Universal statistics for directed polymers and the KPZ equation from the replica Bethe Ansatz

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with : Pasquale Calabrese (Univ. Pise)

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Thomas Gueudre (LPTENS)

P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)

P. Calabrese, P. Le Doussal, Phys. Rev. Letters 106 250603 (2011) and J. Stat. Mech. P06001 (2012) T. Gueudre, P. Le Doussal, arXiv:1208.5669.

- many models in "KPZ class" exhibit universality related to random matrix theory:
Tracy Widom distributions: largest eigenvalue of GUE,GOE..

- provide solution directly continuum model (at all times)

Kardar Parisi Zhang equation

Phys Rev Lett 56 889 (1986)

growth of an interface of height h(x,t)

$$\partial_t h = \nu \partial_x^2 h + \frac{\lambda_0}{2} (\partial_x h)^2 + \eta(x,t)$$

Universal distribution of conductance in 2D localized phase

Somoza, Ortuno, Prior (2007)

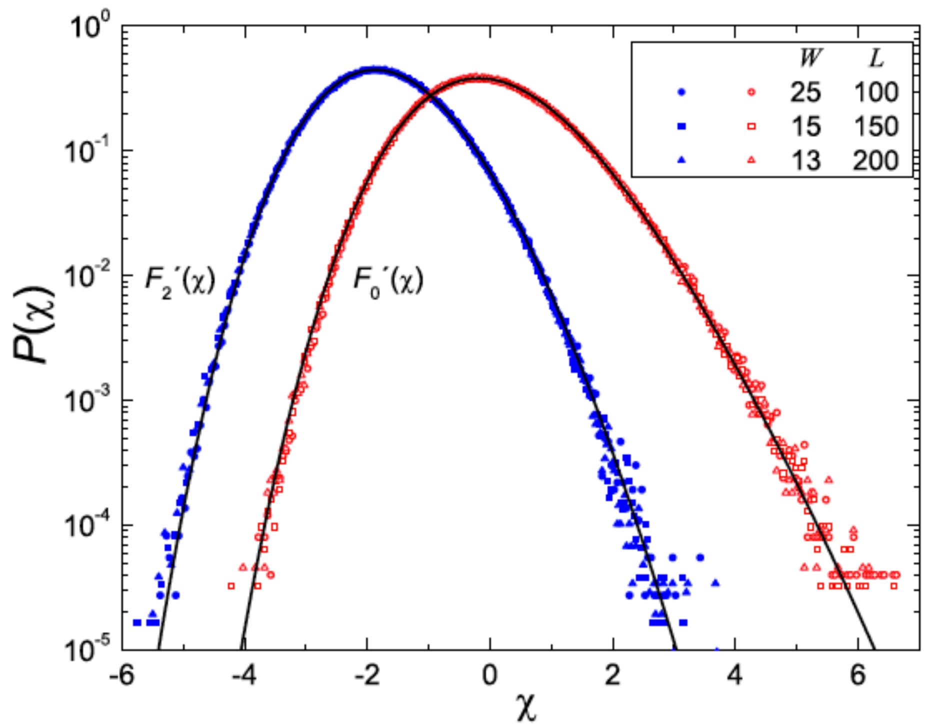
$$H = \sum_i \epsilon_i c_i^+ c_i - t \sum_{<ij>} c_i^+ c_j + c_j^+ c_i$$

$$\ln g = -\frac{2L}{\xi} + \alpha\left(\frac{L}{\xi}\right)^{1/3} \chi_2$$

ξ localization length

L system size

χ random variable with Tracy Widom distribution



i

FIG. 1 (color online). Histograms of $\ln g$ versus the scaled variable χ for several sizes and disorders of the Anderson model with narrow (solid symbols) and wide (empty symbols) leads. The continuous lines correspond to $F_2'(\chi)$ and $F_0'(\chi)$.

Mapping to directed polymers with non-positive weights

$$G_{ij}(E) = < i|\frac{1}{E-H}|j >$$

$$H = \sum_i \epsilon_i c_i^+ c_i - t \sum_{<ij>} c_i^+ c_j + c_j^+ c_i$$

$$g = \sum_{i\in a, j\in b} G_{ij} G_{ji}$$

$$G_{ij}(E) = \sum_{\gamma\in\Gamma_{ij}} \prod_{\ell\in\gamma} \frac{t}{\epsilon_\ell - E}$$

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Nguyen, Spivak, Shklovski (85)

$$\epsilon_i = \eta_i W$$

$$\eta_i = \pm 1$$

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Γ_{ij} restricted to directed paths from i to j

NSS model

$$\sim (\frac{t}{W})^L \sum_{\gamma \in \Gamma_{ij}^{directed}} \prod_{\ell \in \gamma} \eta_\ell$$

Directed Polymer + random sign weights

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$$Z \sim (\frac{t}{W})^L \sum_{\gamma\in\Gamma_{ij}^{directed}} \prod_{\ell\in\gamma} \eta_\ell$$

Directed Polymer + random sign weights

complex weights

Derrida et al. (93)

Kardar Medina (92)

A. Dobrinevski, PLD, K. Wiese
PRE 83 061116 (2011)

$\overline{\ln|Z|} \sim \ln|\overline{Z}|$ phase I

$\sim \frac{1}{2}\ln \overline{ZZ^*}$ phase III

$\overline{(ZZ^*)^n} \sim \overline{Z_{DP}^n}$ phase II similar to positive weights .. d=1+1 expect TW

M Mueller (2011) hard core bosons: DP w. positive weights

Outline

- directed polymer, discrete and continuum, KPZ equation
- quantum mechanics + replica , high T, Lieb Liniger model
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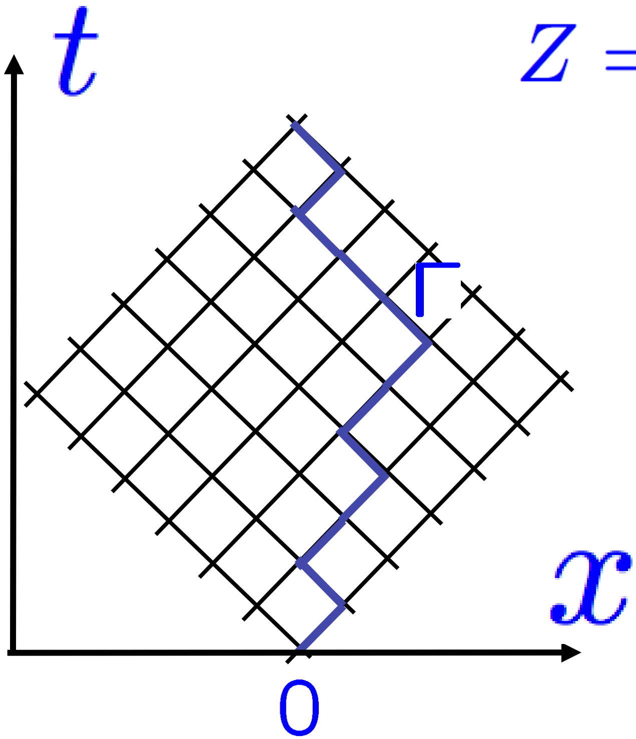
- directed polymer, discrete and continuum, KPZ equation
- quantum mechanics + replica , high T, Lieb Liniger model
- Bethe Ansatz
- moments of partition sum Z^n for DP with fixed endpoint = KPZ with droplet initial condition + numerical checks
- generating function of Z^n can be expressed as a Fredholm determinant, obtain distrib. free energy
- large time limit recovers Tracy Widom GUE
- DP 1 free endpoint=KPZ flat init. cond. Fredholm Pfaffian and TW for GOE
- DP near a wall TW for GSE

directed polymer: 1) lattice model



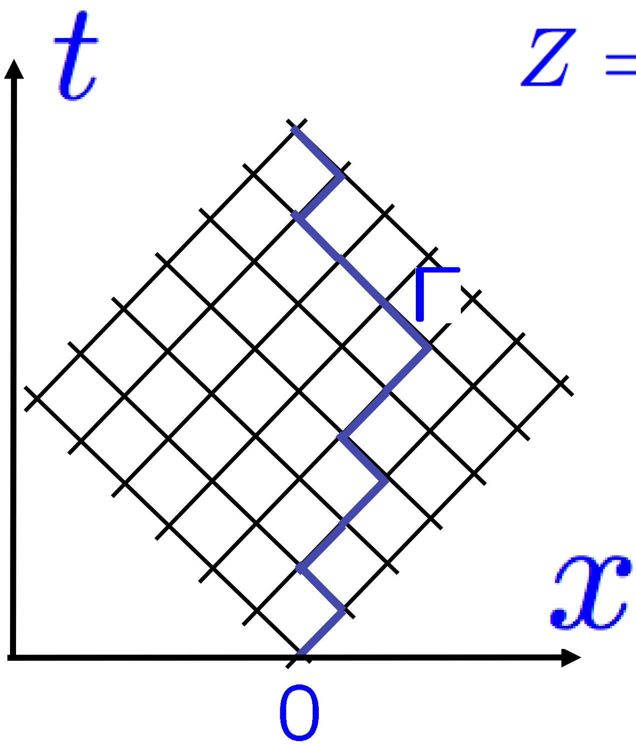
$$Z = \sum_{paths\,\Gamma} e^{-E_\Gamma/T}$$

V_i random variables

$$E_\Gamma = \sum_{i \in \Gamma} V_i$$

$$F = -T \ln Z$$

directed polymer: 1) lattice model



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$$\overline{E_0} = e_0 t \qquad (\overline{E_0^2}^c)^{1/2} = \sigma t^\theta \qquad \overline{x(t)^2} \sim t^{2\zeta}$$

Johansson 2000
T=0 proof

$$\theta = \frac{1}{3} \qquad \zeta = \frac{2}{3}$$

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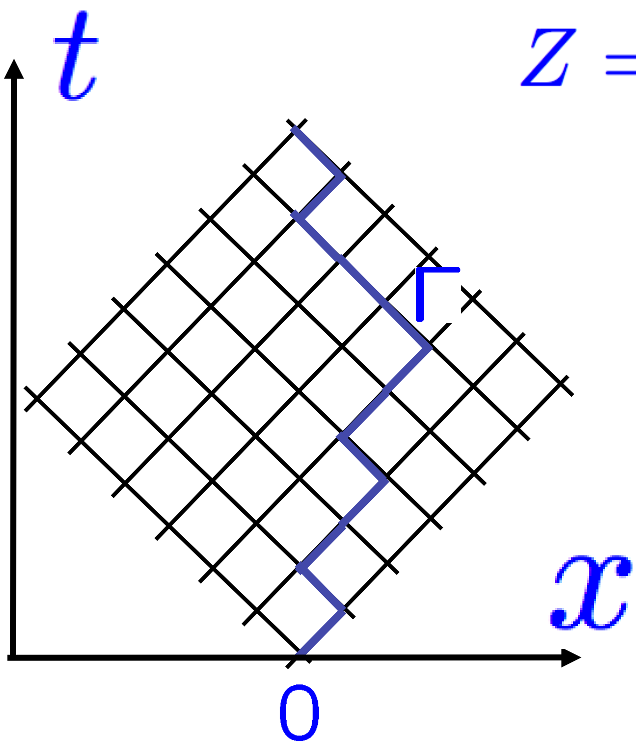

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$$\tilde{T} = T/\Delta F \sim t^{-1/3}$$

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directed polymer: 2) continuum model

$$Z(x,y,t) = \int_{x(0)=x}^{x(t)=y} Dx e^{-\frac{1}{T}\int_0^t d\tau [\frac{\kappa}{2}(\frac{dx}{d\tau})^2 + V(x(\tau),\tau)]}$$

$$\overline{V(x,t)V(x',t)} = \delta(t-t')R(x-x')$$

Feynman Kac

$$\partial_t Z = \frac{T}{2\kappa}\partial_x^2 Z - \frac{V(x,t)}{T} Z$$

$$Z(x,y,t=0) = \delta(x-y)$$

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$$R(x) \to \delta(x)$$

r_f

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Cole Hopf

$$\lambda_0 h(x,t) = T \ln Z(x,t)$$

KPZ

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STS symmetry

$$\theta = 2\zeta - 1$$

if white noise

$$\overline{\eta(x,t)\eta(x',t')} = D\delta(t-t')\delta(x-x')$$

$$P[\{h(x)\}] \sim e^{-\frac{\nu}{2D}\int dx h'(x)^2}$$

$$h \sim x^{1/2} \sim x^{\frac{\theta}{\zeta}}$$

$$\zeta = 2\theta = 2/3$$

Quantum mechanics and Replica..

$$\mathcal{Z}_n := \overline{Z(x_1, y_1, t)..Z(x_n, y_n, t)} = \langle x_1, ..x_n | e^{-tH_n^{rep}} | y_1, ..y_n \rangle$$

$$\partial_t \mathcal{Z}_n = -H_n^{rep} \mathcal{Z}_n$$

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$$x = T^3\kappa^{-1}\tilde{x} \quad , \quad t = 2T^5\kappa^{-1}\tilde{t}$$

high T limit:

$$\tilde{R}(z) \rightarrow 2\bar{c}\delta(z)$$

$$\bar{c} = \int du R(u)$$

$$T^3(\bar{c}\kappa)^{-1} \gg r_f$$

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drop the tilde..

$$H_{LL} = -\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{1\leq i<j\leq n}\delta(x_i-x_j) \qquad c = -\bar{c}$$

Attractive Lieb-Lineger (LL) model (1963)

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bosons or fermions?

Attractive Lieb-Lineger (LL) model (1963)

Bethe ansatz: ground state

Kardar 87

n bosons+attraction = bound state

$$\psi_0(x_1,..x_n) \sim \exp(-\frac{\bar{c}}{2}\sum_{i<j}|x_i - x_j|) \qquad E_0(n) = -\frac{\bar{c}^2}{12}n(n^2-1)$$

$$\overline{Z(x_1,0,t)..Z(x_n,0,t)} \approx_{t\to\infty} \psi_0(x_1,..x_n)e^{-tE_0(n)}$$

$$\overline{Z^n} = \overline{e^{n\ln Z}} = e^{\sum_p \frac{1}{p!}n^p\overline{(\ln Z)^p}^c} \sim e^{\frac{\bar{c}^2}{12}n^3 t}$$

can it be continued in n ?

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can it be continued in n ?

NO !

$$F = -\ln Z = \bar{F} + \lambda f \qquad \lambda = (\frac{\bar{c}^2}{4}t)^{1/3}$$

$$P(f) \sim_{f\to-\infty} \exp(-\frac{2}{3}(-f)^{3/2})$$

information about the tail of FE distribution

$$\overline{Z^n} = \int df e^{-n\lambda f - \frac{2}{3}(-f)^{3/2}} \sim e^{\frac{1}{3}\lambda^3 n^3}$$

FE distribution on a cylinder

Brunet Derrida (2000)

cylinder x+L = x $$E(n,L) = - \lim_{t\to+\infty} \frac{1}{t} \frac{\overline{Z^n(x,t)}}{\overline{Z(x,t)}^n}$$

- Kardar $L = +\infty$

violates $$\frac{\partial^2}{\partial n^2} E(n,L) \leq 0$$

cannot be continued in n

- ground state on cylinder $$E(n,L) = -\frac{1}{L^{3/2}} G(-nL^{1/2})$$

$$\sim n^3 \qquad nL \gg 1$$

large deviation of FE distribution on cylinder

Q: distribution of free energy ln Z? <=> distribution of h(x,t) in KPZ

DP of finite length t

$$Z(x,t) = e^{\frac{\lambda_0}{2\nu} h(x,t)}$$

Here= CONTINUUM model (DP or KPZ) = BA + sum over all excited states

fixed t , hence $L = +\infty$ is ok

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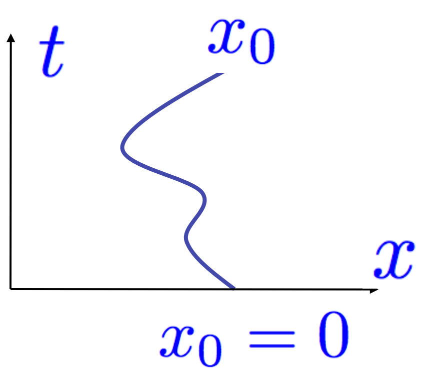
1) DP fixed endpoints



Johansson (2000) T=0

$$E_0 = e_0 t + \sigma \omega t^{1/3} \qquad P(V = q) \sim p^q$$

$$Prob(\omega > -s) = F_2(s)$$

Tracy Widom= largest eigenvalue of GUE

KPZ=narrow wedge, droplet initial condition

$$h(x, t = 0) = -w|x| \qquad w \to \infty$$

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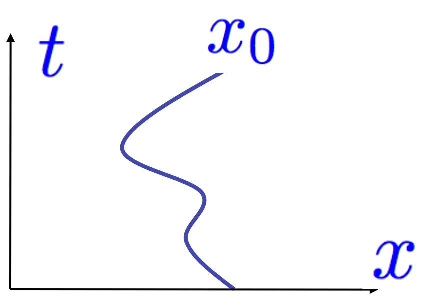

$x_0 = 0$

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2) DP one fixed one free endpoint

$$e^{\frac{\lambda_0}{2\nu} h(x,t)} = \int dy Z(x, t|y, 0) e^{\frac{\lambda_0}{2\nu} h(y,t=0)}$$

KPZ = flat initial condition $w \to 0$

$$h(x, t = 0) = 0$$

PNG model (Spohn, Ferrari,..)

$$t \to +\infty \quad F_1(s)$$

- Continuum DP fixed endpoint/KPZ Narrow wedge

1) BA + replica

- P. Calabrese, P. Le Doussal, A. Rosso EPL 90 20002 (2010)
- V. Dotsenko, EPL 90 20003 (2010) J Stat Mech P07010 Dotsenko Klumov P03022 (2010).

2) WASEP

- T Sasamoto and H. Spohn PRL 104 230602 (2010) Nucl Phys B 834 523 (2010) J Stat Phys 140 209 (2010).
- G. Amir, I. Corwin, J. Quastel Comm.Pure.Appl.Math. 64 466 (2011)

- Continuum DP one free endpoint/KPZ Flat

- P. Calabrese, P. Le Doussal, ArXiv: 1104.1993 (2011).

Universal Fluctuations of Growing Interfaces: Evidence in Turbulent Liquid Crystals

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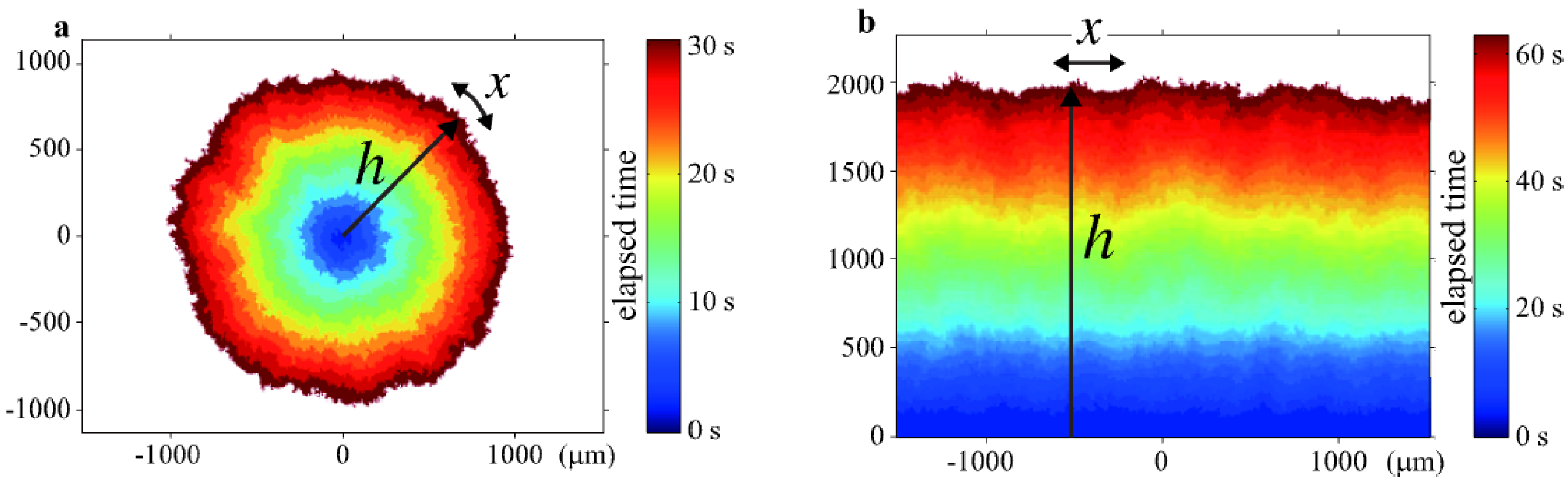
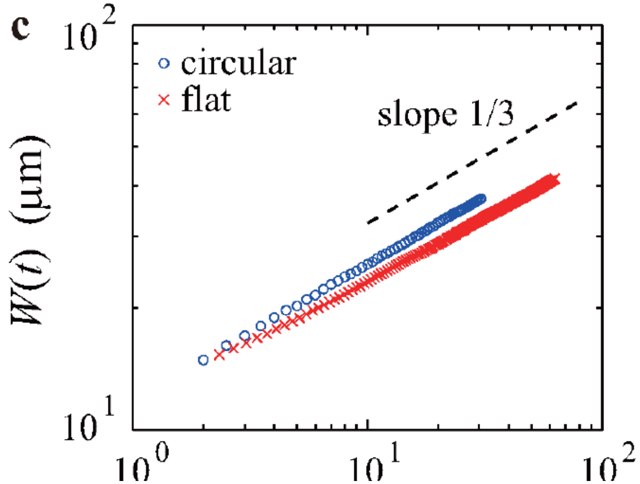
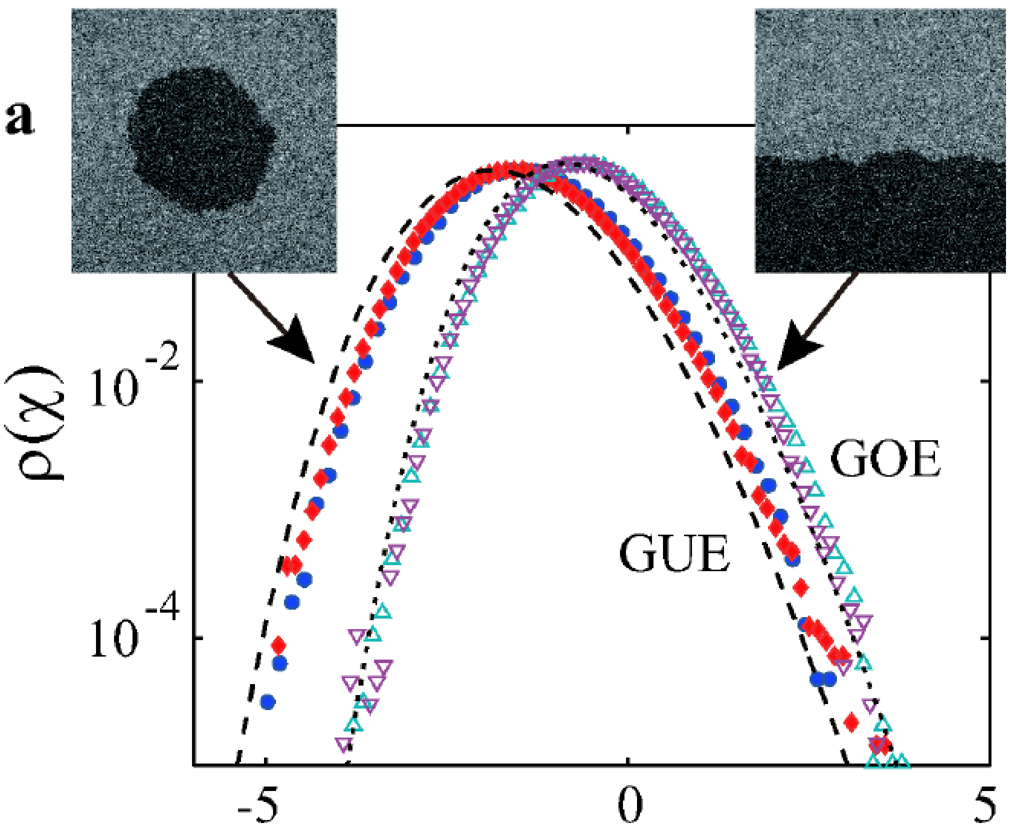
(Received 28 January 2010; published 11 June 2010)

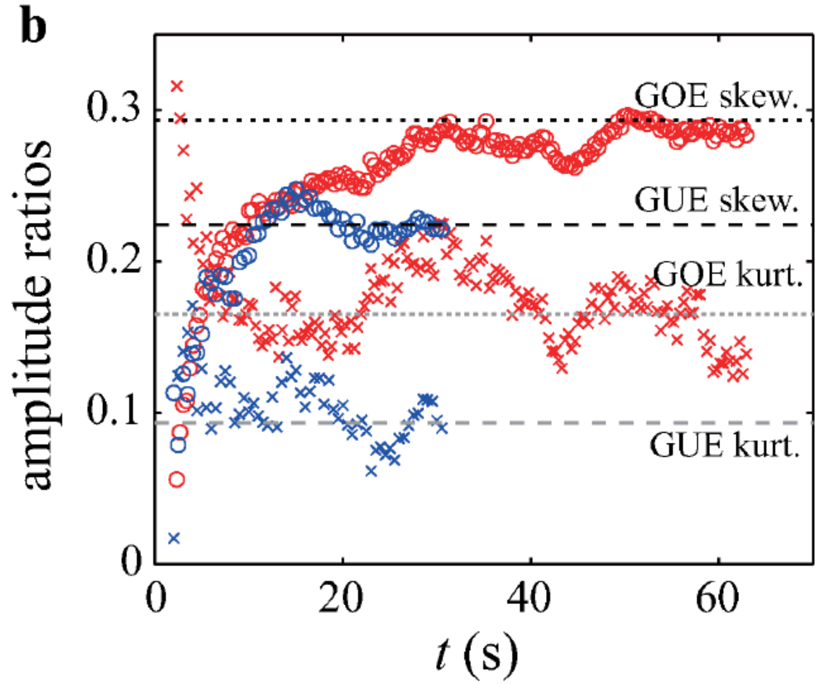


Experimental evidence. We study the convection of nematic liquid crystal, confined in a thin container and driven by an electric field[19,20], and focus on the interface between two turbulent states, called dynamic scattering modes 1 and 2 (DSM1 and DSM2)[20,21]. The latter consists of a large quantity of topological defects and can be created by nucleating a defect with a ultraviolet laser pulse. Whereas



$$h \simeq v_\infty t + (\Gamma t)^{1/3} \chi,$$





Bethe ansatz details

$$-\psi'' - \bar{c}\delta(x)\psi(0) = E\psi$$

$$[\psi'/\psi]_{0^-}^{0^+} = -\bar{c}$$

$$\psi(x) = \cos(kx) - \frac{\bar{c}}{2k}\mathrm{sgn}(x)\sin(kx)$$

$$\psi(0) = 1$$

n=2 $\quad H_2 = -\partial_{x_1}^2 - \partial_{x_2}^2 - \bar{c}\delta(x_1 - x_2)$

$$\psi_{\lambda_1,\lambda_2}(x_1, x_2) = sym_{x_1,x_2}\, e^{i\lambda_1 x_1 + i\lambda_2 x_2}\left(1 - \frac{ic}{\lambda_2 - \lambda_1}\mathrm{sgn}(x_2 - x_1)\right)$$

$$E = \lambda_1^2 + \lambda_2^2$$

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Periodic BC = Bethe equations

$$e^{i\lambda_1 L} = \frac{\lambda_1 - \lambda_2 - i\bar{c}}{\lambda_1 - \lambda_2 + i\bar{c}}$$

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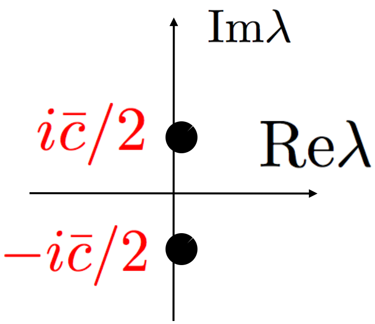
Periodic BC = Bethe equations $\quad e^{i\lambda_1 L} = \frac{\lambda_1 - \lambda_2 - i\bar{c}}{\lambda_1 - \lambda_2 + i\bar{c}}$

solutions:

- 2 1-string $\quad (\lambda_1, \lambda_2) = (k_1, k_2) \in R^2$

$$\lambda_j = \frac{2\pi n_j}{L} + o(\frac{1}{L})$$

- 1 2-string $\quad \lambda_{1,2} = k \pm i\frac{\bar{c}}{2} + O(ie^{-\bar{c}L})$



$$\overline{Z^n} = \langle x_0 \ldots x_0 | e^{-tH_{LL}} | x_0 \ldots x_0 \rangle$$

$$= \sum_{\mu} \frac{|\langle x_0 \ldots x_0 | \mu \rangle|^2}{||\mu||^2} e^{-tE_\mu}$$

all eigenstates are of the form

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^{n} e^{i\lambda_{P_\ell} x_\ell}$$

$$A_P = \prod_{n \geq \ell > k \geq 1} \left(1 - \frac{ic \ \mathrm{sgn}(x_\ell - x_k))}{\lambda_{P_\ell} - \lambda_{P_k}}\right)$$

$$\overline{Z^n} = \langle x_0 \dots x_0 | e^{-tH_{LL}} | x_0 \dots x_0 \rangle$$

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Bethe equations + large L

All possible partitions of n into j=1,..ns strings each with mj particles

$$n = \sum_{j=1}^{n_s} m_j$$

$$\lambda^{j,a} = k_j + \frac{i\bar{c}}{2}(j + 1 - 2a) \qquad a = 1, \dots, m_j$$

$$E_\mu = \sum_{j=1}^{n_s} (m_j k_j^2 - \frac{\bar{c}^2}{12} m_j (m_j^2 - 1)) \qquad K_\mu = \sum_{j=1}^{n_s} m_j k_j$$

(Kardar) ground state ns=1, m1=n, k1=0

what is needed?

$$\overline{Z^n} = \sum_{\mu} \frac{|\langle x_0 \dots x_0|\mu\rangle|^2}{||\mu||^2} e^{-tE_\mu}$$

$$\langle 0 \cdots 0|\mu\rangle = \Psi_\mu(0,..0) = n!$$

norm of states: Calabrese-Caux (2007)

$$||\mu||^2 = \frac{n!(L\bar{c})^{n_s}}{(\bar{c})^n} \frac{\prod_{j=1}^{n_s} m_j^2}{\Phi[k,m]}$$

$$\Phi[k,m] = \prod_{1\leq i<j\leq n_s} \frac{(k_i-k_j)^2+(m_i-m_j)^2c^2/4}{(k_i-k_j)^2+(m_i+m_j)^2c^2/4}$$

integer moments of partition sum

$$n = \sum_{j=1}^{n_s} m_j$$

$$\overline{\hat{Z}^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\ldots m_{n_s})_n}$$

$$\int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi[k,m] \prod_{j=1}^{n_s} e^{m_j^3 \frac{\bar{c}^2 t}{12} - m_j k_j^2 t},$$

$$\Phi[k,m] = \prod_{1 \leq i < j \leq n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 c^2/4}{(k_i - k_j)^2 + (m_i + m_j)^2 c^2/4}$$

numerical check of second moment

$$Z_{\hat{x},\hat{t}+1} = (Z_{\hat{x}-\frac{1}{2},\hat{t}} + Z_{\hat{x}+\frac{1}{2},\hat{t}})e^{-\beta V_{\hat{x},\hat{t}+1}}$$

$$\kappa = 4T \qquad \tilde{x} = 4\hat{x}/T^2$$

$$\tilde{t} = 2\hat{t}/T^4$$

unit gaussian on the lattice $\bar{c} = 1$

$$\lambda = (\frac{\bar{c}^2}{4}t)^{1/3}$$

$$\overline{z^2} = 1 + \sqrt{2\pi}\lambda^{3/2}e^{2\lambda^3}(1 + \mathrm{erf}(\sqrt{2}\lambda^{3/2}))$$

$$z = Z/\overline{Z}$$

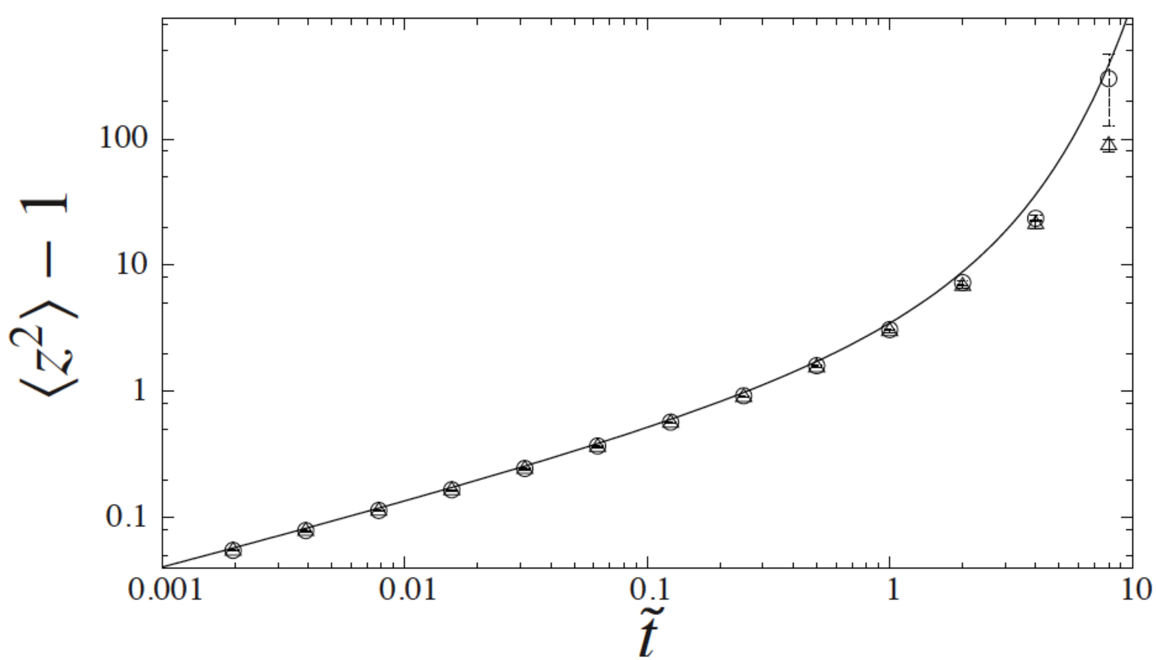

FIG. 1: $\overline{z^2} - 1$ ($4\ 10^6$ samples) for $\hat{t} = 128$ (triangle), $\hat{t} = 256$ (circle) function of $\tilde{t}$ compared to formula (11) with $\bar{c} = 1$.

Numerical check, small time expansion

$$\overline{(\ln z)^2}^c = \sqrt{2\pi}\lambda^{3/2} + (4 + 5\pi - \frac{32\pi}{3\sqrt{3}})\lambda^3 + ..$$

$$\overline{(\ln z)^3}^c = (\frac{32}{3\sqrt{3}} - 6)\pi\lambda^3 + ..$$

$$\lambda = (\frac{\bar{c}^2}{4}t)^{1/3}$$

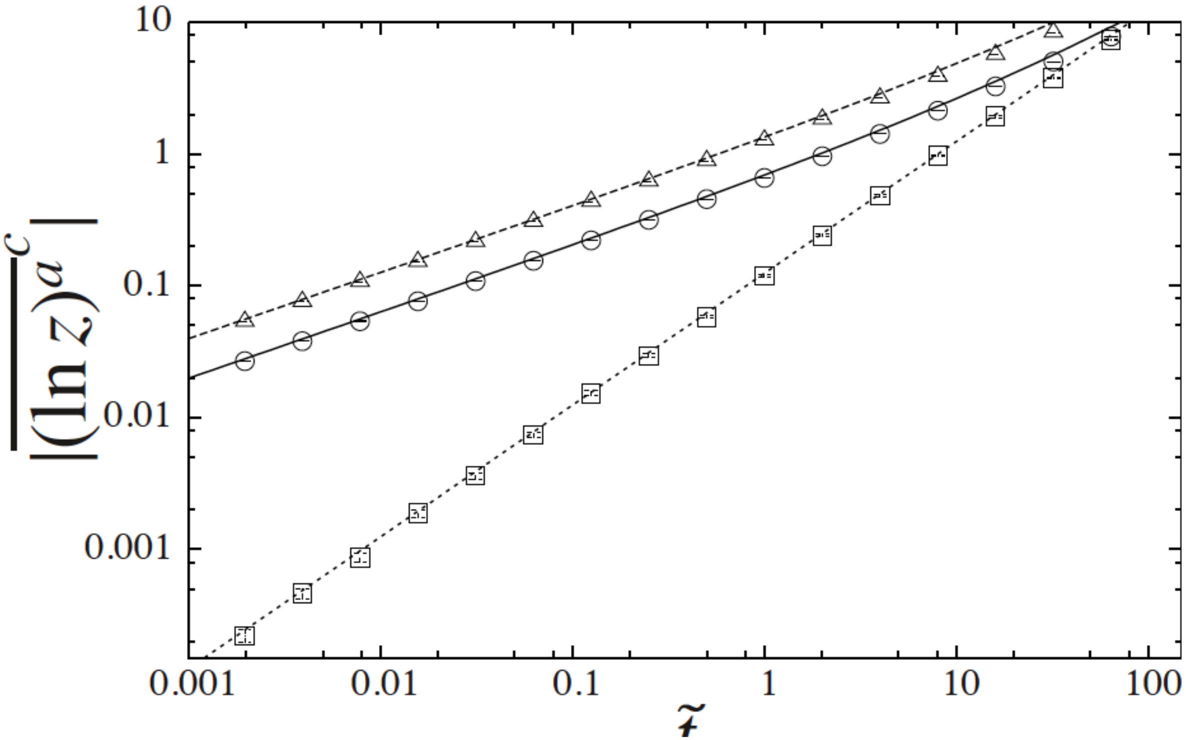

FIG. 2: From top to bottom the cumulants ($4\ 10^6$ samples) $\overline{(\ln z)^2}^c$ (dashed line, triangle), $-\overline{(\ln z)}$ (solid line, circle), and $\overline{(\ln z)^3}^c$ (dotted line, square) for $\hat{t} = 256$ as compared with the the analytical formula (12) with $\bar{c} = 1$.

generating function of moments

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

$$F = \lambda f$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$\lim_{\lambda \to \infty} g(x) = \overline{\theta(f - x)} = Prob(f > x)$$

generating function of moments

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reorganise sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

Interactions between strings

generating function of moments

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(-e^{\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

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reorganise sum over number of strings

$$g(x) = 1 + \sum_{n_s=1}^{\infty} \frac{1}{n_s!} Z(n_s, x)$$

Airy trick

$$\int_{-\infty}^{\infty} dy Ai(y) e^{yw} = e^{w^3/3}$$

$$Z(n_s, x) = \sum_{m_1, \ldots m_{n_s}=1}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}}$$

$$\prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1 \le i < j \le n_s} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 \lambda^3}{(k_i - k_j)^2 + (m_i + m_j)^2 \lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

Interactions between strings

One string contribution ns=1

$$Z(1,x) = \int_{v>0} \frac{dv\ v^{1/2}}{2\pi\lambda^{3/2}} dy Ai(y) \sum_{m=1}^{\infty} (-1)^m e^{\lambda m y - v m + \lambda x m}$$

$$v \to \lambda v$$

$$y \to y + v - x$$

$$Z(1,x) = -\int_{v>0} \frac{dv\ v^{1/2}}{2\pi} dy Ai(y + v - x) \frac{e^{\lambda y}}{1 + e^{\lambda y}}$$

$$\frac{e^{\lambda y}}{1+e^{\lambda y}} \to \theta(y)$$

$$\lim_{\lambda \to \infty} Z(1,x) = -\int_{w>0} \frac{dw}{3\pi} w^{3/2} Ai(w - x)$$

independent string approximation

$$g_{ind}(x) = \exp(Z(1,x)) \qquad Prob_{ind}(f > x) = g_{ind}(x)$$

correct tail for large negative f (exponent and prefactor..)

full solution

$$Z(n_s, x) = \sum_{m_1,\ldots m}^{\infty} \frac{(-1)^{\sum_j m_j}}{(4\pi\lambda^{3/2})^{n_s}} \prod_{j=1}^{n_s} \int \frac{dk_j}{m_j} \prod_{1\le i<j\le n_s} \frac{(k_i-k_j)^2+(m_i-m_j)^2\lambda^3}{(k_i-k_j)^2+(m_i+m_j)^2\lambda^3} \prod_{j=1}^{n_s} e^{\frac{1}{3}\lambda^3 m_j^3 - m_j k_j^2 + \lambda x m_j}$$

$$det\left[\frac{1}{i(k_i-k_j)\lambda^{-3/2}+(m_i+m_j)}\right]$$

$$= \prod_{i<j} \frac{(k_i-k_j)^2+(m_i-m_j)^2\lambda^3}{(k_i-k_j)^2+(m_i+m_j)^2\lambda^3} \prod_{i=1}^{n_s} \frac{1}{2m_i}$$

Result: Fredholm determinant

$$Z(n_s, x) = \int_{v_i > 0} \prod_{i=1}^{n_s} dv_i \quad det[K_x(v_i, v_j)]$$

$$\lambda = (\frac{\bar{c}^2}{4} t)^{1/3}$$

$$K_x(v, v') = \Phi_x(v + v', v - v')$$

$$\Phi_x(u, w) = -\int \frac{dk}{2\pi} dy Ai(y + k^2 - x + u) \frac{e^{\lambda y - ikw}}{1 + e^{\lambda y}}$$

Result: Fredholm determinant

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$$g(x) = Det[1 + P_0 K_x P_0]$$

P_s projector on $[s, +\infty[$

$$= e^{Tr \ln(1+K)} \quad = 1 + TrK + O(TrK^2)$$

$n_s = 1$ $\quad \int_{v>0} K_x(v, v)$

$n_s = 2$

Large time limit and F2(s)

$$\lambda = \left(\frac{\bar{c}^2}{4} t\right)^{1/3}$$

$$\lambda = +\infty$$

$$Prob(f > x) = g(x) = \det(1 + P_{-\frac{x}{2}} \tilde{K} P_{-\frac{x}{2}})$$

$$\tilde{K}(v, v') = -\int_{y>0} \frac{dk}{2\pi} dy Ai(y + k^2 + v + v') e^{-ik(v-v')}$$

Airy function identity

$$\int dk Ai(k^2 + v + v') e^{ik(v-v')} = 2^{2/3} \pi Ai(2^{1/3} v) Ai(2^{1/3} v')$$

$$Prob(f > x = -2^{2/3} s) = Det(1 - P_s K_{Ai} P_s) = F_2(s)$$

$$K_{Ai}(v, v') = \int_{y>0} Ai(v + y) Ai(v' + y)$$

Strong universality at large time

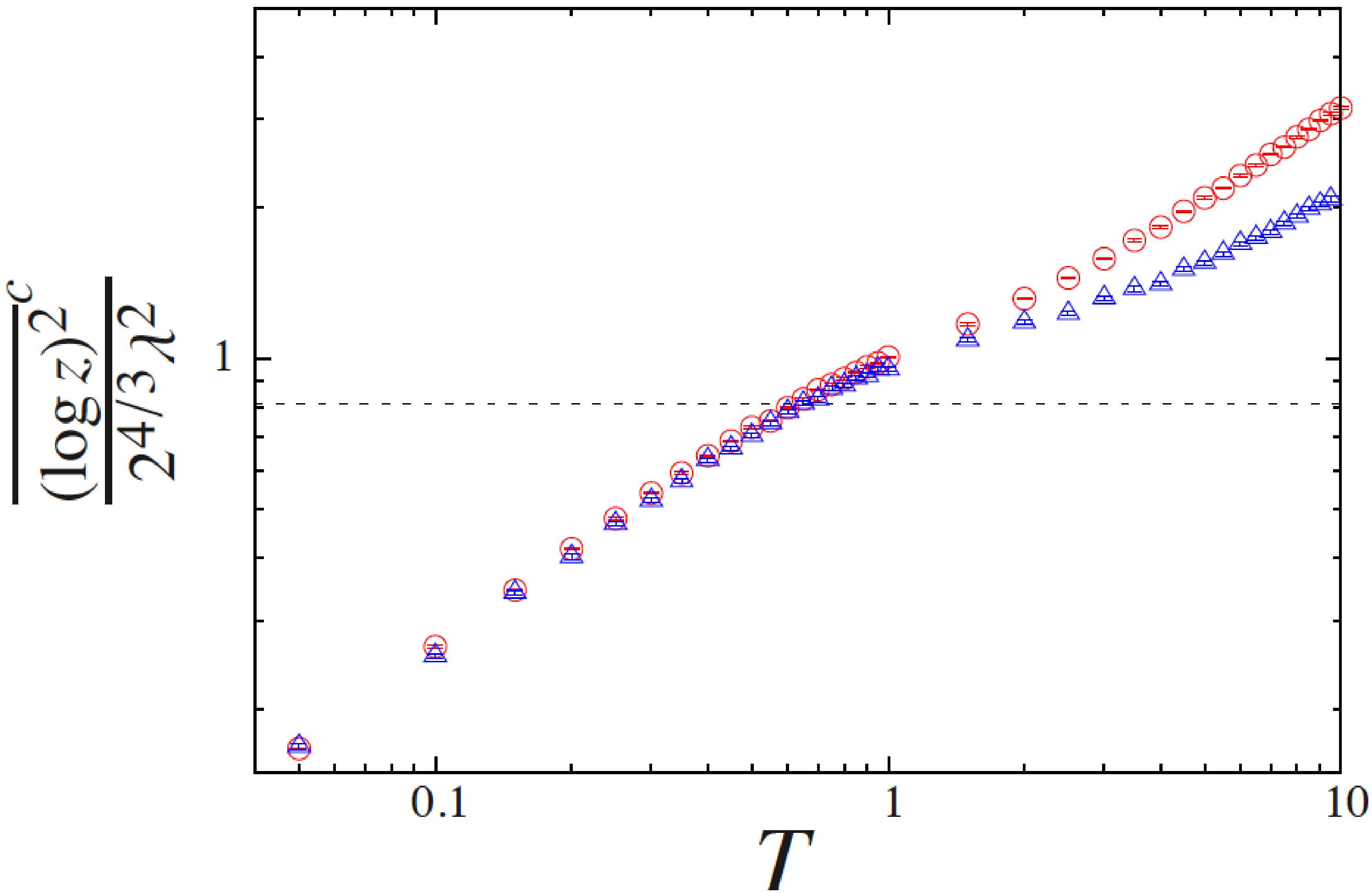

FIG. 3: $\overline{(\ln z)^2}^c/(2^{4/3}\lambda^2)$ plotted as a function of T, for increasing polymer length $\hat{t}$. Triangles correspond to $\hat{t} = 4096$, Circles to $\hat{t} = 256$ and the dotted line to the TW variance 0.81319... Averages are performed over 20000 samples.

An exact solution for the KPZ equation with flat initial conditions

P. Calabrese, P. Le Doussal, PRL (2011)

$$Z(n_s) = \sum_{m_i \geq 1} \prod_{j=1}^{n_s} \int_{k_j} \prod_{q=1}^{m_j} \frac{-2}{2ik_j + q} e^{\frac{\lambda^3}{3} m_j^3 - 4 m_j k_j^2 \lambda^3 - \lambda m_j s}$$

$$\times \mathrm{Pf}\left[\begin{pmatrix} \frac{2\pi}{2ik_i}\delta(k_i + k_j)(-1)^{m_i}\delta_{m_i,m_j} + \frac{1}{4}(2\pi)^2\delta(k_i)\delta(k_j)(-1)^{\min(m_i,m_j)}\mathrm{sgn}(m_i - m_j) & \frac{1}{2}(2\pi)\delta(k_i) \\ -\frac{1}{2}(2\pi)\delta(k_j) & \frac{2ik_i + m_i - 2ik_j - m_j}{2ik_i + m_i + 2ik_j + m_j} \end{pmatrix} \right]$$

$$Z(n_s) = \prod_{j=1}^{n_s} \int_{v_j > 0} \mathrm{Pf}[\mathbf{K}(v_i, v_j)]_{2n_s, 2n_s}$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J} + \mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s) \qquad \mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1 + v_i + s + 4k^2) Ai(y_2 + v_j + s + 4k^2) [\frac{e^{-2i(v_i - v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)}) + \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})]$$

$$K_{12} = \frac{1}{2} \int_y Ai(y + s + v_i)(e^{-2e^{\lambda y}} - 1)\ \delta(v_j)$$

$$K_{22} = 2\delta'(v_i - v_j)\,,$$

$$f_k(z) = \frac{-2\pi k z\, {}_1F_2\left(1; 2-2ik, 2+2ik; -z\right)}{\sinh\left(2\pi k\right) \Gamma\left(2-2ik\right) \Gamma\left(2+2ik\right)}\,, \qquad (19)$$

$$F(z_i, z_j) = \sinh(z_2 - z_1) + e^{-z_2} - e^{-z_1} + \int_0^1 du$$

$$\times J_0(2\sqrt{z_1 z_2 (1-u)})[z_1 \sinh(z_1 u) - z_2 \sinh(z_2 u)]\,.$$

$$g_\lambda(s) = \mathrm{Pf}[\mathbf{J}+\mathbf{K}] = \sum_{n_s=0}^{\infty} \frac{1}{n_s!} Z(n_s) \qquad \mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$K_{11} = \int_{y_1,y_2,k} Ai(y_1+v_i+s+4k^2)Ai(y_2+v_j+s+4k^2)\Big[\frac{e^{-2i(v_i-v_j)k}}{2ik} f_{k/\lambda}(e^{\lambda(y_1+y_2)}) + \frac{\pi\delta(k)}{2} F(2e^{\lambda y_1}, 2e^{\lambda y_2})\Big]$$

$$K_{12} = \frac{1}{2}\int_y Ai(y+s+v_i)(e^{-2e^{\lambda y}}-1)\ \delta(v_j)$$

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$$\times J_0(2\sqrt{z_1 z_2(1-u)})[z_1\sinh(z_1 u) - z_2\sinh(z_2 u)]\,.$$

$$\lim_{\lambda\to+\infty} f_{k/\lambda}(e^{\lambda y}) = -\theta(y)$$

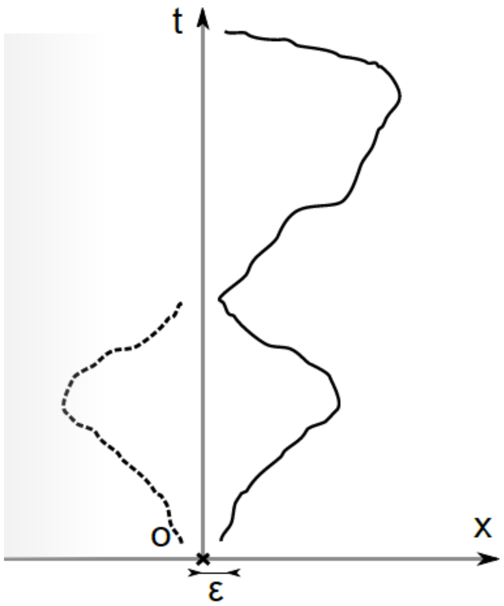
$$\lim_{\lambda\to+\infty} F(2e^{\lambda y_1}, 2e^{\lambda y_2}) = \theta(y_1+y_2)(\theta(y_1)\theta(-y_2) - \theta(y_2)\theta(-y_1))$$

$$\lim_{\lambda\to+\infty} Z(n_s) = (-1)^{n_s}\int_{x_1,..x_{n_s}} \det[\mathcal{B}_s(x_i,x_j)]_{n_s\times n_s}.$$

$$g_\infty(s) = F_1(s) = \det[I-\mathcal{B}_s] \qquad \text{GOE Tracy Widom}$$

$$\mathcal{B}_s = \theta(x)Ai(x+y+s)\theta(y)$$

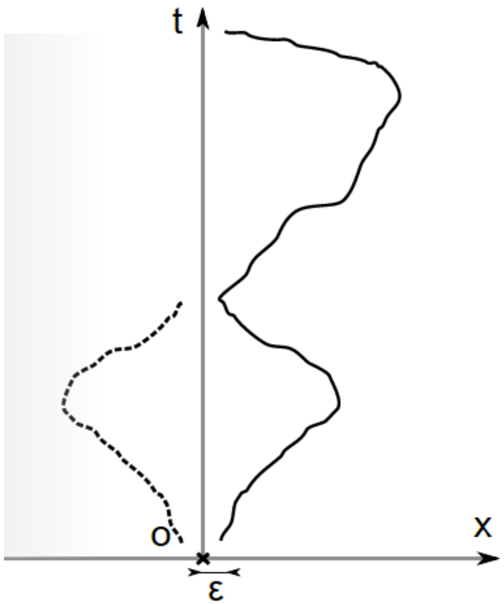
DP near a wall = KPZ equation in half space



$Z(x,0,t) = Z(0,y,t) = 0$

$\nabla h(0,t)$ fixed

DP near a wall = KPZ equation in half space



$$g_{FS}(s) < g_{HS}(s)^2$$

$$g(s) = \sqrt{\mathrm{Det}[I + \mathcal{K}]}$$

$$\mathcal{K}(v_1, v_2) = -2\theta(v_1)\theta(v_2)\partial_{v_1} f(v_1, v_2)$$

$$f(v_1, v_2) = \int \frac{dk}{2\pi} \int_y Ai(y + s + v_1 + v_2 + 4k^2)\, f_{k/\lambda}(e^{\lambda y}) \frac{e^{-2ik(v_1 - v_2)}}{2ik}$$

$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right)$$
$$- {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$Z(x, 0, t) = Z(0, y, t) = 0$$

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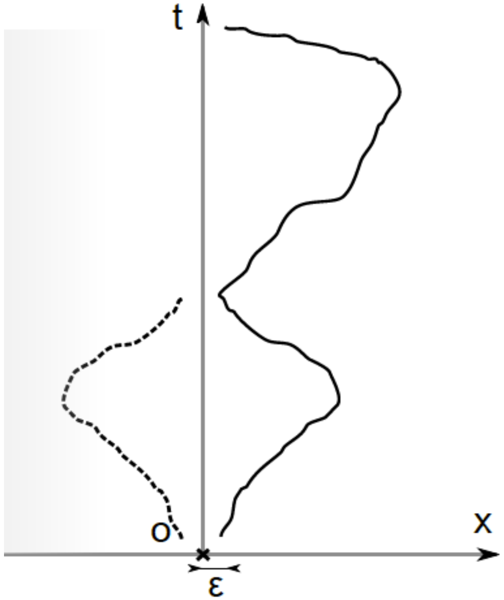

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$$f(v_1, v_2) = \int \frac{dk}{2\pi} \int_y Ai(y + s + v_1 + v_2 + 4k^2) f_{k/\lambda}(e^{\lambda y}) \frac{e^{-2ik(v_1 - v_2)}}{2ik}$$

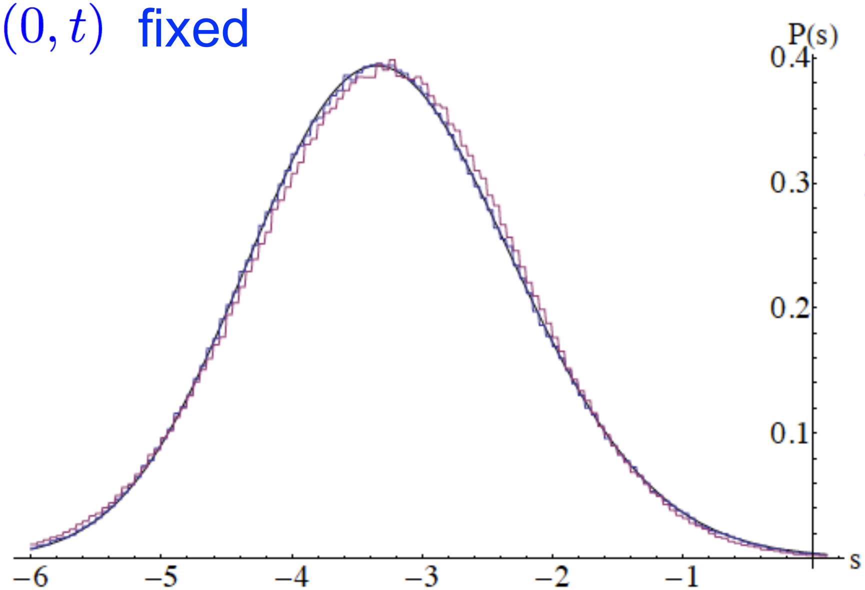
$$f_k[z] = \frac{2\pi k}{\sinh(4\pi k)} \left(J_{-4ik}(\frac{2}{\sqrt{z}}) + J_{4ik}(\frac{2}{\sqrt{z}}) \right)$$
$$- {}_1F_2\,(1; 1 - 2ik, 1 + 2ik; -1/z)$$

$$\lim_{\lambda \to \infty} f_{k/\lambda}[e^{\lambda y}] = -\theta(y)(1 - \cos(2ky))$$

$$Z(x, 0, t) = Z(0, y, t) = 0$$

$\nabla h(0, t)$ fixed

$$\lambda = (\frac{\bar{c}^2 t}{8T^5})^{1/3} = (\frac{D\lambda_0^2 t}{8(2\nu)^5})^{1/3}$$



$$\ln Z = \frac{\lambda_0}{2\nu} \tilde{h}(0, t) = v_\infty t + 2^{2/3} \lambda \chi_4$$

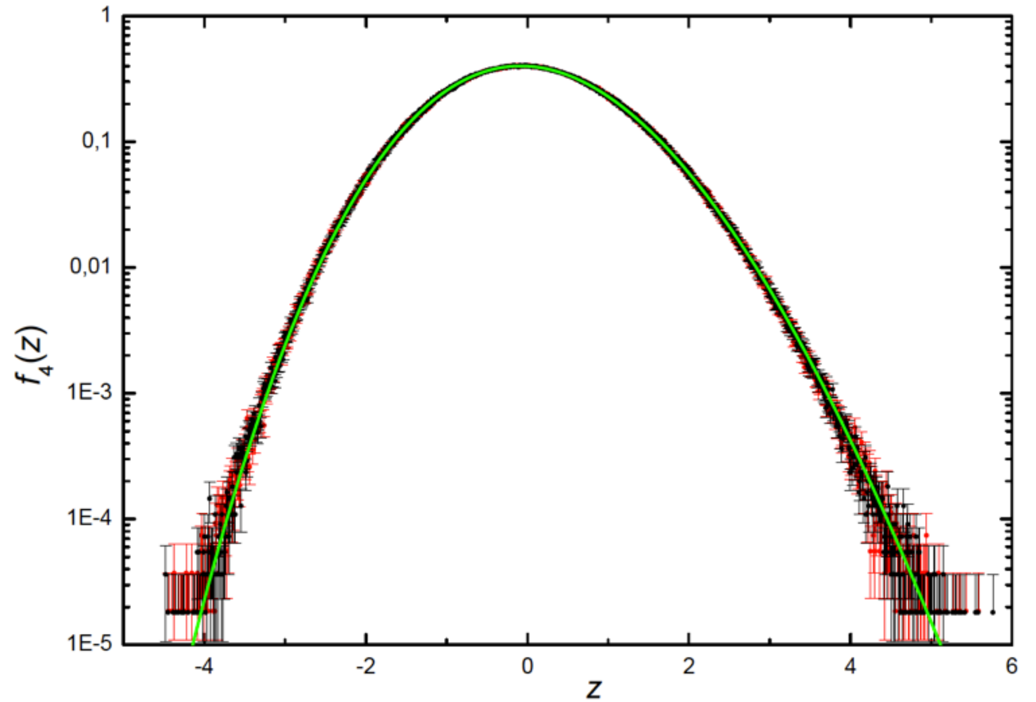
χ_4 distributed as $F_4(s)$

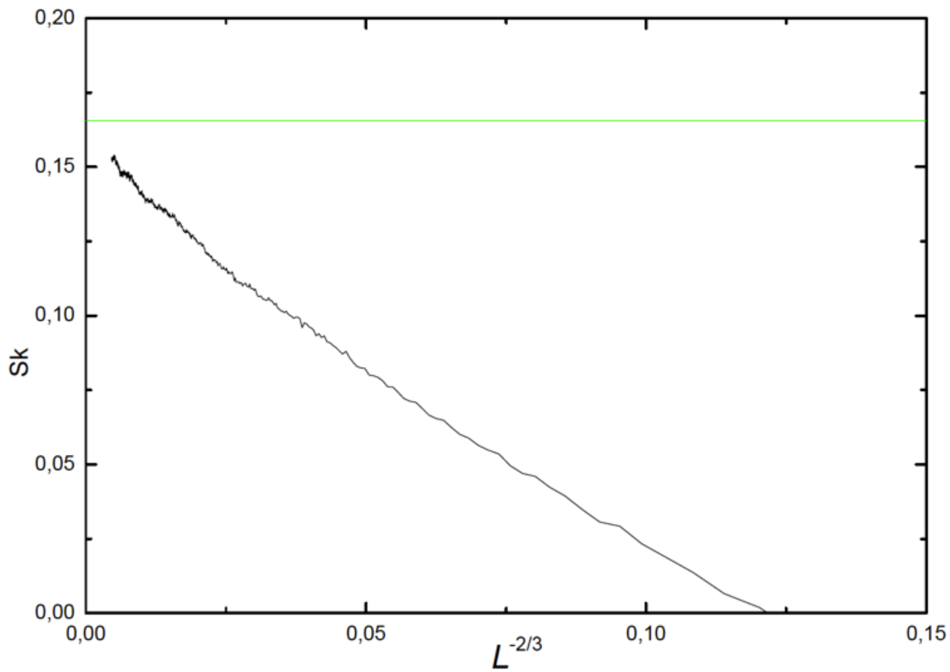
Gaussian Symplectic Ensemble

Ortuno Somoza

log of conductance point to point
near sample edges

box distribution W=10
L=100-3200





conclusion

- solved continuum model delta disorder (DP)/noise (KPZ)

it describes:

- any DP model high T (crossover Brownian to glass)
- any KPZ class growth weak noise/large diffusivity

(crossover Edwards-Wilkinson to KPZ)

strong universality

- solution using BA for all t : generating function related to some Fredholm determinant for all t

- obtain free energy/KPZ height distribution for all t

- obtain convergence to Tracy Widom distrib. large t: KPZ is in KPZ class!

- DP fixed endpoints/KPZ droplet initial condition to GUE
- DP one free endpoint/KPZ flat initial condition to GOE
- DP fixed endpoint near wall/KPZ half-space to GSE

- predict new crossover in 2D strongly localized systems ?

conclusion

- continuum delta model describes DP high T
  strong universality

- solution using BA of DP fixed endpoints for all t
  (KPZ droplet init. cond).

- generating function is a Fredholm determinant for all t

- obtain free energy/KPZ height distribution for all t
  GUE confirmed large t = KPZ in KPZ class..

- solution using BA of DP one free endpoint for all t
  (KPZ flat init. cond).