon_{\boldsymbol{k}}\, c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}$$ band structure



⇒ Mott metal-insulator transition at $U_c \sim$ bandwidth   Mott '49

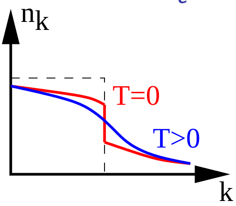
Fermi liquid: quasiparticle excitations   Landau '56

$U = 0$



Fermi gas

$0 < U \lesssim U_c$



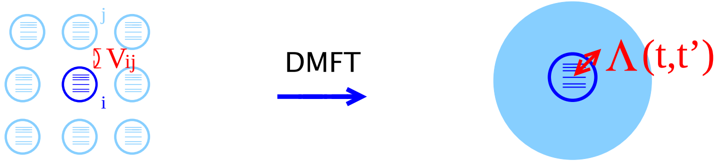
Fermi liquid

Dynamical mean-field theory for nonequilibrium

DMFT in equilibrium: “integrate out the lattice”

lattice problem

single-site problem



- Exact for dimension $d = \infty$ Metzner & Vollhardt ’89, Georges et al. RMP ’96
- Mapped onto single-site problem + self-consistency
  Brandt & Mielsch ’89, Georges & Kotliar ’92
- Conserving approximation; no lattice finite-size effects

DMFT for nonequilibrium:

Schmidt & Monien ’02
Turkowski & Freericks ’05
Freericks, Turkowski & Zlatić ’06
Eckstein & Kollar ’08

- Similar, but $G(t, t')$ instead of $G(t - t')$

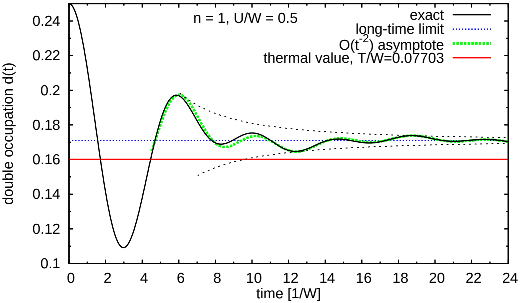
Integrable case

Relaxation in an integrable system

Integrable $1/r$ Hubbard chain: Quench from $U = 0$ to $U = 0.5$

Gebhard & Ruckenstein, PRL **68** '92 Kollar & Eckstein, PRA **78** ('08)

$$d(t) = \langle n_{i\uparrow} n_{i\downarrow} \rangle_t$$



$$\lim_{t\to\infty} d(t) = \frac{n^2}{4} + \frac{n^2(2n-3)}{6} U + O(U^2) \quad = d_{\mathrm{GGE}} \neq d_{\mathrm{therm}}$$

Validity of GGE for integrable system

Integrable system: $H_{\text{eff}} = \sum_{\alpha} \epsilon_\alpha n_\alpha$, $[n_\alpha, n_\beta] = 0$, $n_\alpha = 0, 1$

General observable: $A = \sum_{\alpha\beta} A_{\alpha\beta}\, c^\dagger_{\alpha_1} \cdots c^\dagger_{\alpha_m} c_{\beta_m} \cdots c_{\beta_1}$

Kollar & Eckstein, PRA **78** ('08)

- Ensemble average with GGE:

$$\langle A \rangle_{\text{GGE}} = \sum_{\{\alpha_i\},P} (\pm 1)^P A_{\alpha, P\alpha}\ \langle n_{\alpha_1} \rangle_0 \cdots \langle n_{\alpha_m} \rangle_0$$

- Stationary value from time evolution:

$$\langle A \rangle_{\text{final}} = \sum_{\{\alpha_i\},P} (\pm 1)^P A_{\alpha, P\alpha}\ \langle n_{\alpha_1} \cdots n_{\alpha_m} \rangle_0$$

Validity of statistical prediction: $\langle A \rangle_{\text{GGE}} = \langle A \rangle_{\text{final}}$?

- depends on observable, initial state, system size, ...
- $1/r$ Hubbard chain: $A = \sum_i n_{i\uparrow} n_{i\downarrow} = \sum_{\alpha\beta} A_{\alpha\beta} c^\dagger_\alpha c_\beta$ ✔

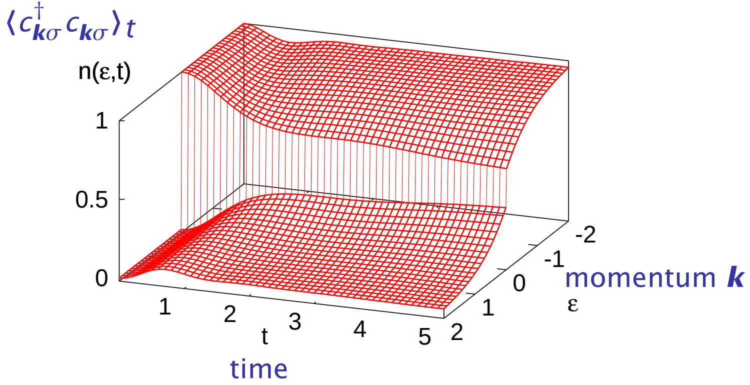
Nonintegrable case

Interaction quench in the Hubbard model

Hubbard model in DMFT: (bandwidth $= 4$, density $n = 1$)

Eckstein, Kollar, Werner PRL '09, PRB '10

Small interaction quench from to $U = 2$



Slow relaxation: *Prethermalization plateaus*

due to vicinity of free system ($U = 0$)

Moeckel & Kehrein '08

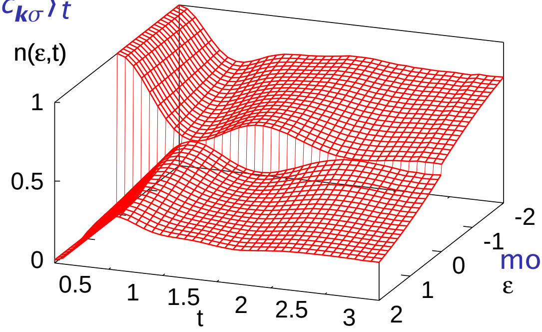
Interaction quench in the Hubbard model

Hubbard model in DMFT: (bandwidth $= 4$, density $n = 1$)

Eckstein, Kollar, Werner PRL '09, PRB '10

Large interaction quench from to $U = 5$

$\langle c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} \rangle_t$



momentum $\boldsymbol{k}$

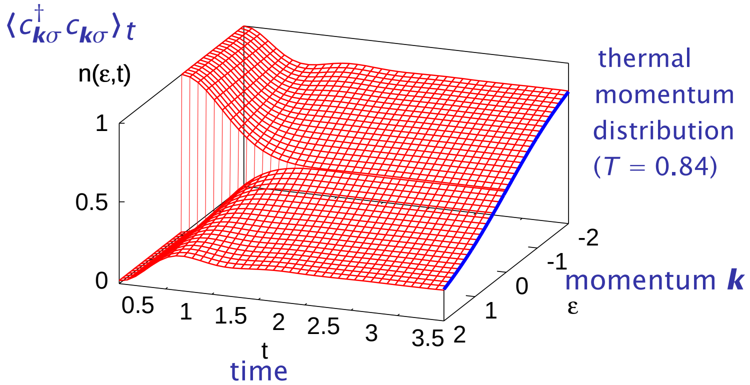
time

Collapse-and-revival oscillations
due to vicinity of atomic limit ($U = \infty$)

Interaction quench in the Hubbard model

Hubbard model in DMFT: (bandwidth = 4, density $n = 1$)
Eckstein, Kollar, Werner PRL '09, PRB '10

Intermediate interaction quench from to $U = 3.3$



Fast thermalization at intermediate U:
both prethermalization and oscillations disappear at $U_c^{\mathrm{dyn}} \approx 3.2\,V$

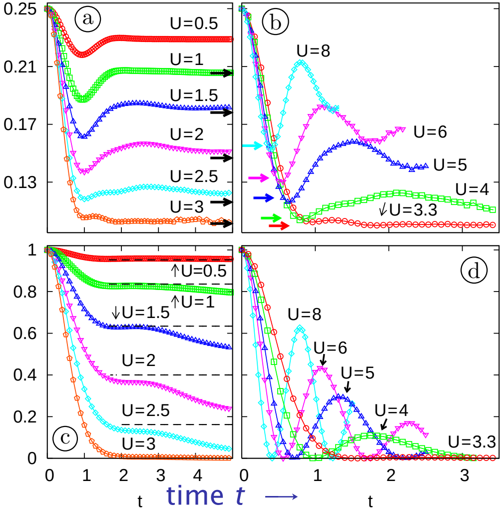
Interaction quench in the Hubbard model

Hubbard model in DMFT: (bandwidth = 4, density $n = 1$)

Eckstein, Kollar, Werner PRL '09, PRB '10

double occupation $\langle n_{i\uparrow} n_{i\downarrow} \rangle_t$

jump in $\langle c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} \rangle_t$ at Fermi surface



time t ⟶

Prethermalization

Vicinity of an integrable point: Prethermalization

Near integrable point:

- $H_0 = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$

  $\Rightarrow$ integrable

- Quench to

  $H_0 + g\, H_1$ with $|g| \ll 1$

  $\Rightarrow$ long-lived state for

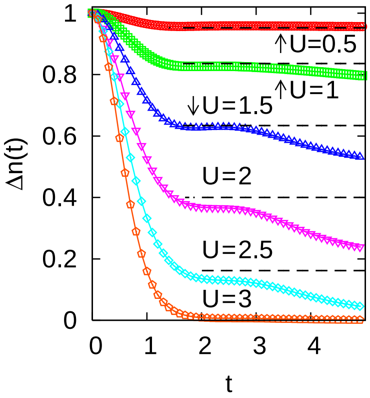
  $$\frac{\text{const}}{g} \ll t \ll \frac{\text{const}}{g^2}$$

"Prethermalization"

Berges et al, PRL **93** ('04)
Moeckel & Kehrein, PRL **100** ('08)
Moeckel & Kehrein, Ann. Phys. **324** ('09)

Interaction quench from 0 to U:

Fermi surface discontinuity



Eckstein, Kollar, Werner, PRL **103** ('09)

Second order unitary perturbation theory

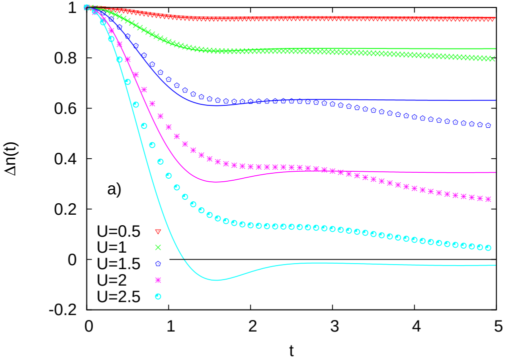
Unitary perturbation theory: trafo, evolve, backtrafo

Moeckel & Kehrein, PRL **100** ('08)
Kollar, Wolf, Eckstein, PRB **84** ('11)

$$\langle A\rangle_t = \langle A\rangle_0 + 4g^2 \int_{-\infty}^{\infty} d\omega \, \frac{\sin^2(\omega t/2)}{\omega^2} J(\omega) + O(g^3) \ \stackrel{t\to\infty}{\longrightarrow} \ A_{\text{pretherm}}$$

$$J(\omega) = \langle H_1 \, (A - \langle A\rangle_0) \, \delta(H_0 - \langle H_0\rangle_0 - \omega) \, H_1 \rangle_0$$

jump in $\langle c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}\rangle_t$ at Fermi surface



Integrable vs. nonintegrable systems

Integrable systems vs. Nearly integrable systems

(Q1) Relaxation to nonthermal (quasi-)steady state

Nonthermal steady state $\overset{?}{\longleftrightarrow}$ Prethermalization plateau

(Q2) Statistical description of (quasi-)steady state

Generalized Gibbs ensemble

$$\rho_{\mathrm{GGE}} \propto \exp\left(-\sum_\alpha \lambda_\alpha I_\alpha\right) \overset{?}{\longleftrightarrow} \rho_{\widetilde{\mathrm{GGE}}} \propto \exp\left(-\sum_\alpha \lambda_\alpha \tilde{I}_\alpha\right)$$

GGE with exact constants of motion I_α

GGE with approximate constants of motion $\tilde{I}_\alpha$?

Result 1:

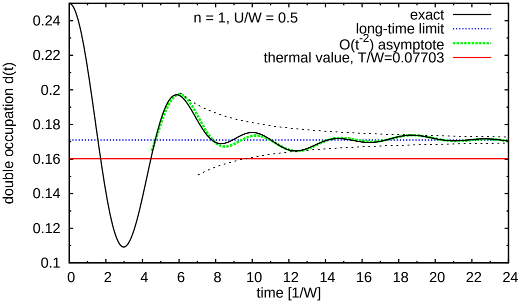
Nonthermal states in integrable systems are prethermalization plateaus that never decay

Q1: Nonthermal vs. prethermalized

Integrable $1/r$ Hubbard chain: Quench from $U = 0$ to $U = 0.5$

Gebhard & Ruckenstein, PRL **68** '92 Kollar & Eckstein, PRA **78** ('08)

$$d(t) = \langle n_{i\uparrow} n_{i\downarrow} \rangle_t$$



$$\lim_{t \to \infty} d(t) = \frac{n^2}{4} + \frac{n^2(2n-3)}{6} U + O(U^2) \quad = d_{\mathrm{GGE}} \neq d_{\mathrm{therm}}$$

Prediction for prethermalization plateau: Kollar, Wolf, Eckstein, PRB **84** ('11)

$$d_{\mathrm{pretherm}} = \frac{n^2}{4} + \frac{n^2(2n-3)}{6} U + O(U^2) \quad \mathbf{!!}$$

Result 2:

Prethermalization plateaus are described by GGE with approximate constants of motion

Q2: Approximate constants of motion

Before quench:

- $H_0 = \sum_\alpha \epsilon_\alpha I_\alpha$ with $I_\alpha = a^\dagger_\alpha a_\alpha$ (bosons or fermions)
- $[I_\alpha, I_\beta] = 0, \quad [H_0, I_\alpha] = 0$
- Basis: $I_\alpha |\boldsymbol{n}\rangle = n_\alpha |\boldsymbol{n}\rangle$
- System in ground state $|\psi_0\rangle$ of H_0

After quench:

- $H = H_0 + g H_1$ with $[H_0, H_1] \neq 0$ and $|g| \ll 1$
- H_1 can be expressed with $a^\dagger_\alpha$ and a_α
- $|\tilde{\psi}_0\rangle$ = ground state of H

Construction of approximate constants of motion

- Canonical trafo: $S = gS_1 + \frac{1}{2}g^2 S_2 + O(g^3)$

Harris & Lange, PR **157** (1963)

$$e^S H e^{-S} = H_0 + g(H_1 + [S_1, H_0]) \\ + g^2(\tfrac{1}{2}[S_2, H_0] + [S_1, H_1] + \tfrac{1}{2}[S_1,[S_1,H_0]]) + O(g^3)$$

- With $\tilde{I}_\alpha = e^{-S} I_\alpha e^S$ and $|\tilde{\boldsymbol{n}}\rangle = e^{-S}|\boldsymbol{n}\rangle$:

$$H = \sum_\alpha \epsilon_\alpha \underbrace{\tilde{I}_\alpha} + \sum_{\tilde{\boldsymbol{n}}} |\tilde{\boldsymbol{n}}\rangle\langle\tilde{\boldsymbol{n}}| \; (E_{\boldsymbol{n}}^{(1)} + E_{\boldsymbol{n}}^{(2)}) + O(g^3)$$

= approx. const. of motion of H

- Generalized Gibbs ensemble:

$$\rho_{\widetilde{\mathrm{GGE}}} = \exp\left(-\sum_\alpha \lambda_\alpha \tilde{I}_\alpha\right) \quad \text{with} \quad \langle \tilde{I}_\alpha \rangle_{\widetilde{\mathrm{GGE}}} \stackrel{!}{=} \langle \tilde{I}_\alpha \rangle_0$$

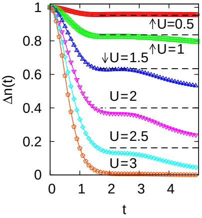
Statistical prediction for pretherm. plateau

GGE prediction $\langle A \rangle_{\widetilde{\text{GGE}}}$ vs. pretherm. plateau $\langle A \rangle_{\text{pretherm}}$:

Kollar, Wolf, Eckstein, PRB **84** ('11)

- Simplest observable: $A = I_\alpha = a_\alpha^\dagger a_\alpha$

  $\Rightarrow \quad \langle I_\alpha \rangle_{\widetilde{\text{GGE}}} = \langle I_\alpha \rangle_{\text{pretherm}} + O(g^3)$ **!!**



- General observable: $A = I_{\alpha_1} \cdots I_{\alpha_m}$

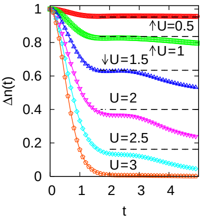
  $\Rightarrow \quad \langle A \rangle_{\widetilde{\text{GGE}}} - \langle A \rangle_{\text{pretherm}}$

$$= \langle I_{\alpha_1} \rangle_{\tilde{\psi}_0} \cdots \langle I_{\alpha_m} \rangle_{\tilde{\psi}_0} - \langle I_{\alpha_1} \cdots I_{\alpha_m} \rangle_{\tilde{\psi}_0} + O(g^3)$$

similar to GGE criteria in Kollar & Eckstein, PRA **78** ('08)

Conclusion

- 'Dressed' GGEs predict prethermalization in nearly integrable systems (e.g., for $n_{\mathbf{k}\sigma}$ and d in small-U quenches)



- Integrable / nearly integrable systems are connected, and described by one statistical theory

- Nonthermal steady states in integrable systems
  evolve into
  prethermalization plateaus away from integrability

Evaluation of the GGE

$$\langle A \rangle_{\widetilde{\mathrm{GGE}}} = \frac{1}{Z} \mathrm{Tr}[A\, e^{-\sum_\alpha \lambda_\alpha \tilde{I}_\alpha}]$$

$$= \frac{1}{Z} \mathrm{Tr}[e^S A e^{-S}\, e^{-\sum_\alpha \lambda_\alpha I_\alpha}] = \langle e^S A e^{-S} \rangle_{\mathrm{GGE}}$$

$$= \underbrace{\langle A \rangle_{\mathrm{GGE}}}_{=(\mathrm{i})} + \underbrace{\langle [S, A] \rangle_{\mathrm{GGE}}}_{=0} + \underbrace{\langle \tfrac{1}{2}[S,[S,A]] \rangle_{\mathrm{GGE}}}_{=(\mathrm{ii})} + O(g^3)$$

$$(\mathrm{i}) = \Big\langle \prod_{i=1}^{m} I_{\alpha_i} \Big\rangle_{\mathrm{GGE}}$$

$$= \prod_{i=1}^{m} \langle I_{\alpha_i} \rangle_{\mathrm{GGE}} \quad \text{common eigenbasis}$$

$$= \prod_{i=1}^{m} \langle \psi_0 | \tilde{I}_{\alpha_i} | \psi_0 \rangle \quad \text{fix initial value}$$

$$= \prod_{i=1}^{m} \langle \tilde{\psi}_0 | I_{\alpha_i} | \tilde{\psi}_0 \rangle + O(g^3) \quad \text{state transformation}$$

Evaluation of the GGE

$$
\begin{aligned}
\text{(ii)} &= \langle \tfrac{1}{2}[S,[S,A]]\rangle_{\mathrm{GGE}} \\
&= \frac{g^2}{Z}\sum_{\boldsymbol{n}} \underbrace{\langle \boldsymbol{n}|\tfrac{1}{2}[S_1,[S_1,A]]|\boldsymbol{n}\rangle}_{=:\,F(\{n_\alpha\})} e^{-\sum_\alpha \lambda_\alpha n_\alpha} + O(g^3) \\
&= g^2 F(\{\langle \mathcal{I}_\alpha\rangle_{\mathrm{GGE}}\}) + O(g^3) \quad \text{Wick's theorem} \\
&= g^2 F(\{\langle \psi_0|\tilde{\mathcal{I}}_\alpha|\psi_0\rangle\}) + O(g^3) \quad \text{fix initial value} \\
&= g^2 F(\{\langle \psi_0|\mathcal{I}_\alpha|\psi_0\rangle\}) + O(g^3) \quad \text{leading order} \\
&= g^2 \langle \psi_0|\tfrac{1}{2}[S_1,[S_1,A]]|\psi_0\rangle + O(g^3) \quad \text{same lin.comb.} \\
&= \langle \psi_0|\tfrac{1}{2}[S,[S,A]]|\psi_0\rangle + O(g^3) \\
&= \langle \psi_0|\tilde{A}|\psi_0\rangle - \langle A\rangle_0 + O(g^3) \\
&= \langle \tilde{\psi}_0|A|\tilde{\psi}_0\rangle - \langle A\rangle_0 + O(g^3) \\
&= \langle \tilde{\psi}_0|\prod_{i=1}^{m}\mathcal{I}_{\alpha_i}|\tilde{\psi}_0\rangle - \prod_{i=1}^{m}\langle \mathcal{I}_{\alpha_i}\rangle_0 + O(g^3)
\end{aligned}
$$