Semi-classical theory for non-equilibrium quantum relaxation in integrable spin chains

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Quantum dynamics in far from equilibrium thermally isolated systems
KITP, UCSB 2012

Overview

- Fast quenches in **finite** transverse Ising & XY chains
- Order parameter profile
- Relaxation / plateau / reconstruction
- Semi classical theory
- Quantitative predictions
- Entanglement entropy
- Local quenches

With B. Blass[1], U. Divakaran[1], F. Iglói[2]

[1] *Theoretical Physics, Saarland University*
[2] *KFKI Budapest*

Model:

Transverse Ising chain (TIC):

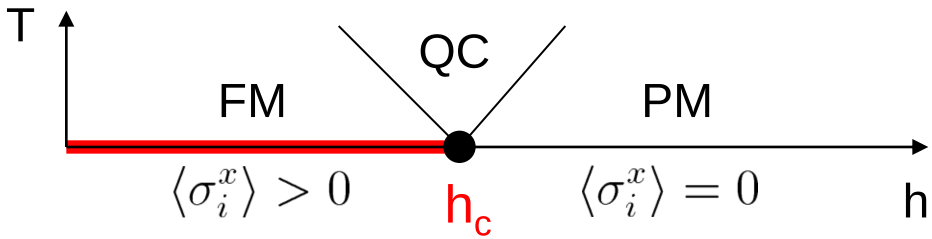
$$H = -J \sum_{i=1}^{L-1} \sigma_i^x \sigma_{i+1}^x - h \sum_{i=1}^{L} \sigma_i^z$$

Generalization: XY-chain in transverse field

$$H = -J \sum_{i=1}^{L-1} \left(\frac{1+\gamma}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-\gamma}{2} \sigma_i^y \sigma_{i+1}^y \right) - h \sum_{i=1}^{L} \sigma_i^z$$

Integrable – free fermion system after Jordan-Wigner trafo

TIC: Quantum phase transition (T=0) at h_c=J:



ξ Correlation length
τ Relaxation time
both finite at T>0.

Equilibrium relaxation: Barouch, McCoy 1970

Non-equilibrium relaxation at T=0: Iglói, HR (2000)

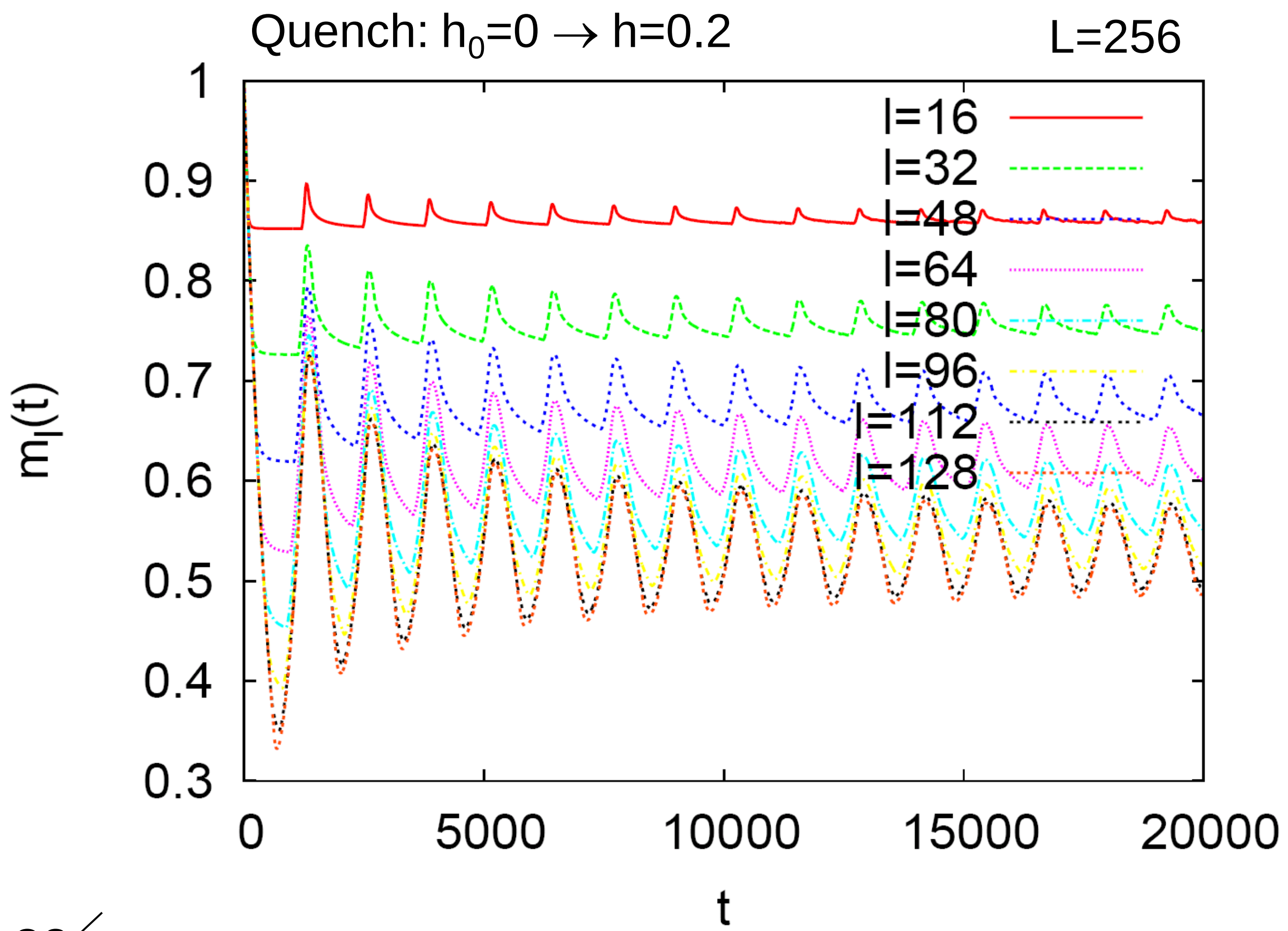
Long range correlations also for $h \neq h_c$

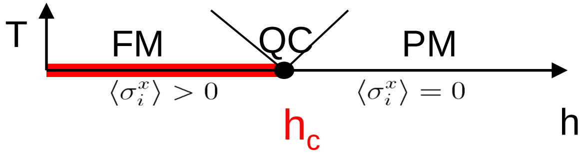
Fast quenches $h_0 \rightarrow h$: Rossini etal. (2010), Iglói, HR (2011).

Effective temperature for T=0 non-equilibrium relaxation of σ^x correlations

Local order parameter (magnetization)

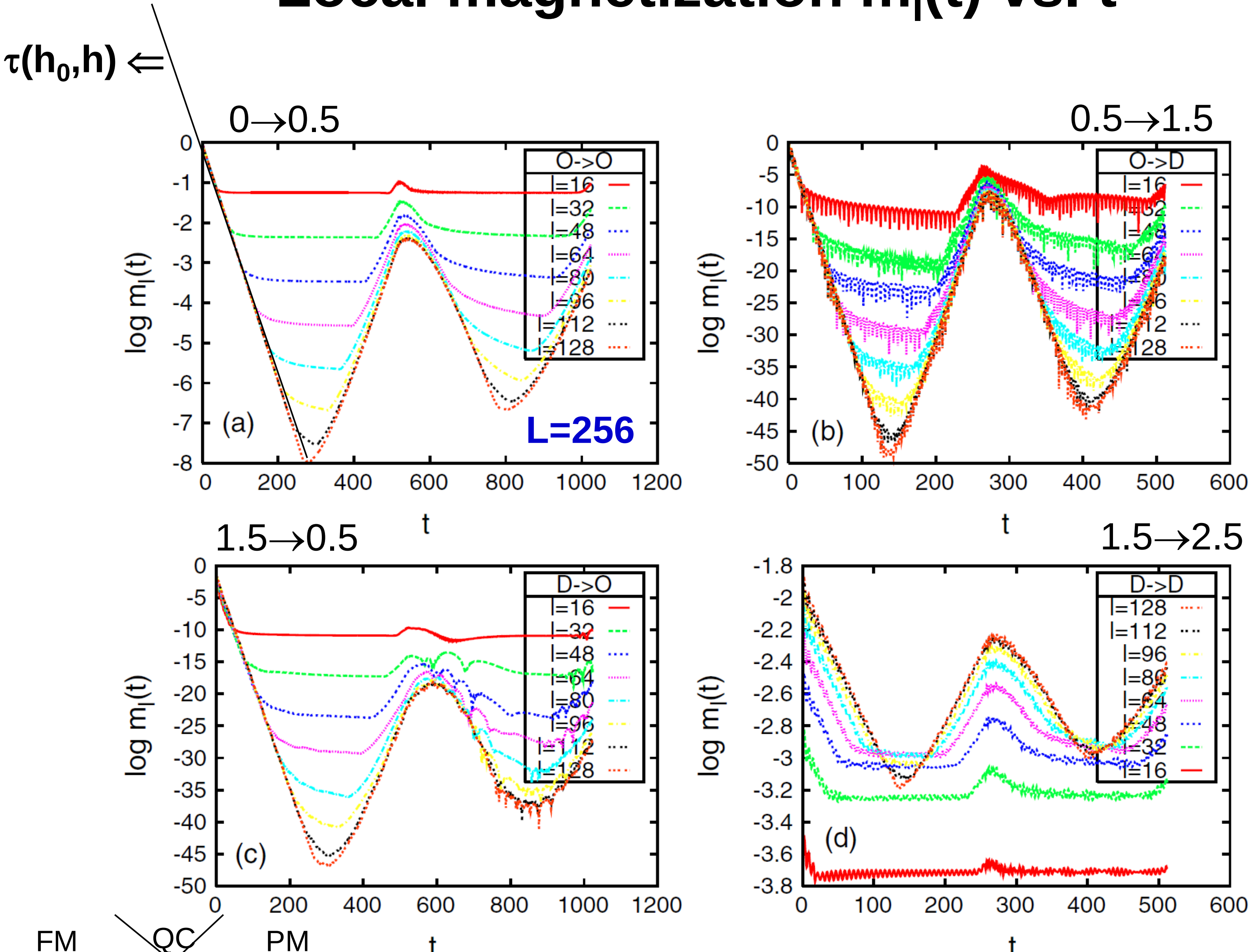
$$m_l(t) = \langle \psi_0 | \sigma_l^x(t) | \psi_1 \rangle = \lim_{b \to 0} \langle \psi_0^{(b)} | \sigma_l^x(t) | \psi_0^{(b)} \rangle$$

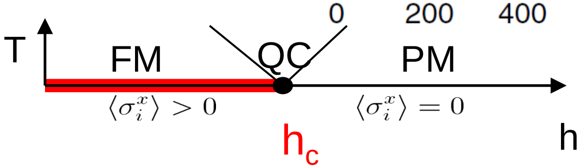




Exact free fermion calculation

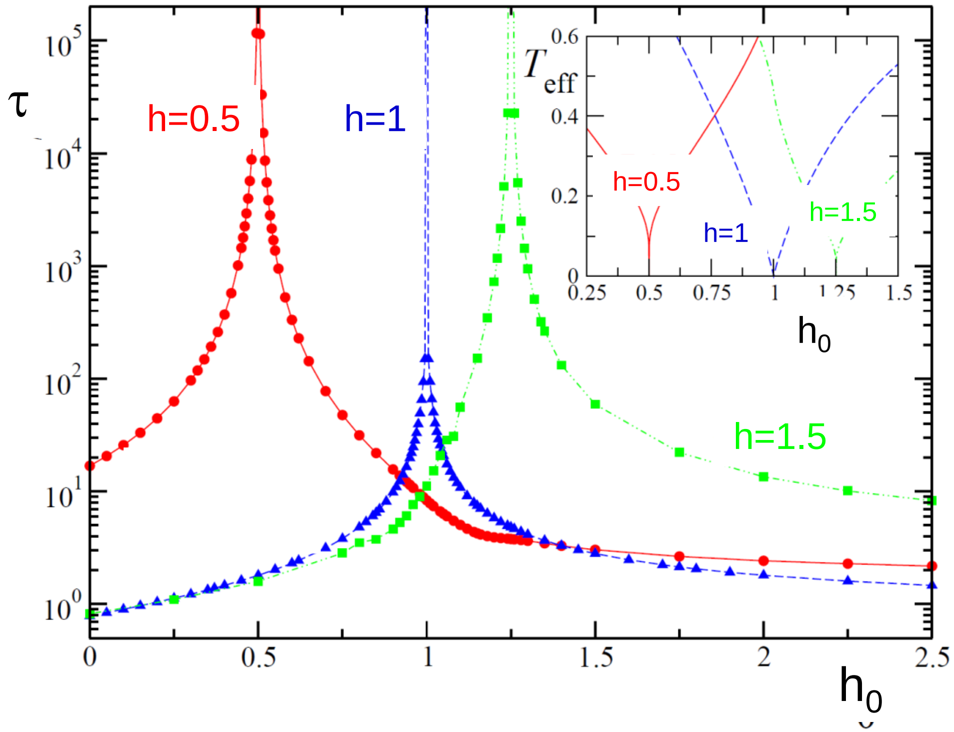
Local magnetization $m_l(t)$ vs. t





Exact free fermion calculation
[Iglói, HR, PRL 106, 035701 (2011)]

Relaxation time τ, comparison with thermal relaxation



$h \to h_0$: $\tau \sim |h-h_0|^{-2}$,
$h \ll 1$: $\tau \sim |h|^{-1}$
$h_0=0$: $\tau \sim h^{-3}$

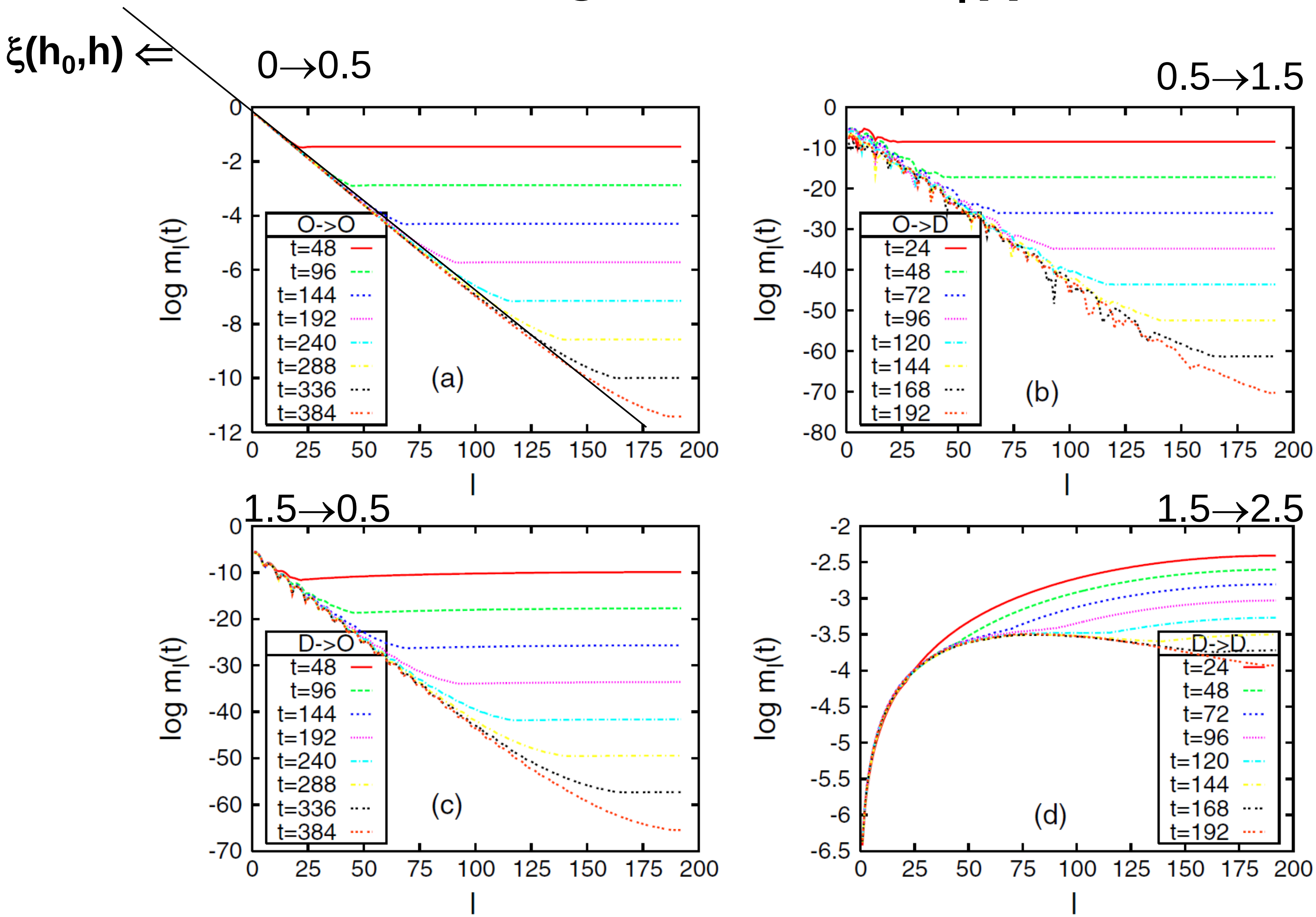
c.f. Rossini et al. 2010 from

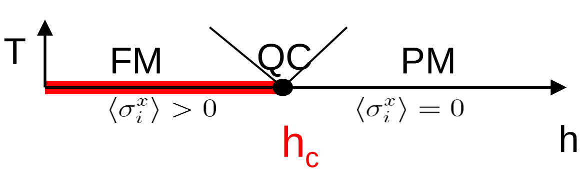
$$G(t) = \lim_{t' \to \infty} \langle \psi_0 | \sigma_i^x(t') \sigma_i^x(t'+t) | \psi_0 \rangle$$

$$C(r) = \lim_{t' \to \infty} \langle \psi_0 | \sigma_i^x(t') \sigma_{i+r}^x(t') | \psi_0 \rangle$$

with p.b.c.

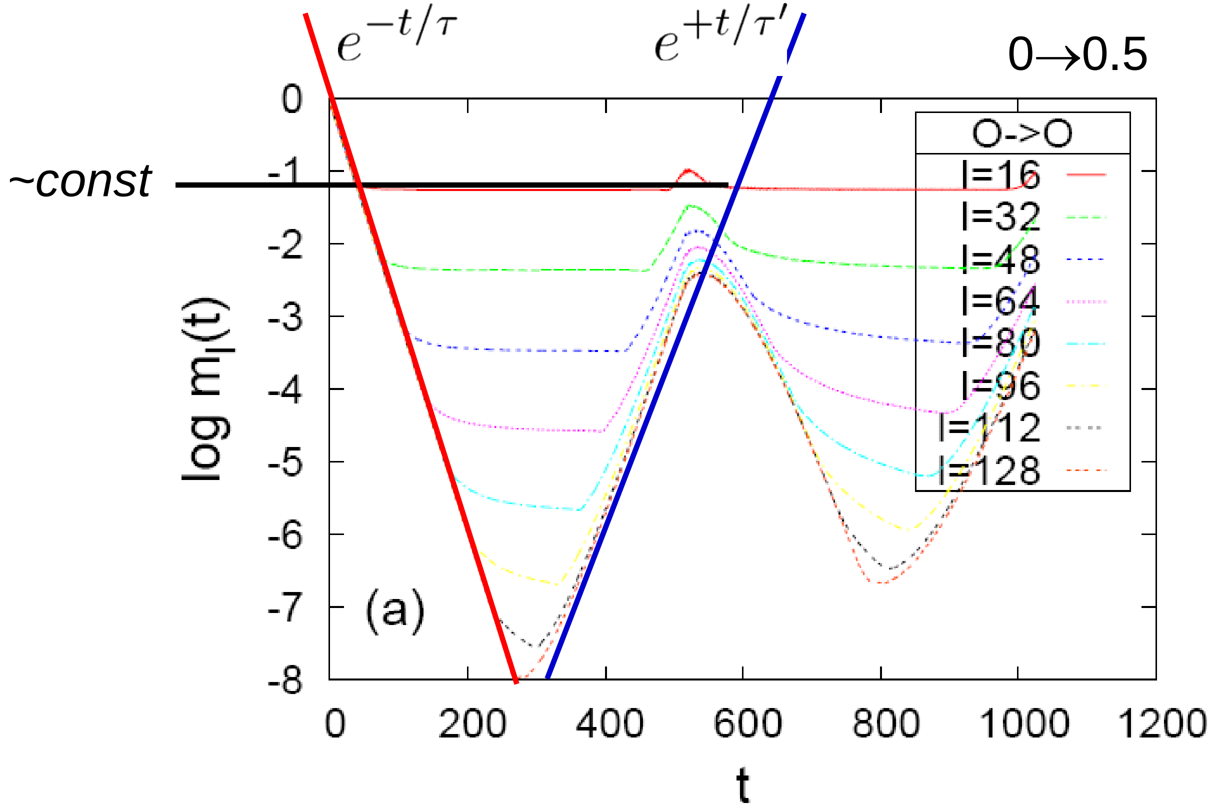
Local magnetization $m_l(t)$ vs. l





[Iglói, HR, PRL 106, 035701 (2011)]

What‘s going on?



- Exponential relaxation
- Quasi-stationary regime
- Exponential recovery

Quasiparticles in the FM phase ($h<<h_c$)

$h<<h_c$: Lowest energy excitations have one (!) kink:

$$|r\rangle = |+++\ldots+--\!-\ldots-\rangle$$

↑ r

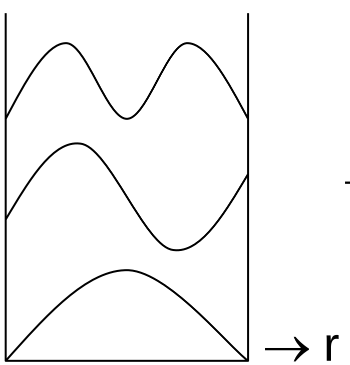
$\sigma_r^z|r\rangle = |r+1\rangle$, $\sigma_r^z|r-1\rangle = |r-2\rangle$

transverse field operator → kink translation

One-kink eigenstates of H: $$|k\rangle = \sum_{r=1}^{L-1} \sin\left(\frac{rk\pi}{L}\right)|r\rangle$$ k=1,…,L

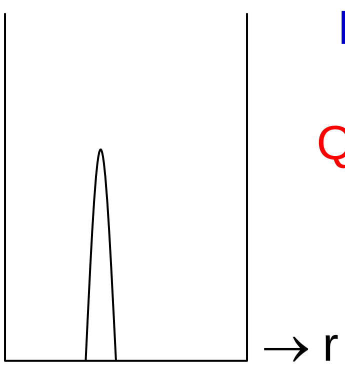
with energy: $$\epsilon_k = \sqrt{J^2 + h^2 - 2Jh\cos(\pi k/L)}$$

Eigenstates: (like free particle in a box)



→

Localized Wave-packets = "kinks at r"



Moving kinks = Quasi-particles

Quasi-particles = kinks (in FM phase: h<J)

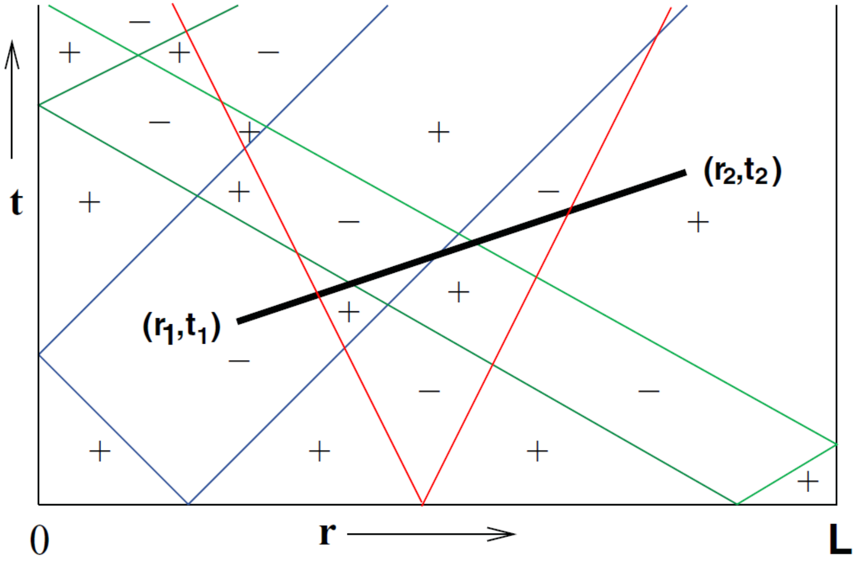
Kinks are created in pairs (+p,-p) move with velocity $\pm v_p$

$$\epsilon_p = \sqrt{J^2 + h^2 - 2Jh\cos(p)}$$

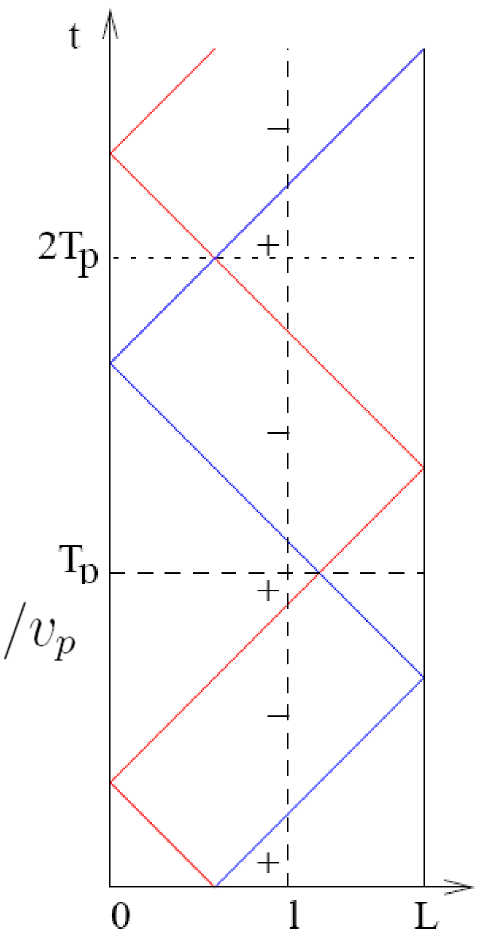
$$v_p = \frac{\partial \epsilon_p}{\partial p} = \frac{Jh\sin(p)}{\epsilon_p}$$

… and will flip spins upon arrival!

E.g. $C(r_1t_1;r_2t_2) = \langle\sigma_{r1}(t_1)\ \sigma_{r2}(t_2)\rangle$:



Finite system:



$$T_p = L/v_p$$

Reflection at the boundaries at i=0 and i=L!

QP passing probability ↔ flip probability

QP (kink) trajectories passing site l flip spin at site l, reduce or increase $m_l(t)$

Probability for passing odd # of times = q(t,l), then

$$m_l(t) = m_l^{\text{eq}} e^{-2q(t,l)L}$$

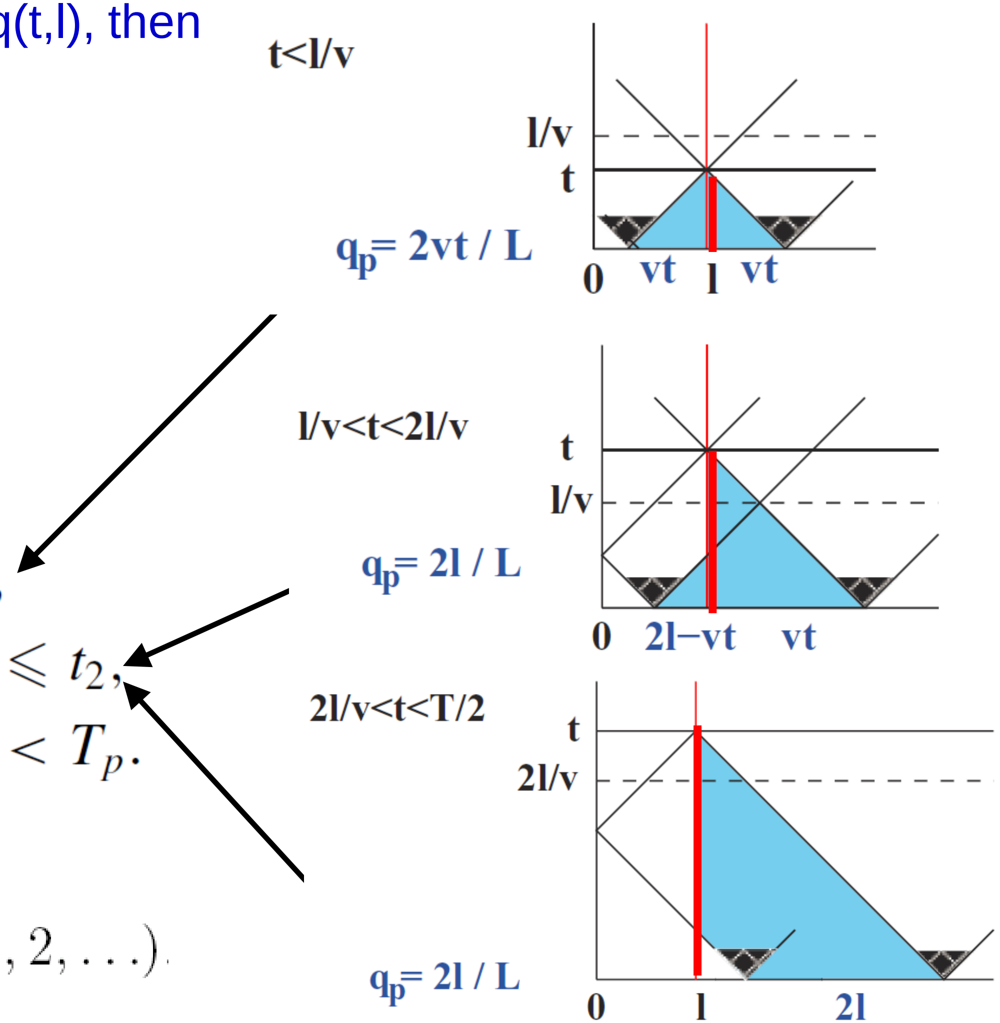
$$q(t,l) = \frac{1}{2\pi}\int_0^\pi dp\, f_p(h_0,h)\, q_p(t,l)$$

$$q_p(t,l) = \frac{1}{L}\int_0^L dx_0\, q_p(x_0,t,l)$$

$$q_p(t,l) = \begin{cases} 2v_p t/L & \text{for } t \leqslant t_1, \\ 2l/L & \text{for } t_1 \leqslant t \leqslant t_2, \\ 2 - 2v_p t/L & \text{for } t_2 \leqslant t < T_p. \end{cases}$$

$$T_p = L/v_p$$

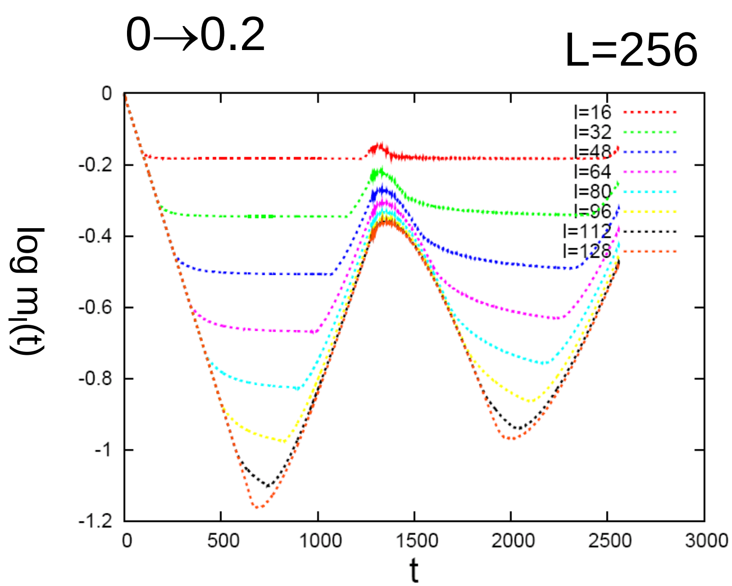
$$q_p(t + nT_p) = q_p(t), \quad (n = 1, 2, \ldots).$$



[HR, Igloi, Phys. Rev. B 84, 165117 (2011)]

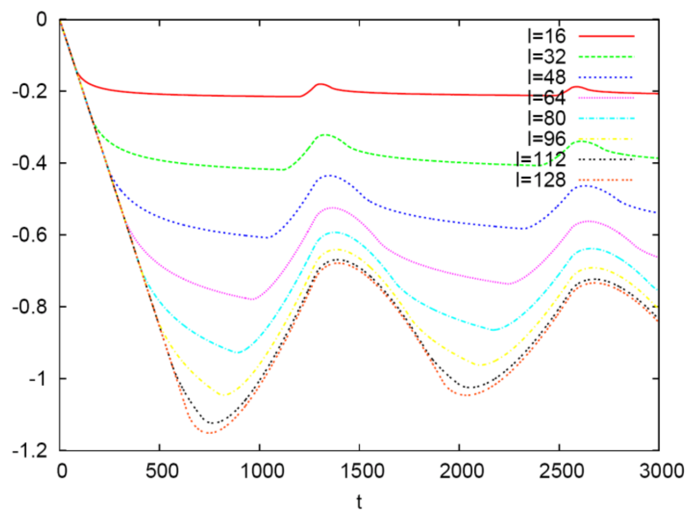
Comparison: exact – semi-classical

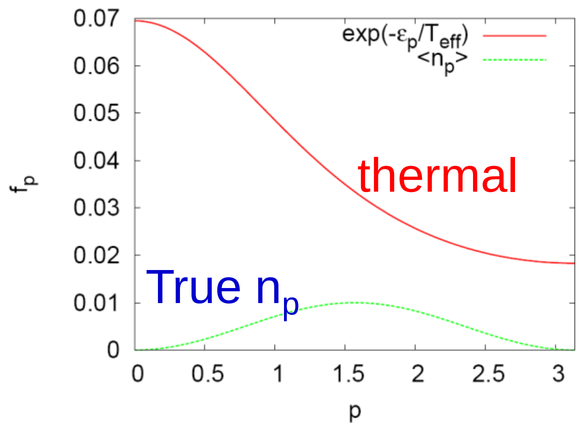
exact



Semi-classical with <u>thermal</u> occ.-prob.

$$f_p(h_0, h) = e^{-\epsilon_p / T_{\rm eff}(h_0,h)}$$





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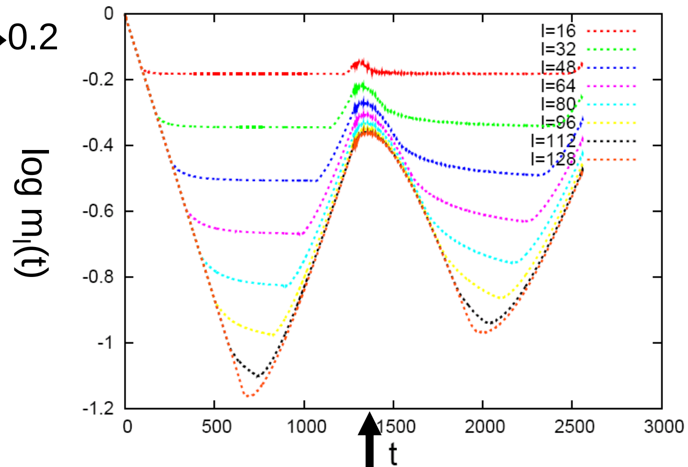
Initial relaxation:
Correct τ via definition of $T_{\rm eff}$

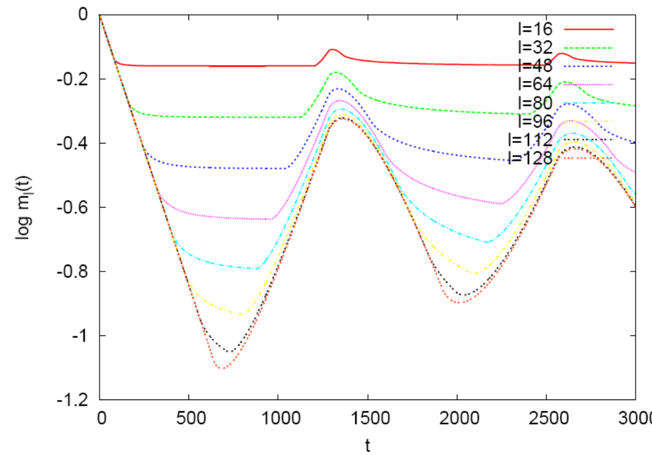
Problem:
QP trajectories too disperse –
True occupation prob. different!

Comparison: exact – semi-classical

"Correct" QP occupation probability: $f_p(h_0, h) = \langle \psi_0 | \eta_p^+ \eta_p | \psi_0 \rangle = \frac{1}{4}(h - h_0)^2 \sin^2(p)$

0→0.2

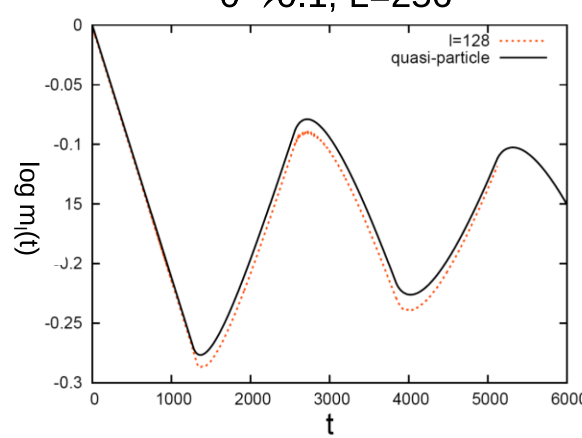


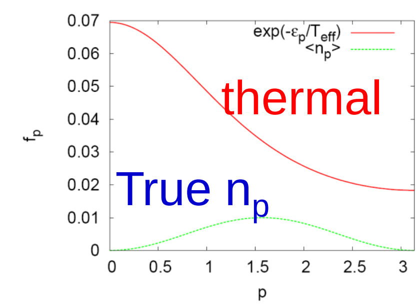


exact

semi classical

0→0.1, L=256



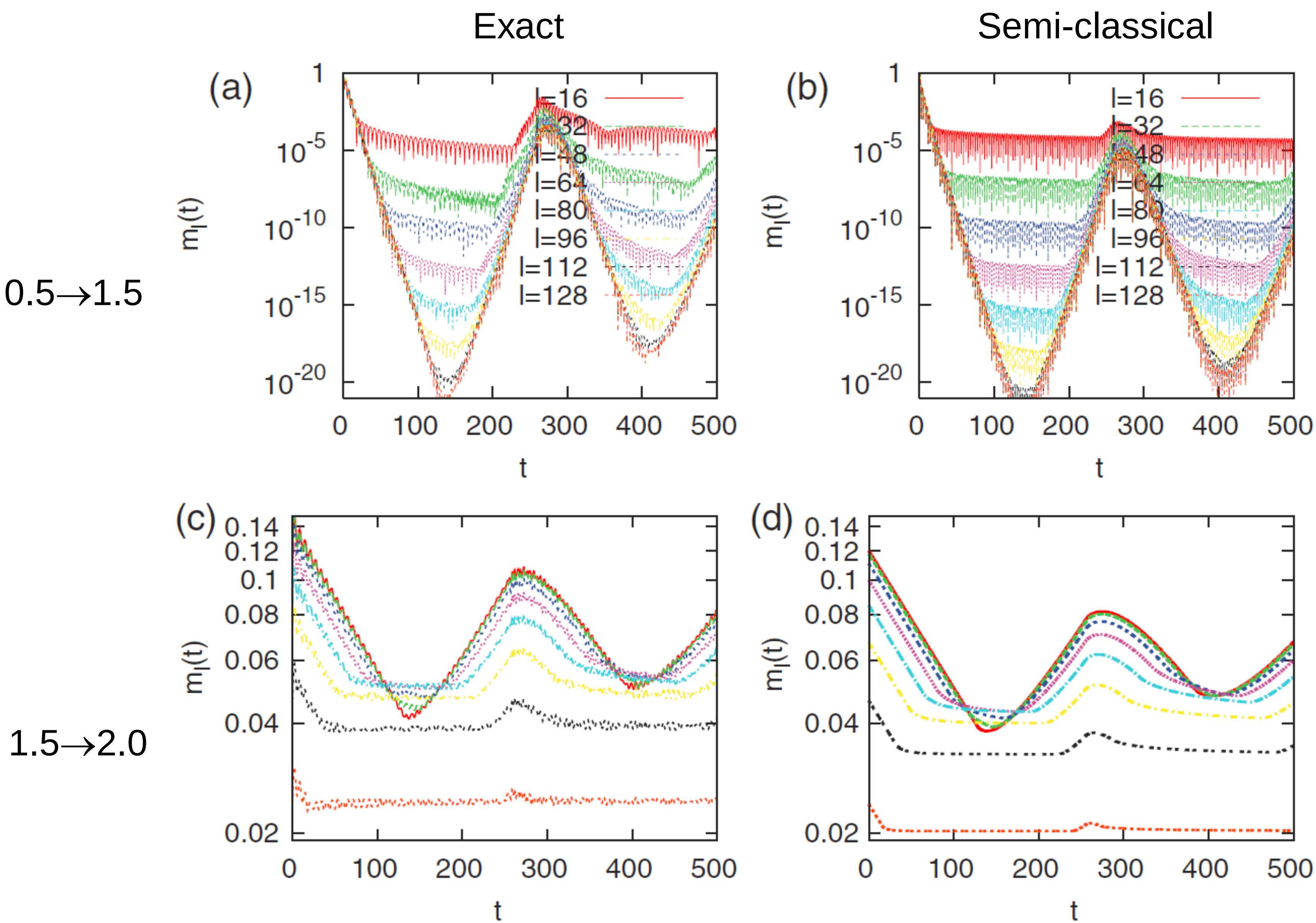


Period:

$T_{period} = L/v_{max} \sim L/h$

$$v_p = \frac{\partial \epsilon_p}{\partial p} = \frac{Jh \sin(p)}{\epsilon_p}$$

Quenches FM→PM / PM→PM



Semi-classical prediction for asymptotics (t, L →∞):

$$f_p(h_0,h) = \langle\psi_0|\eta_p^+\eta_p|\psi_0\rangle = \frac{1}{4}(h-h_0)^2\sin^2(p)$$

f_p time-independent → p-mode has ist own „effective temperature“: $T_{eff}(p)$

$$f_p = \frac{1}{1+\exp(\epsilon_p/T_{\rm eff}(p))}$$

No boundaries:

L→∞:

$$m_l(t) = m_l^{\rm eq}\exp\left(-t\frac{2}{\pi}\int_0^\pi dp\, v_p\, f_p(h_0,h)\,\theta(l-v_pt)\right)$$
$$\times\exp\left(-l\frac{2}{\pi}\int_0^\pi dp\, f_p(h_0,h)\,\theta(v_pt-l)\right),$$

Relaxation time / correlation length

$$\tau_{\rm mag}^{-1}(h_0,h) = \frac{2}{\pi}\int_0^\pi dp\, v_p\, f_p(h_0,h),$$

$$\xi_{\rm mag}^{-1}(h_0,h) = \frac{2}{\pi}\int_0^\pi dp\, f_p(h_0,h).$$

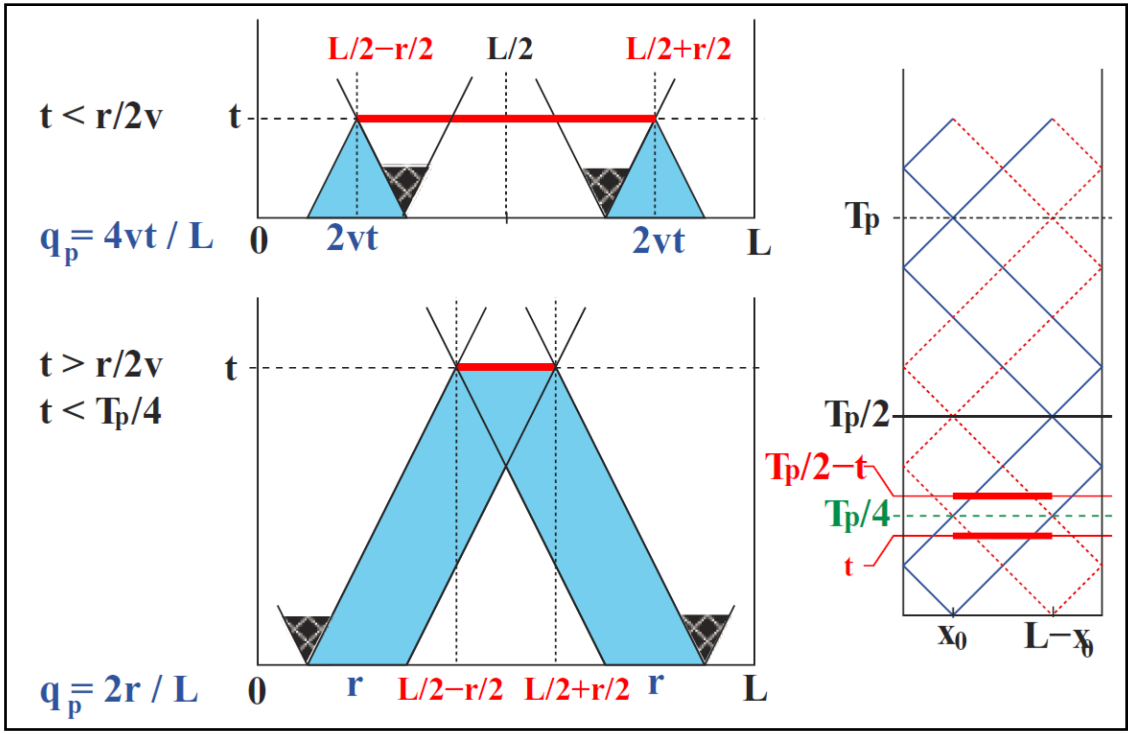
Result of semi-classical theory = exact asymptotics after $f_p \to$ ½ $\ln(1-2f_p)$!

[Calabrese, Essler, Fagotti (2011, 2012)]

Spatial corrleation function $C_t(r)$

$f_p(h_0,h)$ = occupation prob. of QP with momentum p
$q_p(t,r)$ = prob. that QP with momentum p intersects red line 1, 3, 5, … times

$$C_t(r) = C_{\rm eq}(r)\exp\left(-\frac{L}{2\pi}\int_0^\pi dp\, f_p(h_0,h)q_p^c(t,r)\right)$$



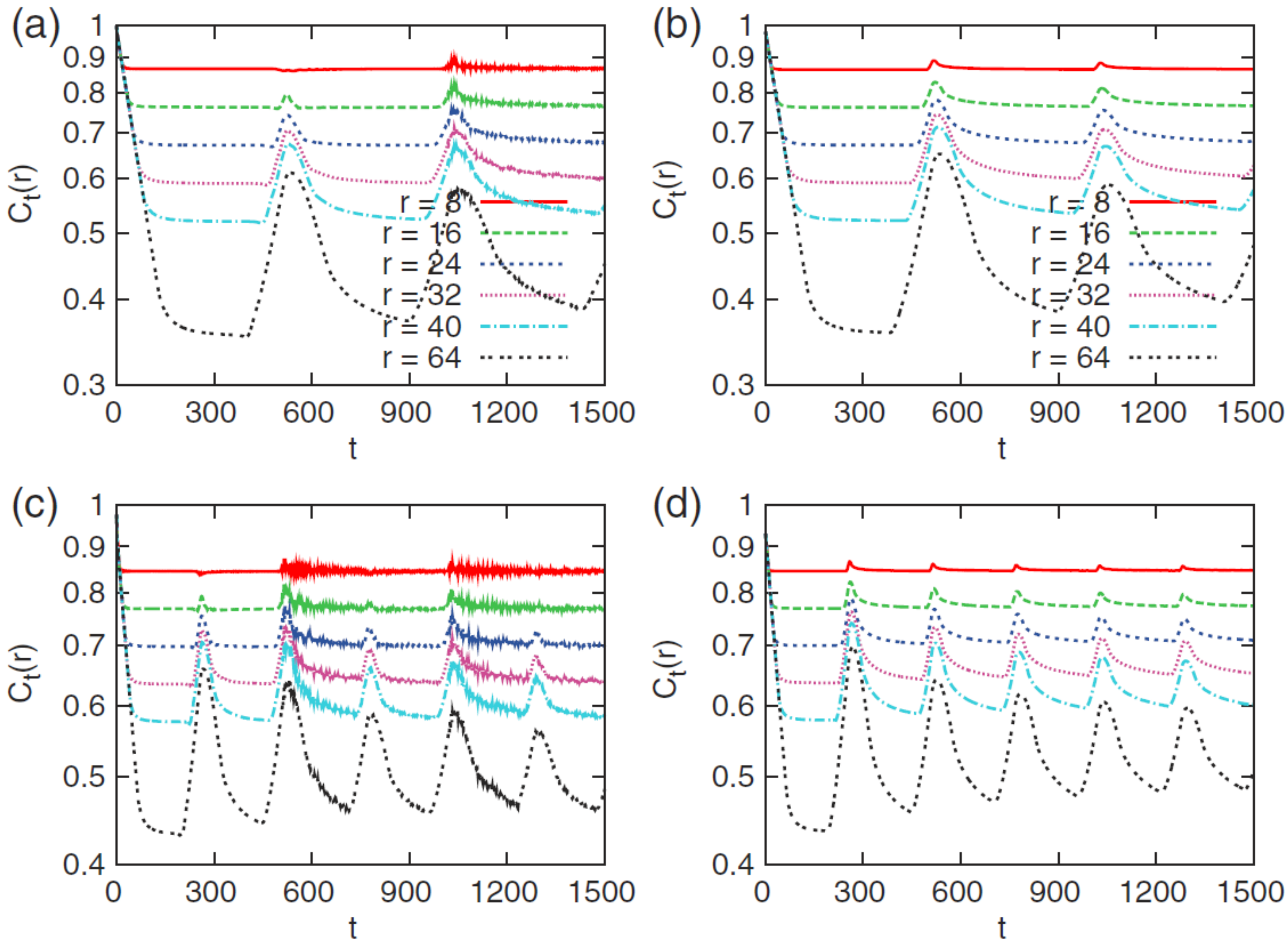
$$q_p^c(t,r) = \begin{cases} 4v_p t/L & \text{for } t \leqslant t_1, \\ 2r/L & \text{for } t_1 \leqslant t \leqslant t_2, \\ 2 - 4v_p t/L & \text{for } t_2 \leqslant t < T_p/2 \end{cases}$$

Spatial correlation function

Exact | Semi-classical

$0 \rightarrow 0.2$

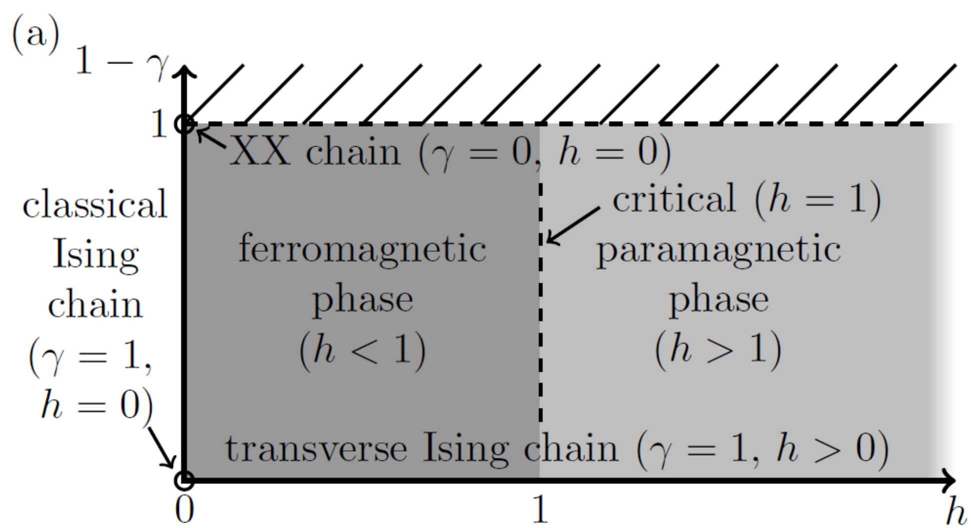
$0.3 \rightarrow 0.5$



$$f_p(h_0, h) = \langle \psi_0 | \eta_p^+ \eta_p | \psi_0 \rangle = \frac{1}{4}(h - h_0)^2 \sin^2(p)$$

XY-chain in a transverse field (TXY)

$$\mathcal{H} = -\frac{1}{2}\sum_l \left[\frac{1+\gamma}{2}\sigma_l^x\sigma_{l+1}^x + \frac{1-\gamma}{2}\sigma_l^y\sigma_{l+1}^y\right] - \frac{h}{2}\sum_l \sigma_l^z$$



$h \ll h_c$, $\gamma \approx 1$: Lowest energy excitations have one (!) kink:

$$|r\rangle = |+++\ldots+---\ldots-\rangle$$

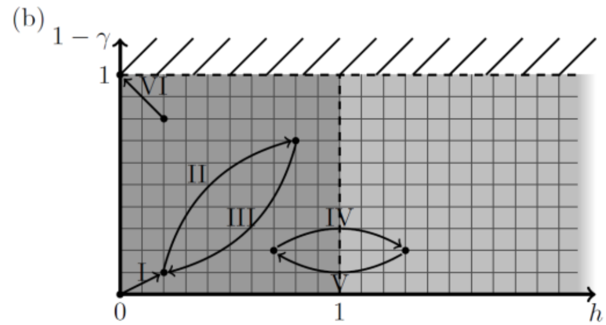
↑ r

$\sigma_r^y\sigma_{r+1}^y\,|r\rangle = |r+2\rangle$

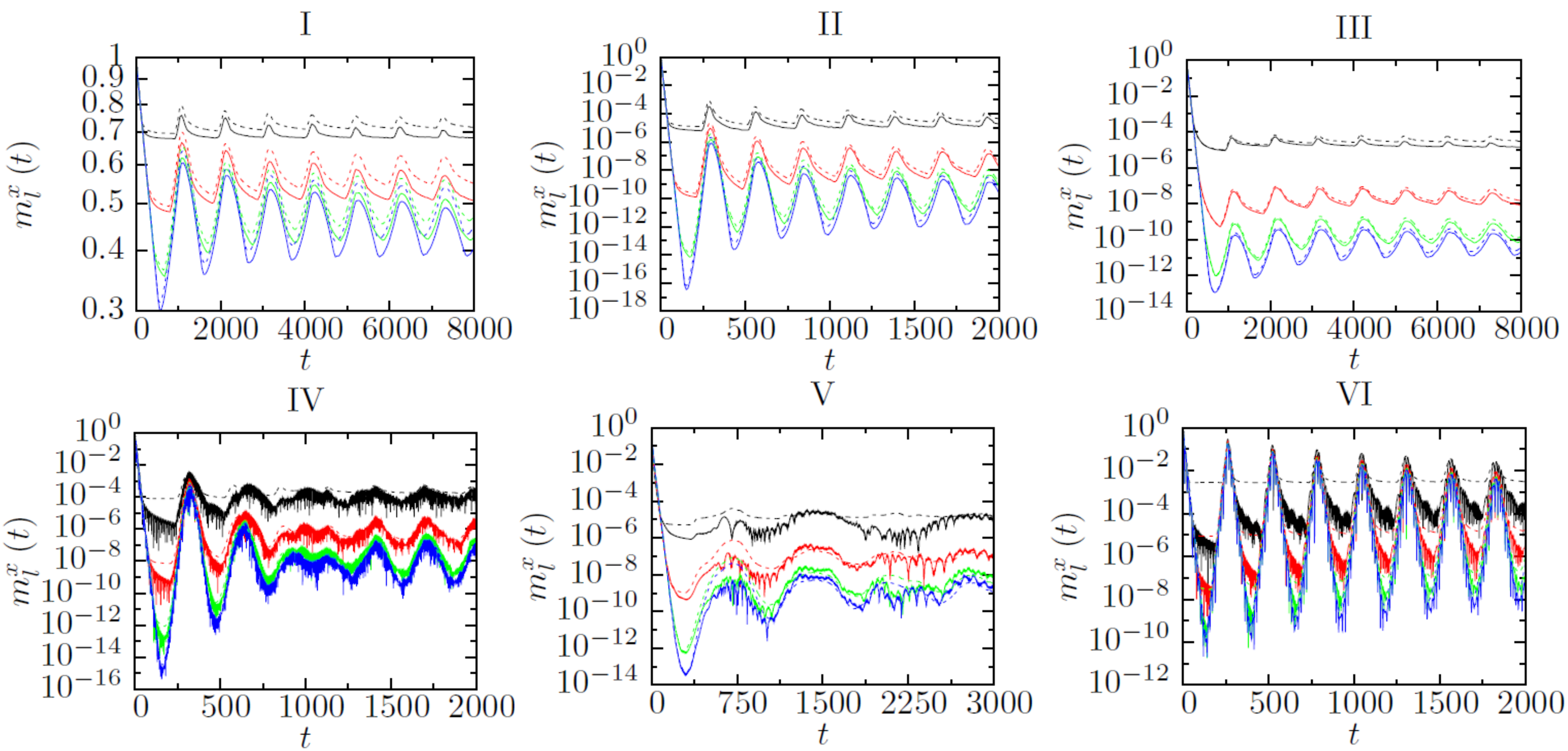
Operator $\sigma_r^y\sigma_{r+1}^y \rightarrow$ kink translation by 2 (!!) lattice units

TXY chain: Quench results

Local order parameter



Semi-classical theory:
as for TIM, just adapt
ε_p, v_p, f_p



full line: exact, broken line: semi-classical

[B. Blaß, HR, F.Iglói, arXiv:1205.3303]

Entanglement Entropy

$$(\rho_\ell(t) = e^{iHt} \rho_\ell \, e^{-iHt})$$

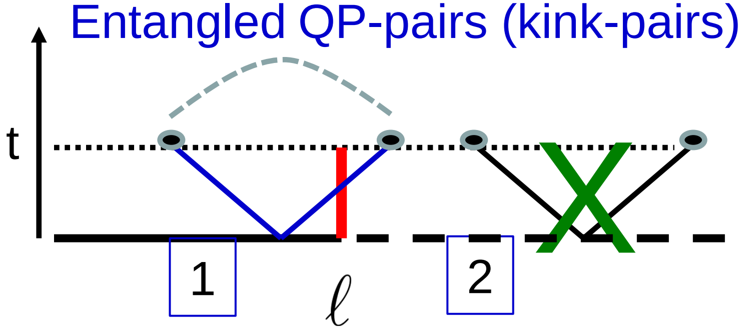
Reduced density matrix of subsystem {i≤ l }

$$\rho_\ell = \mathrm{Tr}_{i>\ell} \, |\Phi_0\rangle\langle\Phi_0|$$

Entanglement entropy of subsystem {i≤ l }

$$S_\ell(t) = \mathrm{Tr}_{i\le\ell} \, \rho_\ell(t) \log \, \rho_\ell(t)$$

Entangled QP-pairs (kink-pairs)



- contribution to entanglement entropy:

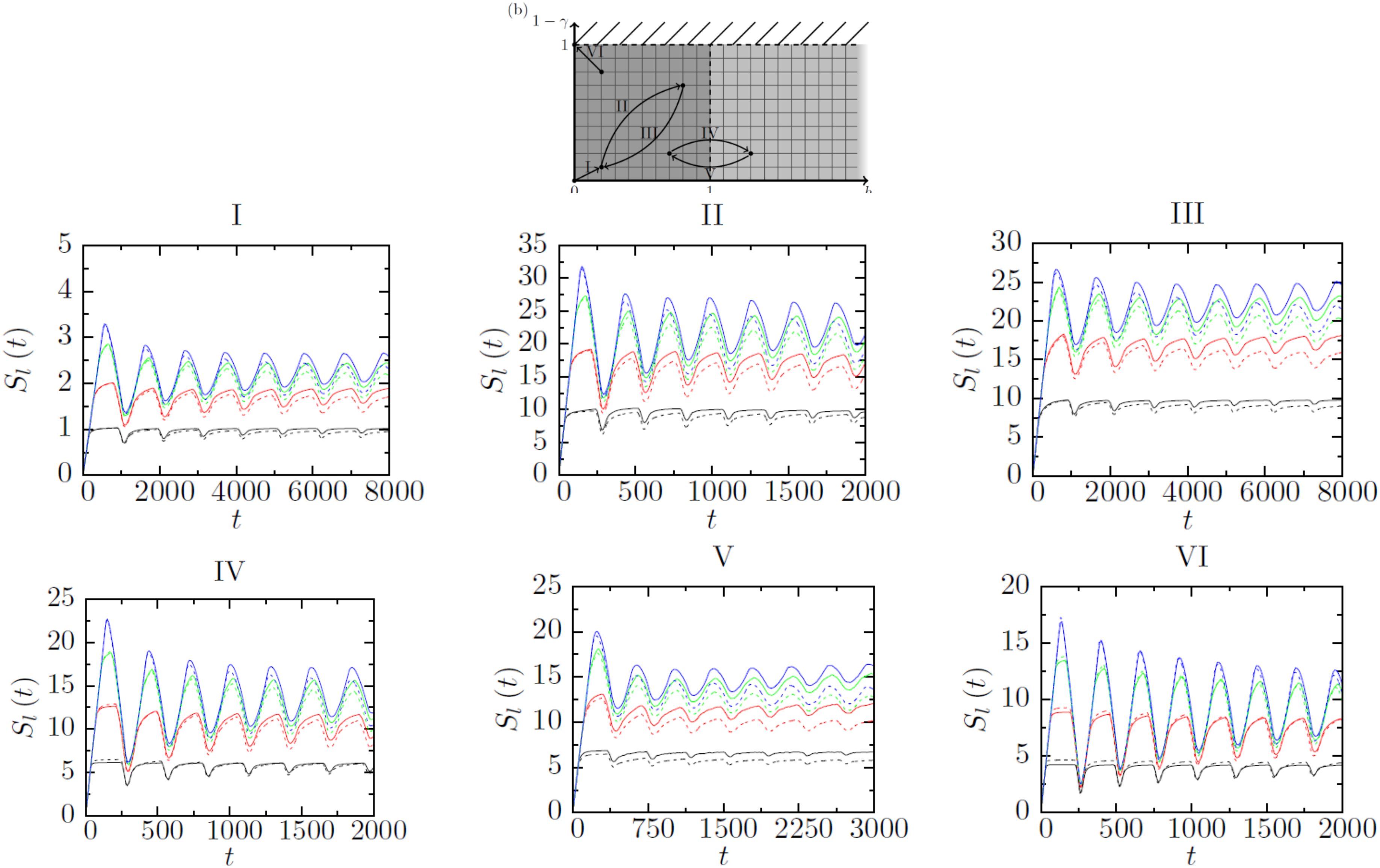
$$s_p = -(1-f_p) \ln (1-f_p) - f_p \ln f_p$$

$$\boxed{S_\ell(t) = \frac{1}{2\pi} \int_0^\pi dp \, s_p \, q_p(t)}$$

$q_p(t)$ = prob. for QP-pair with momentum p, 1 QP in subsystem 1, 1 QP in subsystem 2

$$= \begin{cases} v_p t & \text{for} \quad t < \ell/v_p \\ \ell & \text{for} \quad t > \ell/v_p \end{cases}$$

Entanglement entropy (TXY chain)



t<<T_{period}=L/v_{max}: linear slope = $\frac{1}{2\pi}\int_0^\pi dp\, v_p s_p$

[Blass, HR, Iglói, EPL 99, 30004 (2012)]

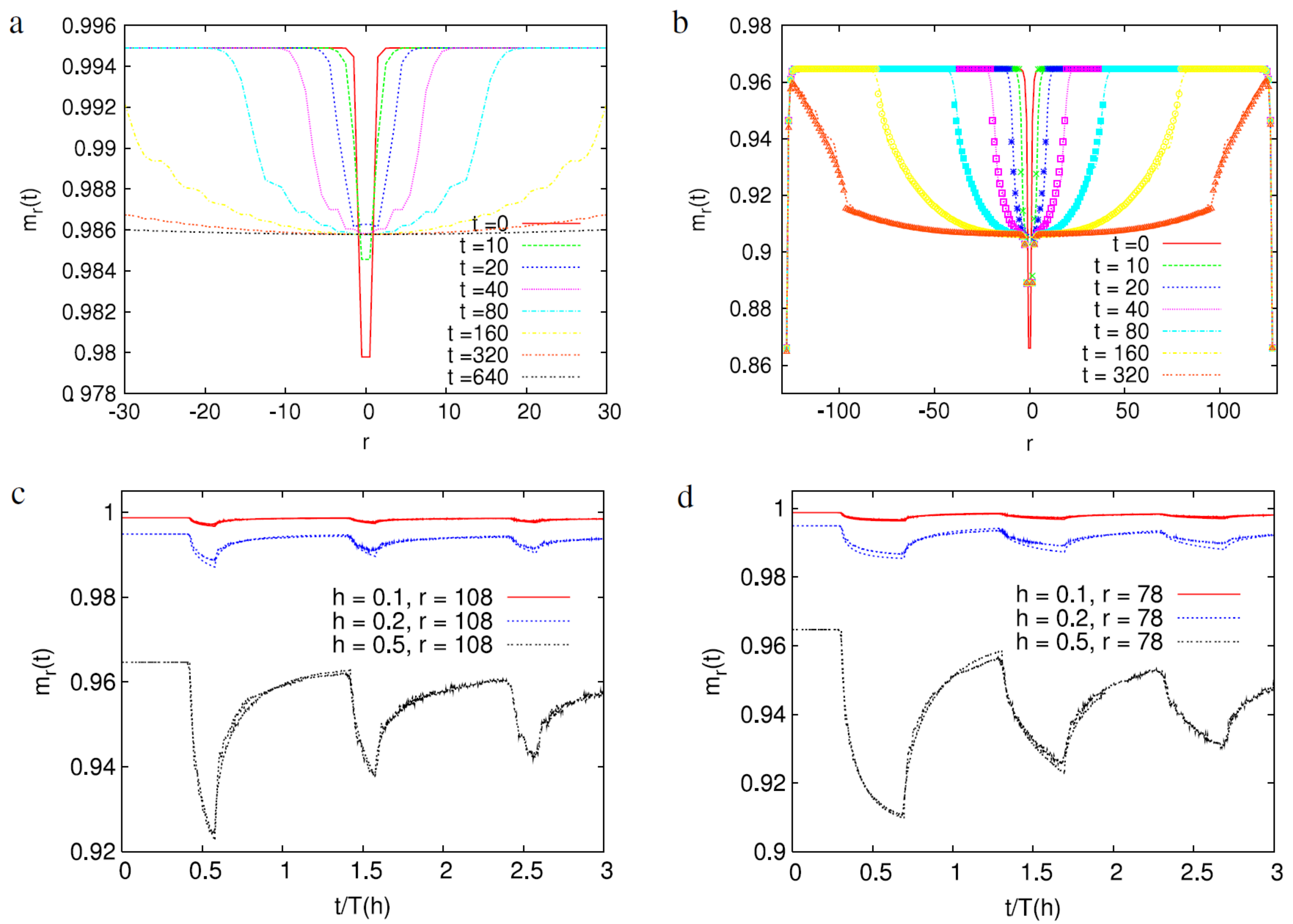
Local quenches in the TIC:

t<0 GS ↓ t<0 GS

t>0

local order param. $m_r(t)$

comp. with Semi-Classical theory



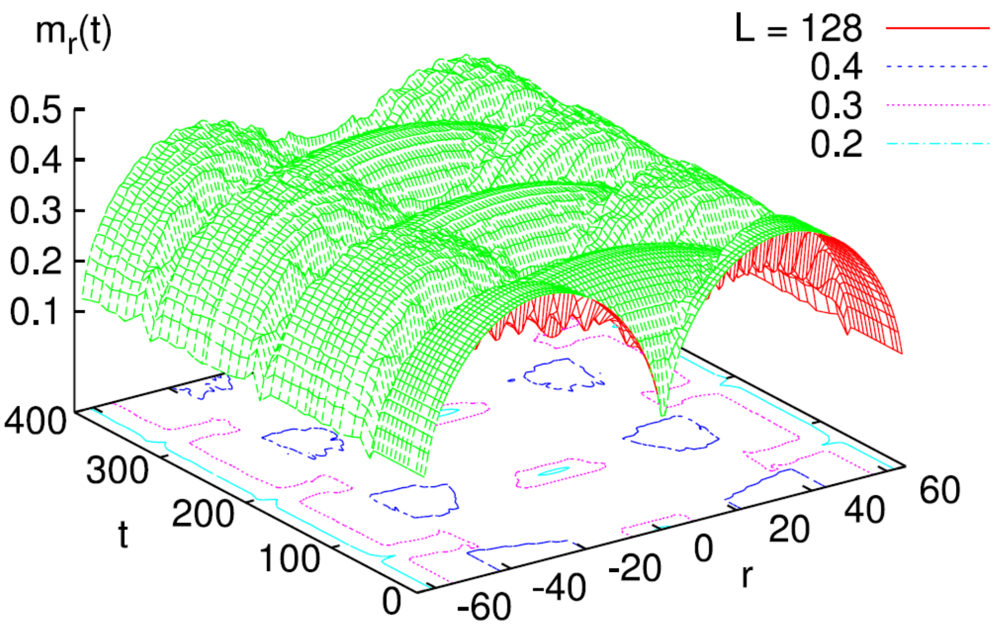
[U. Divakaran, F. Iglói, HR, JSTAT (2011) P10027]

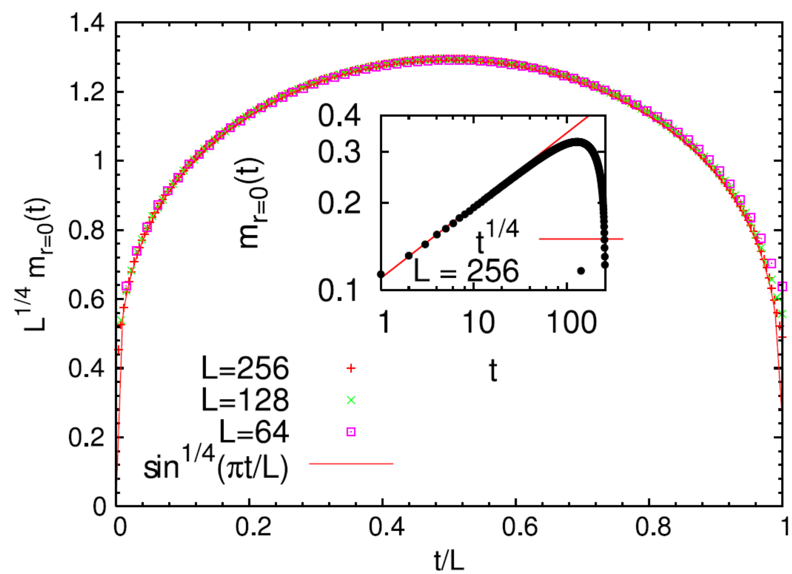
Local quenches:

t<0 GS ↓ t<0 GS

t>0

Critical quench: Magnetization profiles in space-time diagram





Initially:

$$m(l,t=0) \propto L^{-x}\left(\sin\pi\frac{2l}{L}\right)^{x_s-x}$$

At t=L/2:

$$m(l,t=L/2) \propto L^{-2x}\left(\sin\pi\frac{l}{L}\right)^{x_s-x}$$

At l=L/2:

$$m(l=L/2,t) \propto L^{-2x}\left(\sin\pi\frac{t}{L}\right)^{x_s-2x}$$

[U. Divakaran, F. Iglói, HR, JSTAT (2011) P10027]

Conclusions

- TIC / TXYC with boundaries – integrable system:
  σ^x-correlations "look" thermalized for $L \to \infty$
  (via exponential correlations, τ and ξ)
- In finite systems: boundary reflections $\Rightarrow$
  - quasi-periodic time dependence
  - quasi-stationary regimes
- Conserved modes, each with its own effective temperature
  - not Gibbs distributed, not thermalized
- Semi-classical theory based on uniformly moving kink pairs:
  - quantitatively good description for correlations, entanglement, etc.
  - exact asymptotics for quenches within the FM phase

- Semi-classical theory for multiple or periodic quenches?
- QP scattering necessary for thermalization
- Expansion of non-integrable models around integrable ones?