Geometry of Quantum Observables, Integrability-Thermalizability Transition, and Extended Thermodynamics of Integrable and/or Mesoscopic Systems

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Preview

- Proposition:

  The sum of the ensemble variance of the temporal means and the ensemble mean of the temporal variances remains approximately constant across the integrability-to-ergodicity transition

- Example of a Sinai-type billiard
- Example of long-range interacting hard-core bosons
- Ensemble variance of temporal means as a cos^2 of the Hilbert-Schmidt (HS) angle between the observable and integrals of motion

  HS geometry of density matrices for Quantum Information: 164 arXiv articles
  HS geometry of observables: 0 arXiv, hints in Suzuki's quantum extension of Mazur's theorem (Physica 51 (1971))

- IPR as an angle between the original and perturbed integrals of motion
- An application of the HS geometry: Optimal integrals of motion for GGE
- Mesoscopicity and integrability on the same footing $\rightarrow$ "nano-meso"

Preview

We suggest

$$\tan^2[\Theta_A] \equiv \frac{Var_{MC}[Mean_t[A]]}{Mean_{MC}[Var_t[A]]}$$

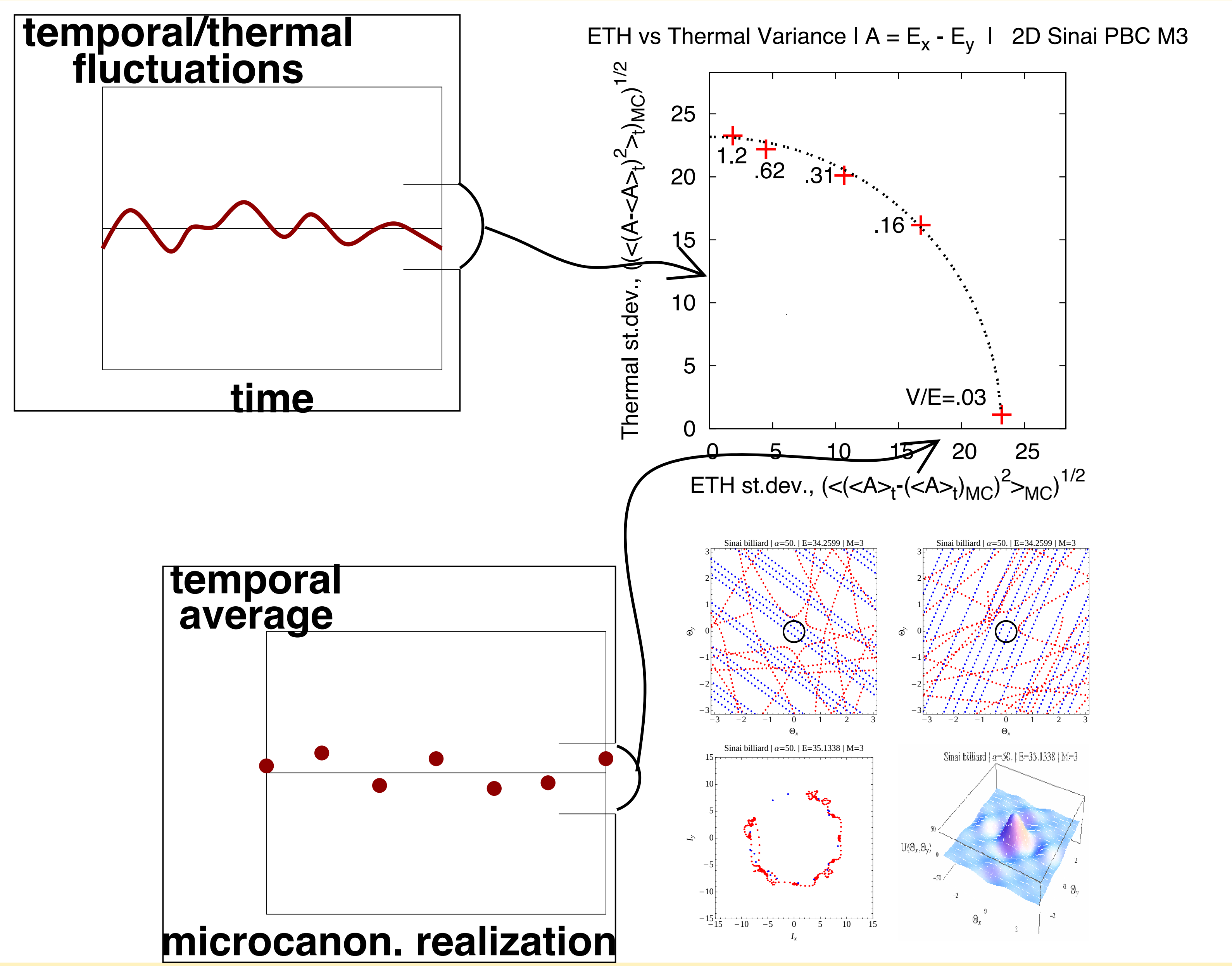
as a measure of the position of an observable A on the (Integral of Motion)-(Thermalizable Observable) continuum.

Sinai billiard

A Sinai-type billiard

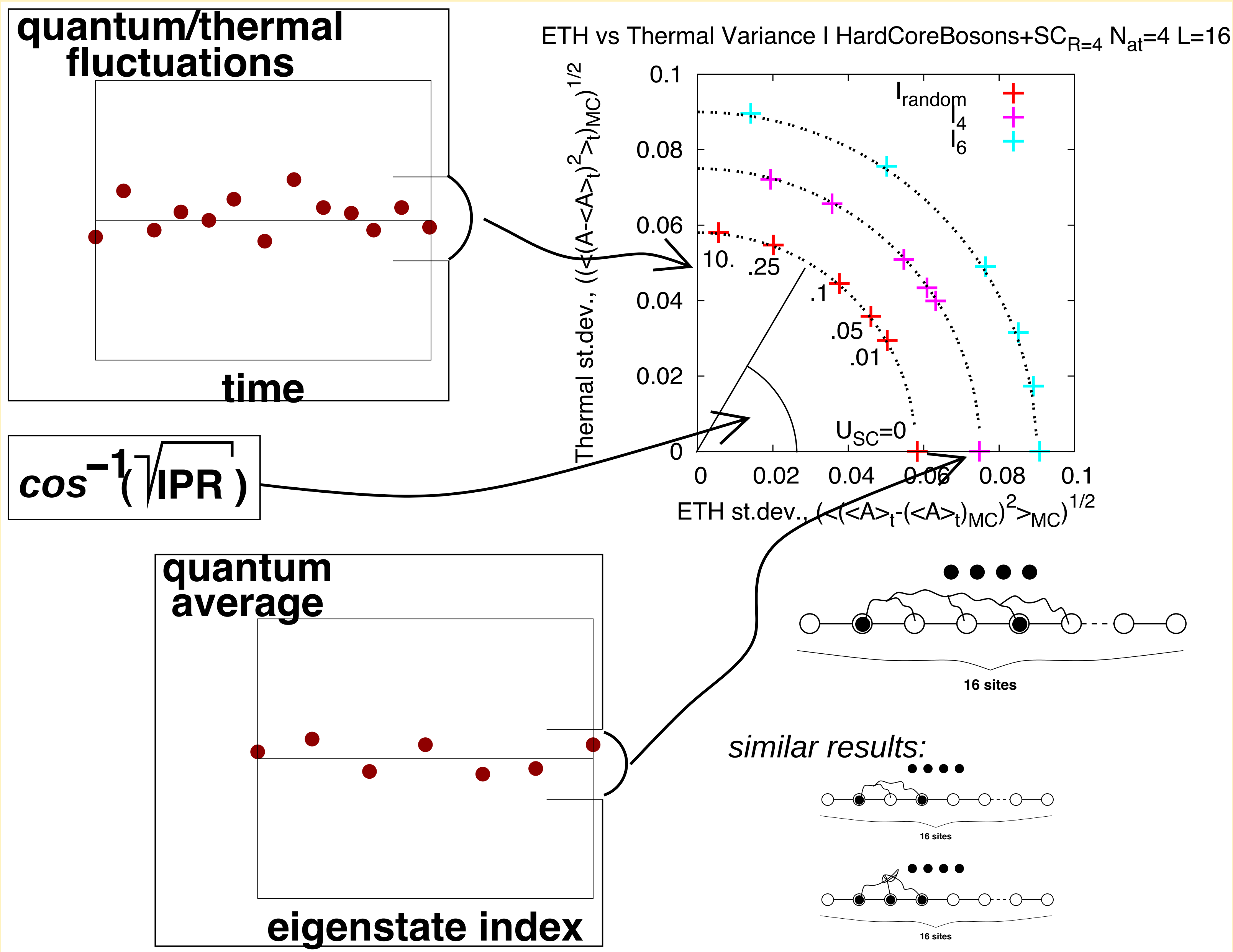
Movies:
_R__trajectory01.gif
_R__trajectory02.gif

A Sinai-type billiard: $Var_{MC}[Mean_t[A]]$ vs. $Mean_{MC}[Var_t[A]]$



Hard-Core Bosons

HCB+NNNNN: $Var_{MC}[Mean_t[A]]$ vs. $Mean_{MC}[Var_t[A]]$



Hilbert-Schmidt inner product

"[T]he angel of geometry and the devil of algebra share the stage, illustrating the difficulties of both."

Hermann Well

The Hilbert-Schmidt (HS) inner product between two matrices:

$$(\hat{A}|\hat{B}) \equiv Tr[\hat{A}^\dagger \hat{B}] \quad .$$

HS product is invariant under unitary transformations:

$$(\hat{U}\hat{A}\hat{U}^{-1}|\hat{U}\hat{B}\hat{U}^{-1}) = (\hat{A}|\hat{B}) \quad .$$

The unitary transformations form a (small) subgroup of the group of HS rotations: the latter preserve the HS norm, defined as

$$||\hat{A}|| \equiv \sqrt{Tr[\hat{A}^2]} \quad .$$

Some definitions

Introduce:

(a) a microcanonical window: $\mathcal{W}_{MC} \equiv \{|\alpha\rangle \quad | \quad E_\alpha \in [E_{min},\, E_{max}]\}$, where $|\alpha\rangle$ and E_α are the eigenstates and eigenenergies of a Hamiltonian $\hat{H}$;

(b) a HS normalized identity operator: $\hat{\mathcal{I}} = (N_{MC})^{-1/2} I$;

(c) a space of the diagonal (w.r.t. $\hat{H}$) observables: $\mathcal{L}_{d,\,\hat{H}} \equiv Span[\{|\alpha\rangle\langle\alpha| \quad | \quad |\alpha\rangle \in \mathcal{W}_{MC}\}]$;

(d) a space of the off-diagonal (w.r.t. $\hat{H}$) observables: $\mathcal{L}_{o-d,\,\hat{H}} \equiv Span[\{2^{-1/2}(|\alpha\rangle\langle\beta| + h.c.)\} \cup i2^{-1/2}(|\alpha\rangle\langle\beta| - h.c.)\} \quad | \quad |\alpha\rangle \in \mathcal{W}_{MC}; |\beta\rangle \in \mathcal{W}_{MC}; \quad \beta > \alpha \quad \}]$.

Then:

ensemble variance of the temporal means (ETH variance) $\equiv$

$$\left(N_{MC}^{-1}\sum_{\alpha}^{\mathcal{W}_{MC}} A_{\alpha,\,\alpha}^2 - \langle\hat{A}\rangle_{MC}\right) / \left(\langle\hat{A}^2\rangle_{MC} - \langle\hat{A}\rangle_{MC}^2\right) = \cos^2(\hat{A}\widehat{\ }\mathcal{L}_{d,\,\hat{H}}) - \cos^2(\hat{A}\widehat{\ }\hat{\mathcal{I}})$$

ensemble mean of the temporal (quantum) variance $\equiv$

$$N_{MC}^{-1}\left(\sum_{\alpha,\,\beta\neq\alpha}^{\mathcal{W}_{MC}} A_{\alpha,\,\beta}^2\right) / \left(\langle\hat{A}^2\rangle_{MC} - \langle\hat{A}\rangle_{MC}^2\right) = \cos^2(\hat{A}\widehat{\ }\mathcal{L}_{o-d,\,\hat{H}})$$

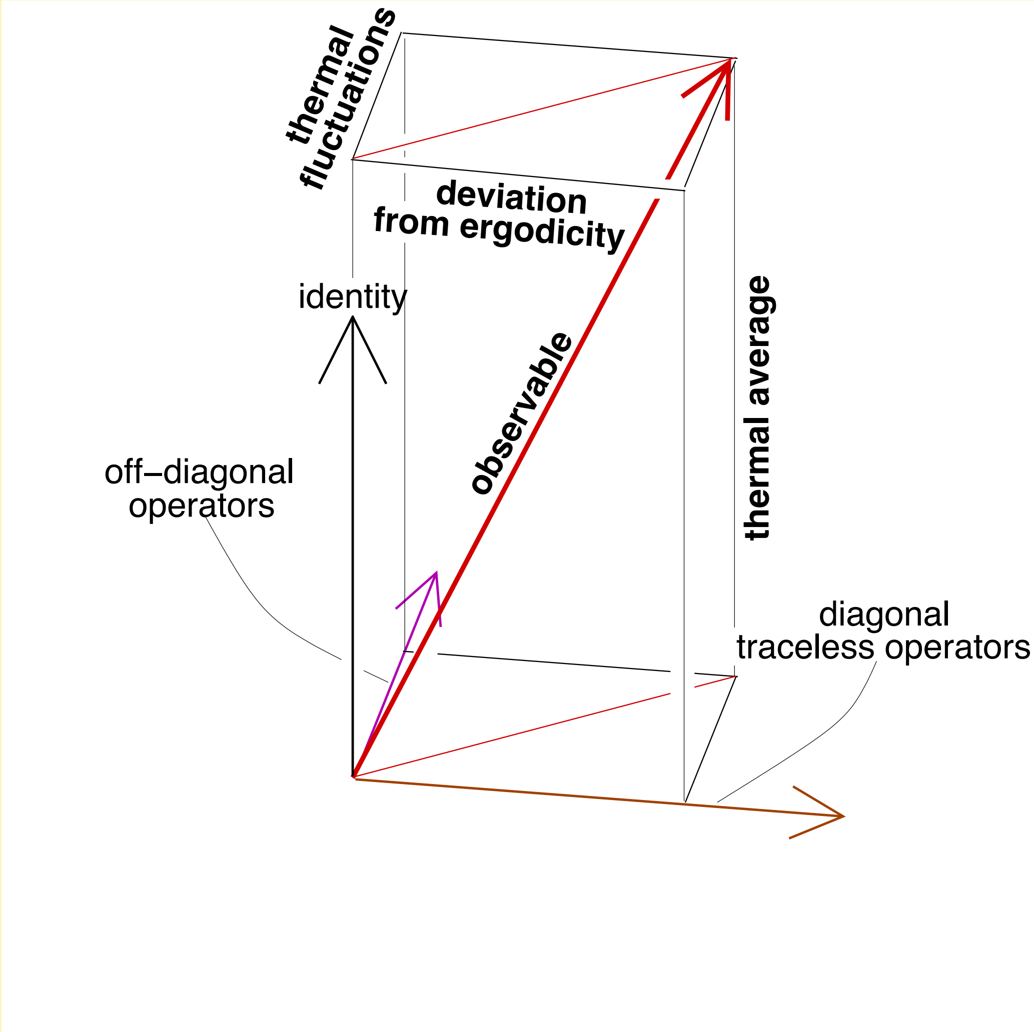
ensemble variance $\equiv$

$$\langle\hat{A}^2\rangle_{MC} - \langle\hat{A}\rangle_{MC}^2 = N_{MC}^{-1}\left(||\hat{A}^2|| - N_{MC}^{-1}||\hat{A}||^2\right)$$

inverse participation ratio (ITR) between the eigenstates of an integrable ($\hat{H}_0$)

$$N_{MC}^{-1}\sum_{\alpha,\,\alpha_o}^{\mathcal{W}_{MC}} |\langle\alpha_0|\alpha\rangle|^4 = \cos^2(\mathcal{L}_{d,\,\hat{H}_0}\widehat{\ }\mathcal{L}_{d,\,\hat{H}})$$

HS Geometry of the Integrability-Thermalizability transition



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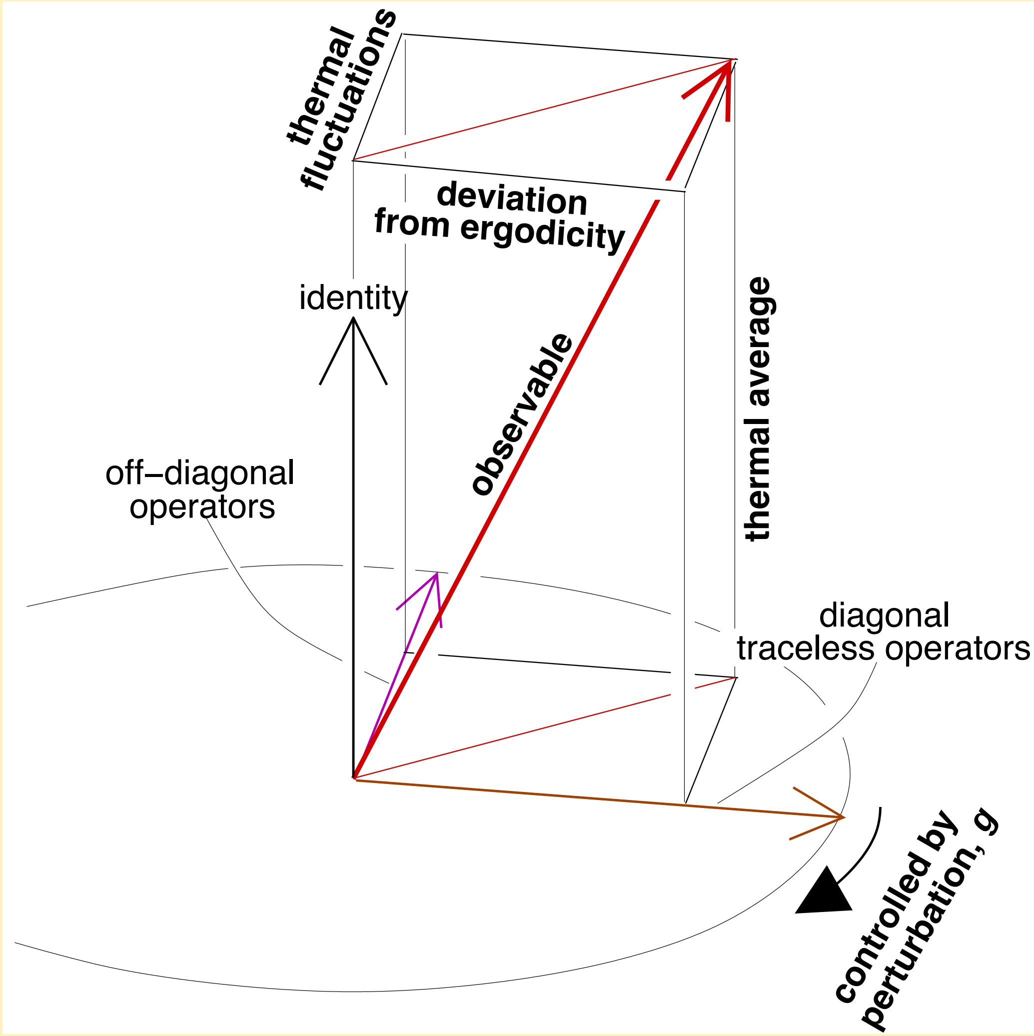

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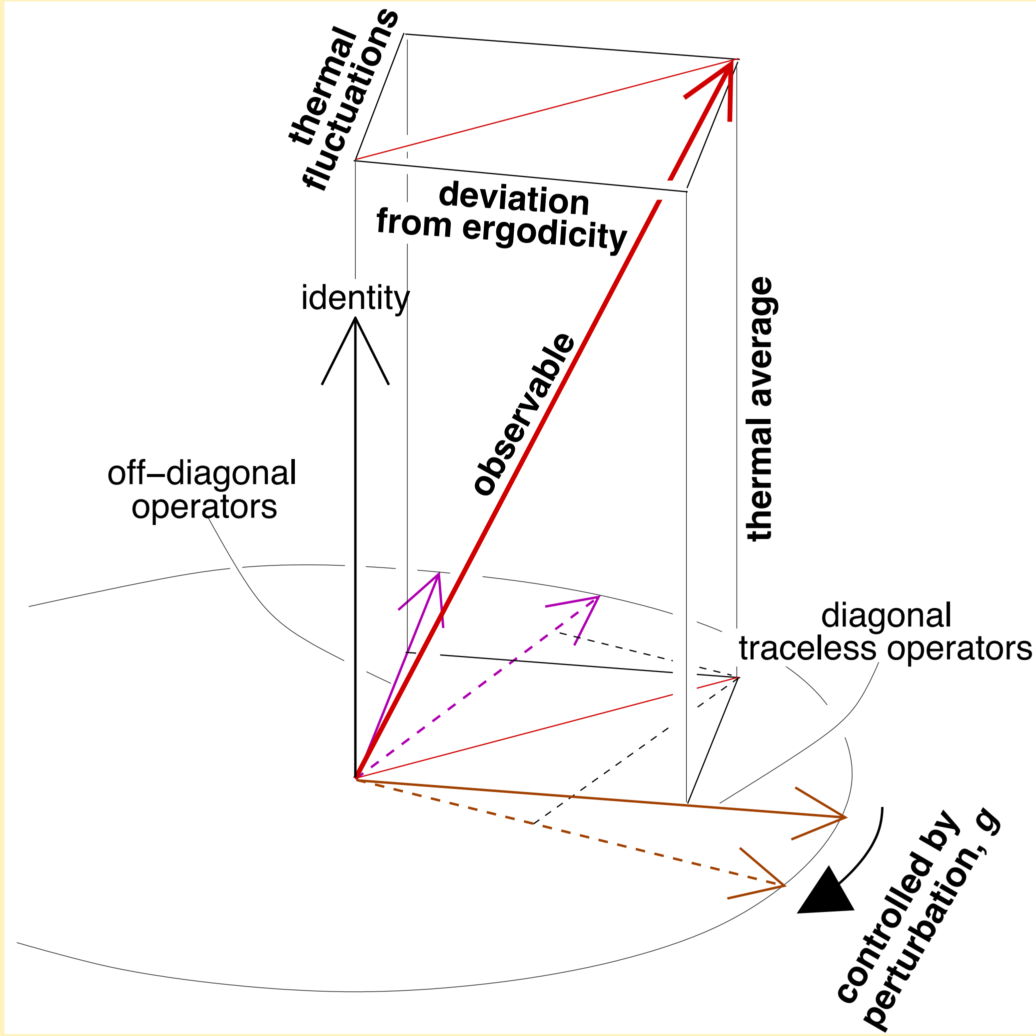

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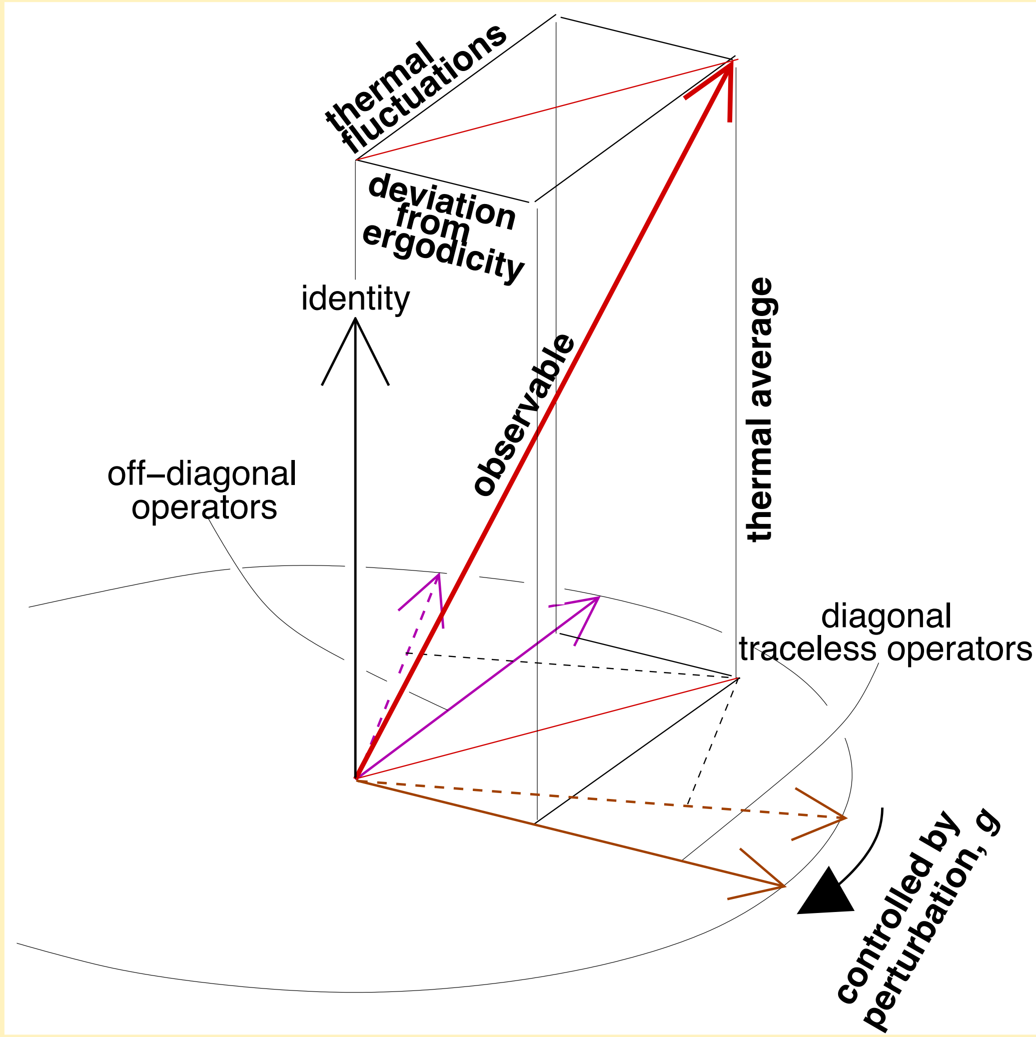
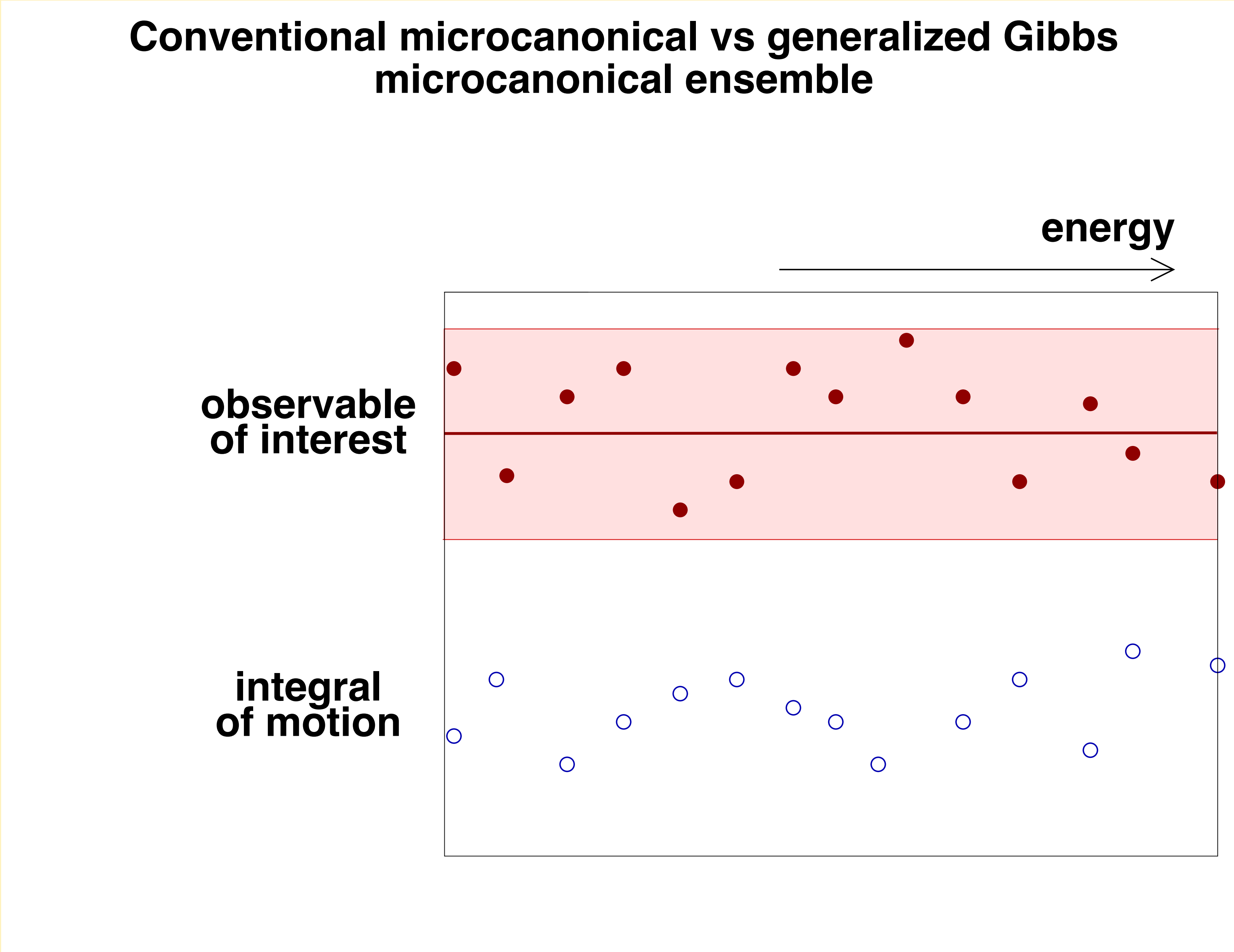
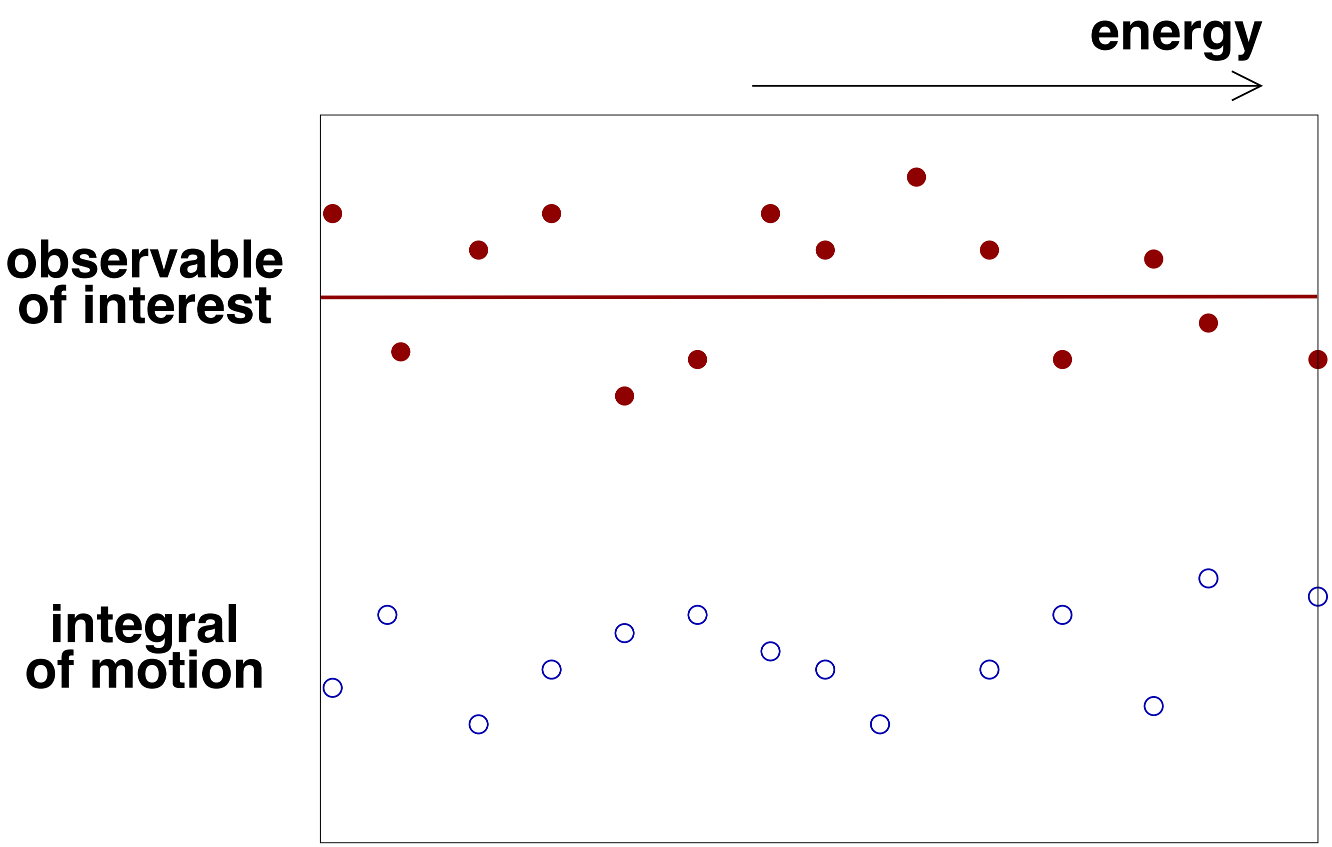


Optimizing the GGE

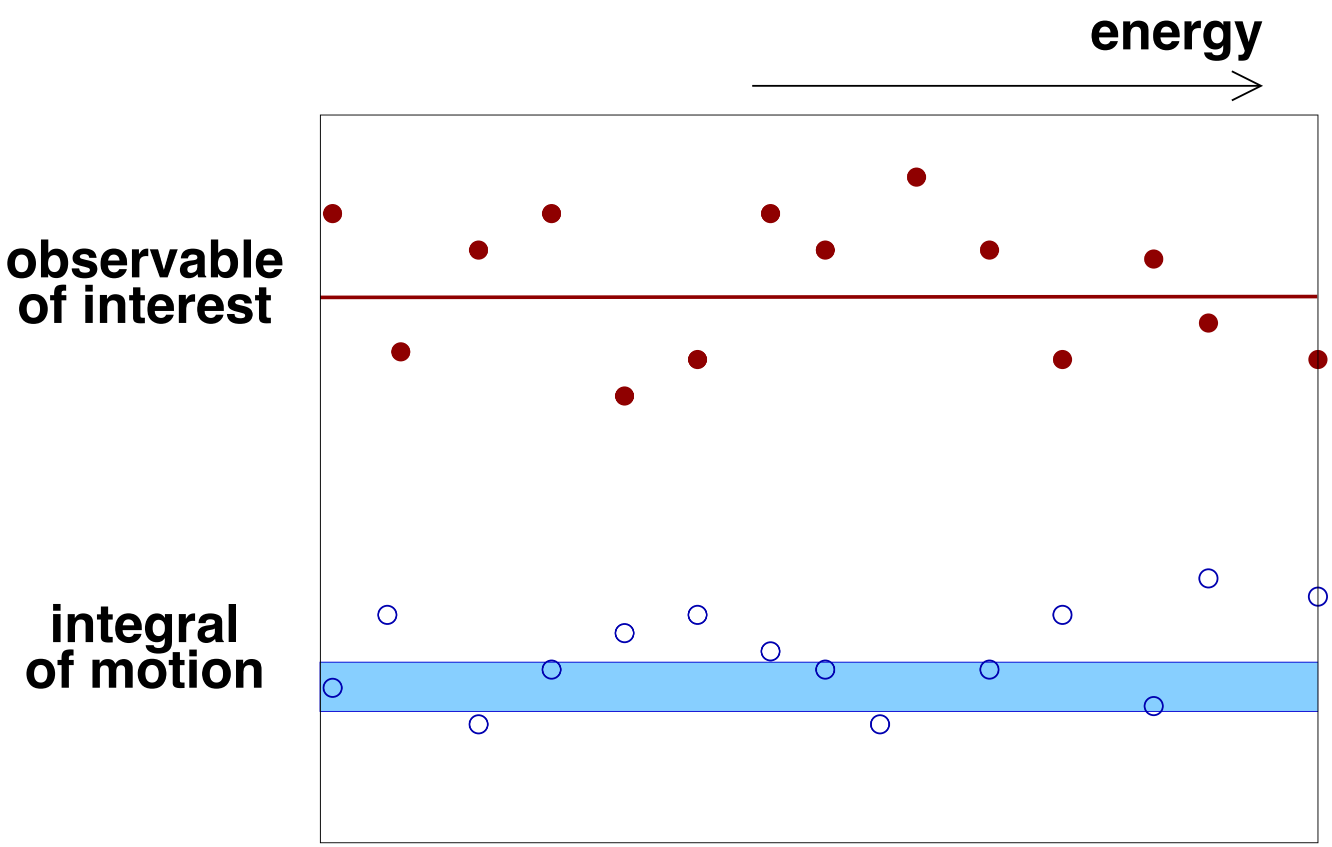
Directly inspired by an informal discussion session at Abdus-Salam, in May 2010, mediated by Alessandro Silva


Conventional microcanonical vs generalized Gibbs
microcanonical ensemble
energy
observable
of interest
integral
of motion

Conventional microcanonical vs generalized Gibbs microcanonical ensemble



Conventional microcanonical vs generalized Gibbs microcanonical ensemble



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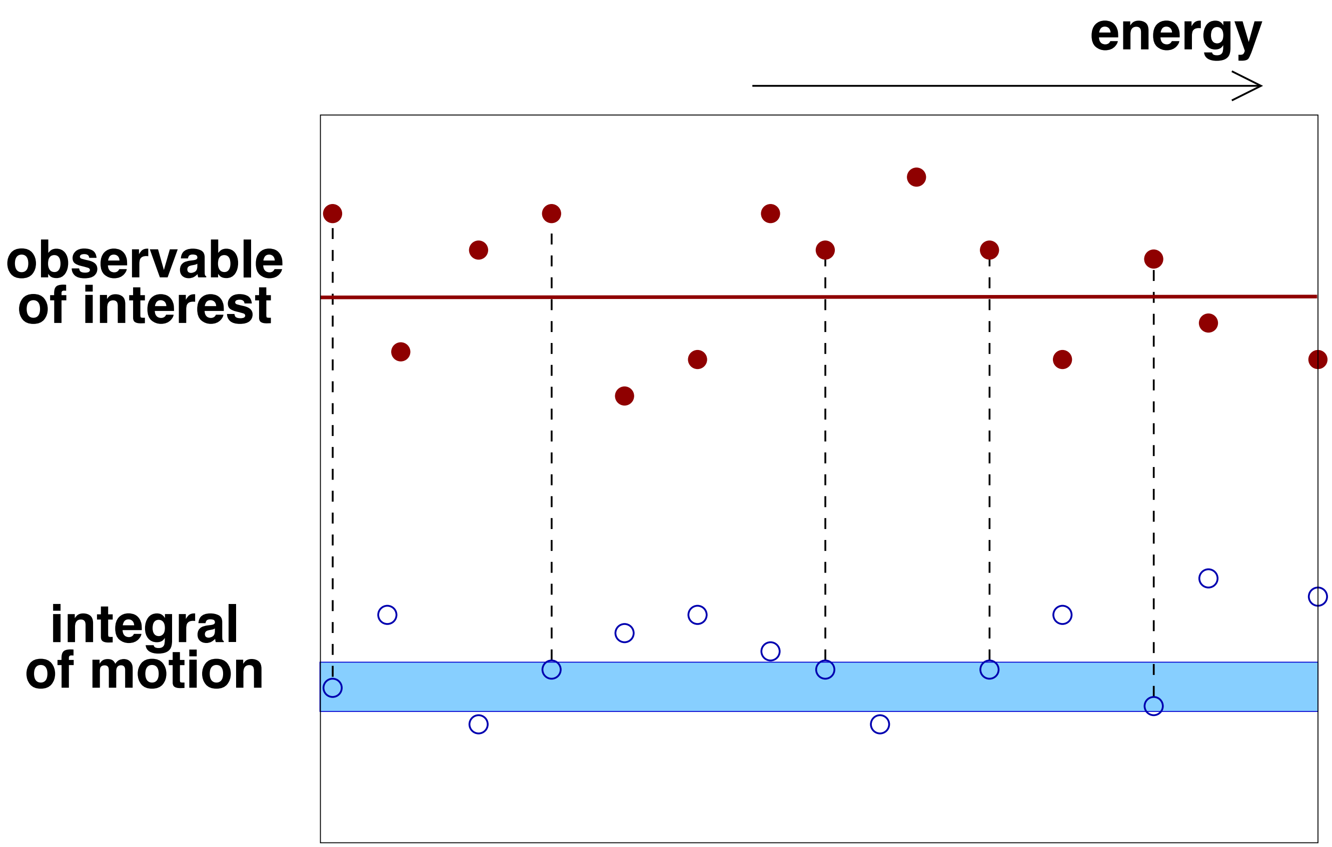

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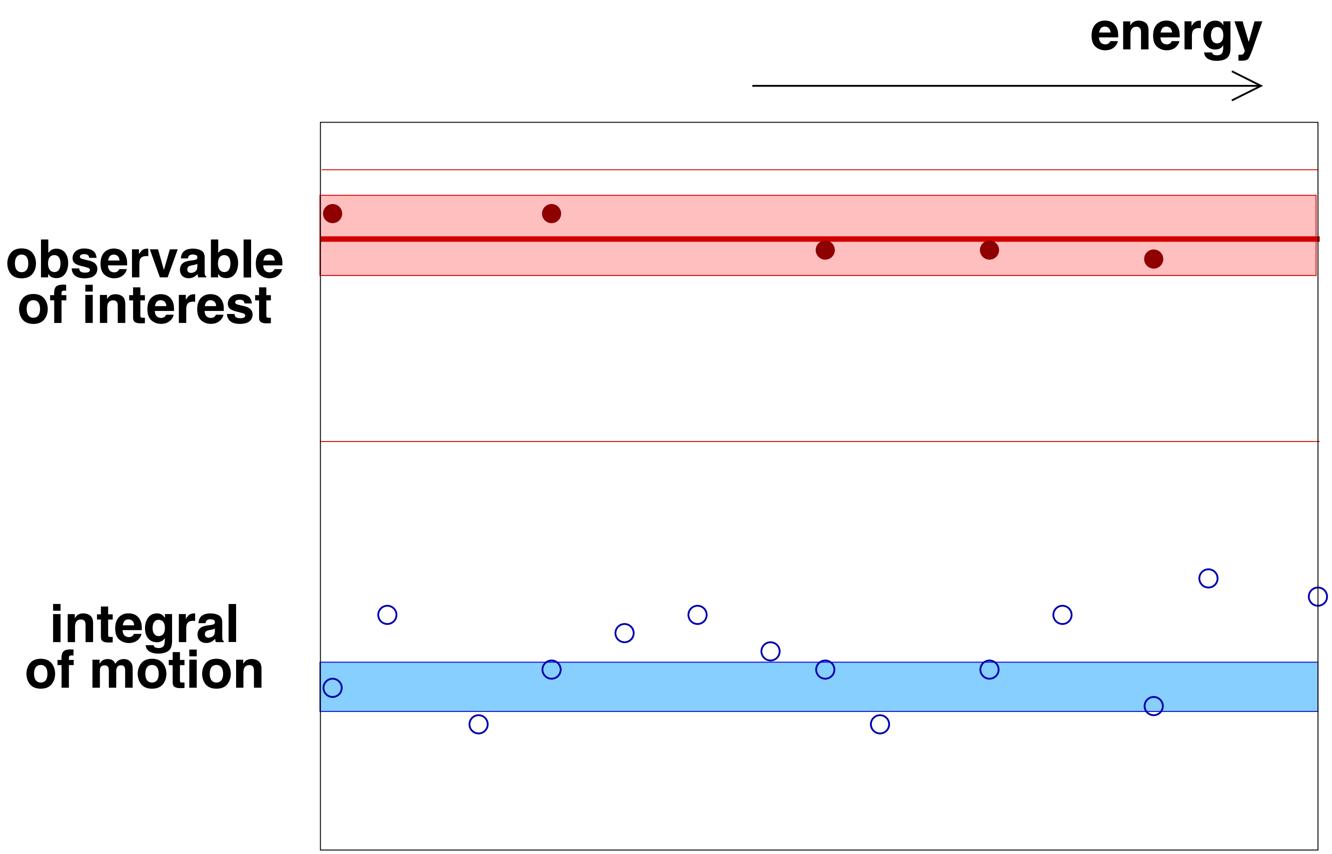
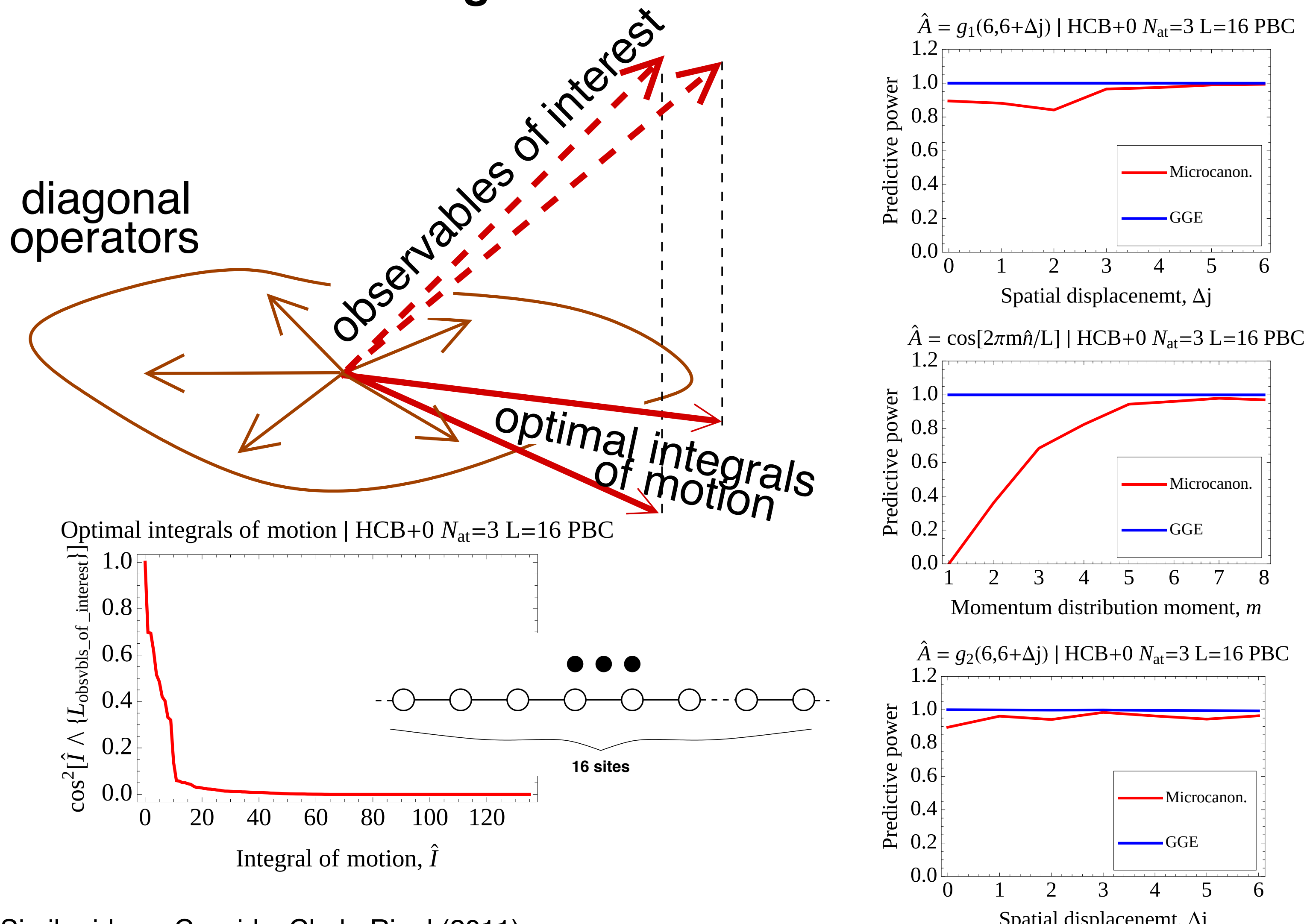


Optimizing the GGE: underlying exact inequality

$$\frac{Var_{\mathsf{GGE}}[\langle\alpha|\hat{A}|\alpha\rangle]}{Var_{\mathsf{MC}}[\langle\alpha|\hat{A}|\alpha\rangle]} \leq \sin^2[\hat{I}_{tl,d} \wedge \hat{A}_{tl,d}] + 2|\cos[\hat{I}_{tl,d} \wedge \hat{A}_{tl,d}]| \underbrace{\sqrt{\frac{Var_{\mathsf{GGE}}[\langle\alpha|\hat{I}|\alpha\rangle]}{Var_{\mathsf{MC}}[\langle\alpha|\hat{I}|\alpha\rangle]}}}_{\mathcal{O}\left(\frac{\Delta I}{\sqrt{Var_{\mathsf{MC}}[\langle\alpha|\hat{I}|\alpha\rangle]}}\right)}$$

where $\Delta I \equiv \max_j (I_{j+1} - I_j) =$ maximal GGE interval for $\hat{I}$,
$\hat{B}_{tl,d} \equiv \sum_\alpha (\langle\alpha|\hat{B}|\alpha\rangle - Mean_{\mathsf{MC}}[\langle\alpha|\hat{B}|\alpha\rangle])|\alpha\rangle\langle\alpha|$

How to choose the optimal set of integrals of motion for GGE



Similar ideas: Cassidy–Clark–Rigol (2011)

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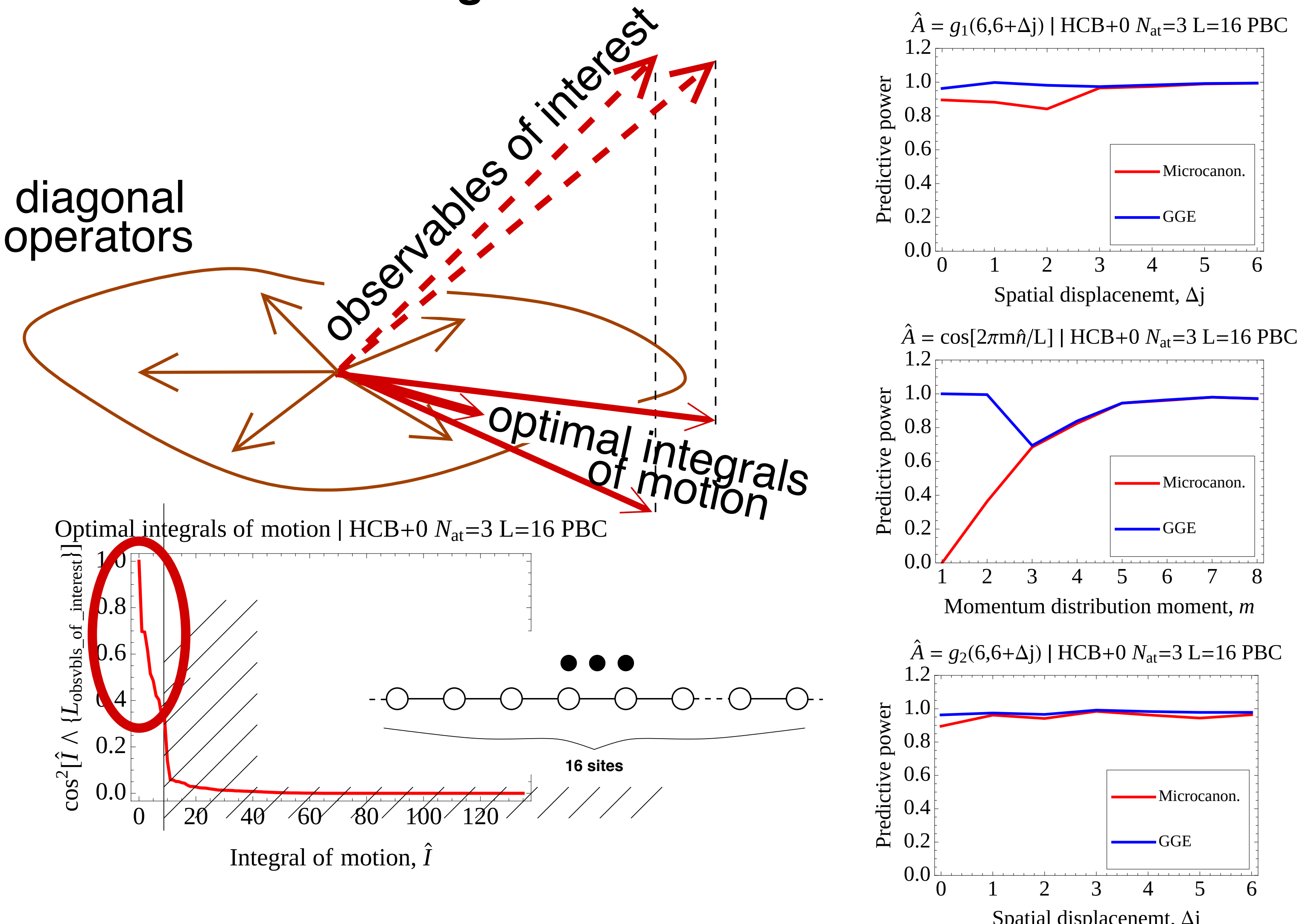

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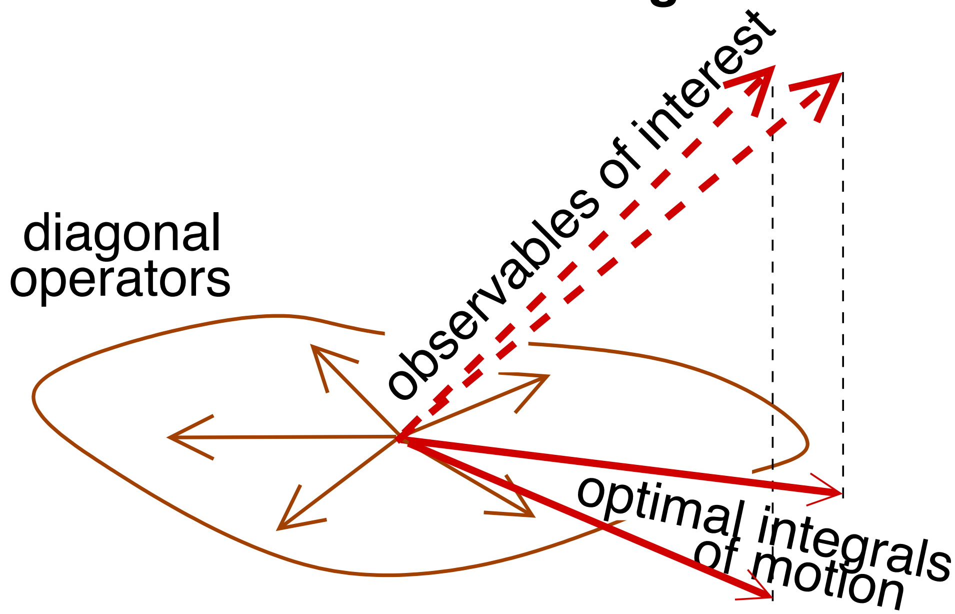
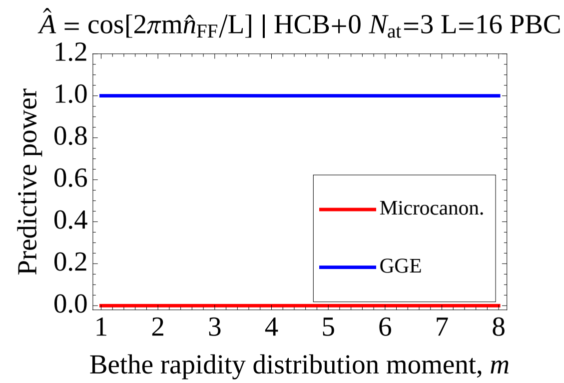
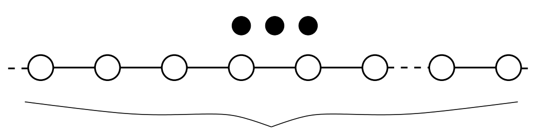
Optimizing the GGE: hard-core bosons

How to choose the optimal set of integrals of motion







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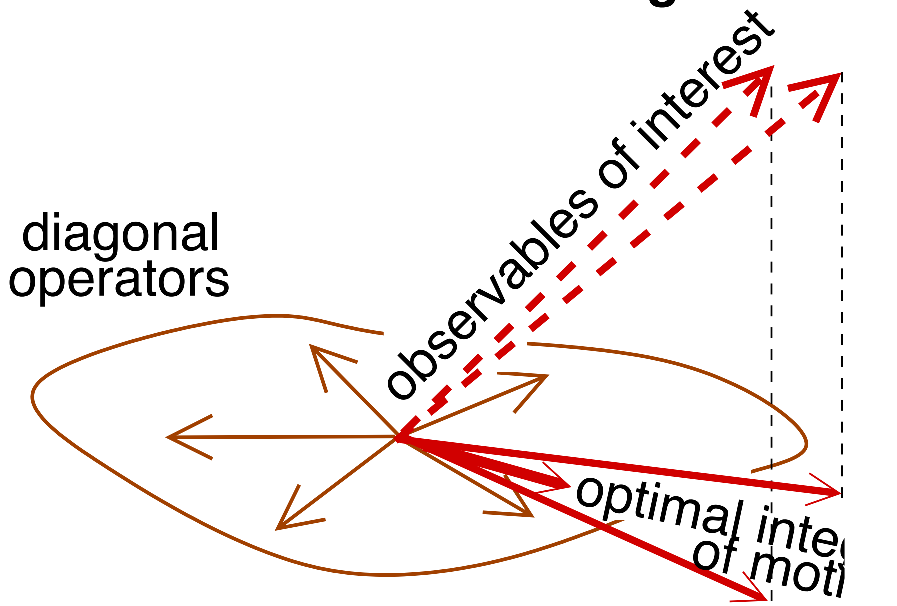
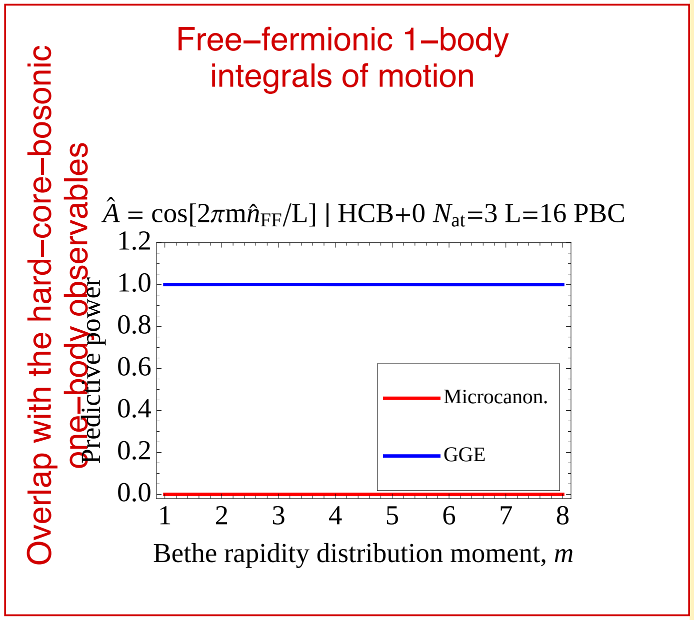
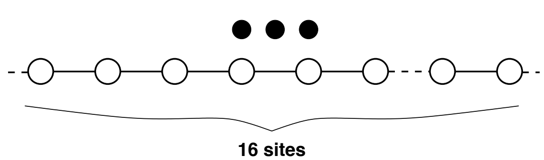


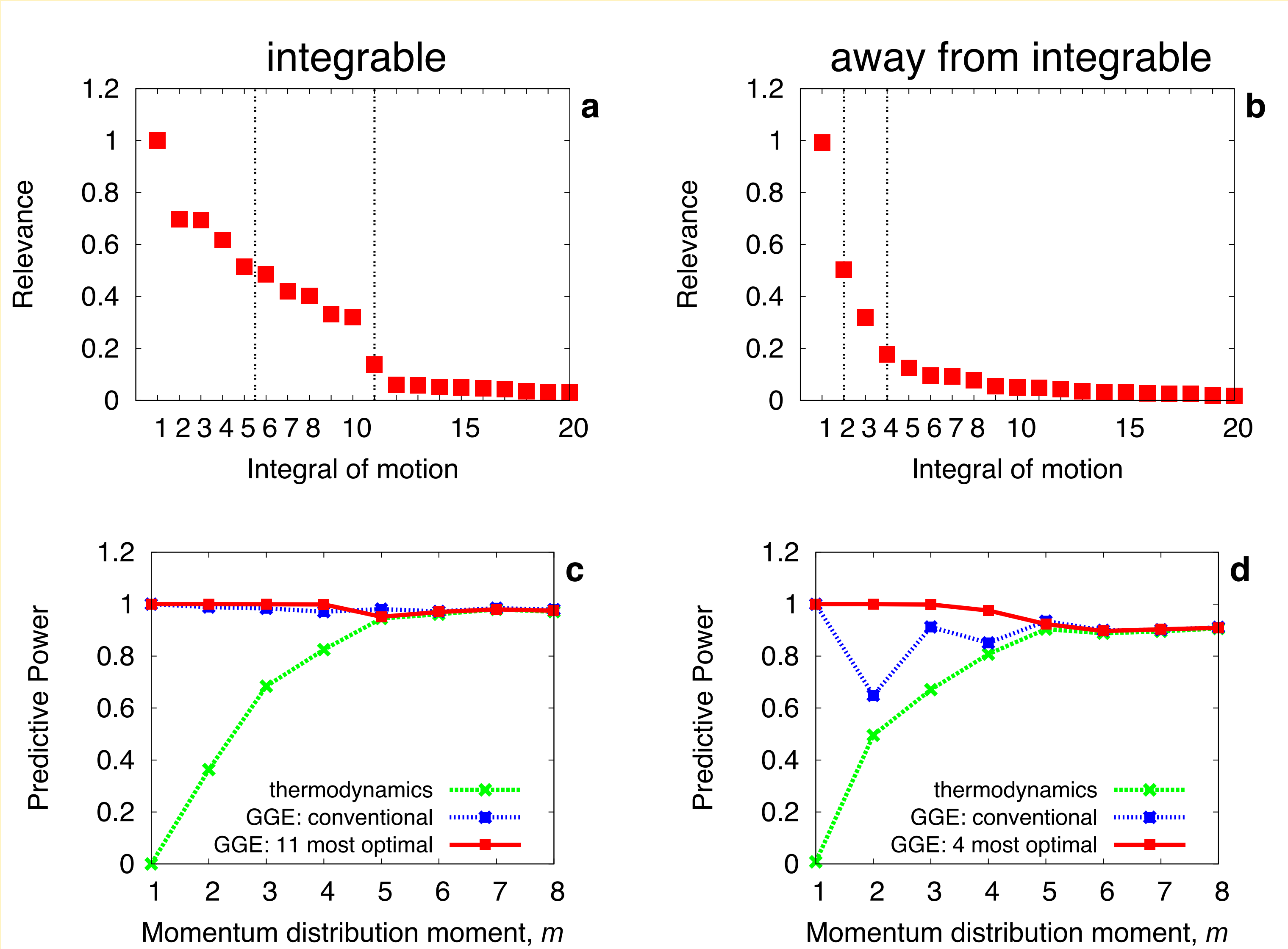




Take home:

in free–fermionic integrable systems, the Hilbert–Schmidt angle between the free–fermionic and "system proper" one–body observables is close to zero

Optimizing the GGE: beyond integrability



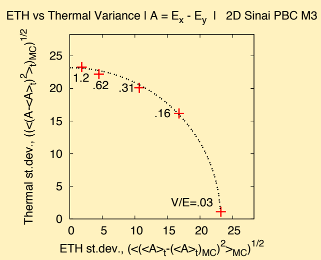
Concusions

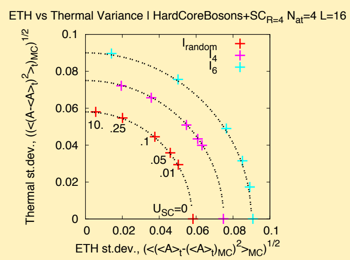
Summary

In this presentation we

- suggested the following proposition:

  The sum of the ensemble variance of the temporal means and the ensemble mean of the temporal variances remains approximately constant across the integrability-to-ergodicity transition

- 



- linked the proposition to the Hilbert-Schmidt (HS) geometry of the observables: ETH variance $= cos^2$(HS angle between the observable and integrals of motion); IPR $= cos^2$(HS angle between the original and perturbed integrals of motion);
- found a way to identify the optimal integrals of motion for GGE;
- found a way to treat the integrability and mesoscopicity under the same umbrella, with possible applications in nano-systems

Support by

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