Full time statistics of equilibration dynamics

A tool for the investigation of quantum criticality and integrability

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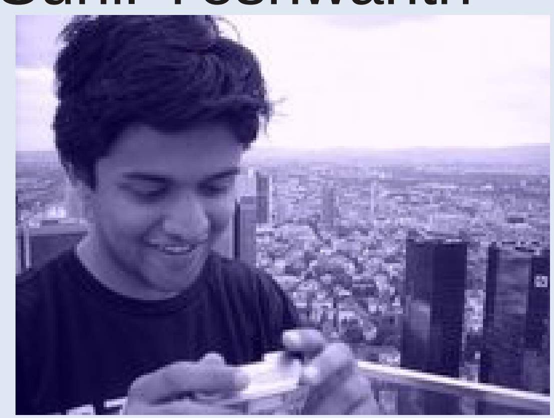


KITP, Santa Barabara,
October 1st 2012

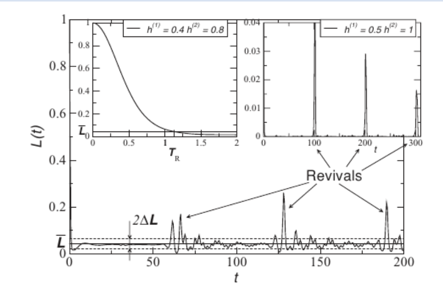
Preliminaries

Equilibration? $\rho(t) \Rightarrow \rho_{eq}$

1. Isolated, ***finite*** system
2. Prepare initial state ρ_0
3. Evolve with *H:* $\rho(t) = e^{-iHt}\rho_0 e^{iHt}$
4. Monitor observable A:

$$a(t) = \langle A(t) \rangle = Tr\, A\rho(t)$$

1. No strong convergence $\|\rho(t) - \rho_{eq}\| = cnst$
2. For finite systems no weak convergence
3. Stochastic convergence



•Observation window $[0, T]$

•Time average $\bar{f} = \int_0^T f(t) \frac{dt}{T}$

$$P_A(a)\,da = \text{Prob}(\langle A(t) \rangle \in [a, a+da],\ t \in [0, T])$$

$$P_A(a) = \overline{\delta(a - a(t))}$$

Equiliration = concentration of *P_A(a)*

Few words on the average state

PRL **98,** 050405 (2007) PHYSICAL REVIEW LETTERS week ending 2 FEBRUARY 2007

Relaxation in a Completely Integrable Many-Body Quantum System: An *Ab Initio* Study of the Dynamics of the Highly Excited States of 1D Lattice Hard-Core Bosons

Marcos Rigol,[1] Vanja Dunjko,[2,3] Vladimir Yurovsky,[4] and Maxim Olshanii[2,3,*]

Conjecture:

$$\rho(t) \to \bar{\rho} \simeq \rho_{GGE} = Z^{-1} e^{\sum_k \lambda_k n_k}$$

Remark:

ρ_{GGE} **is a Gaussian state with covarinace** $\bar{R}$

$$\rho_{GGE} = \rho_{\bar{R}}$$

1) Cramer, Dawson, Eisert, Osborne, PRL (2008)

$$\|\bar{\rho}^S - \rho^S_{GGE}\| < \epsilon$$

Subsystems, Bosons

2) Lanford III, Robinson, Comm. Math. Phys. (1972)

$$\rho(t) \to \rho_{GGE}$$

TD limit, $\epsilon_k = k^2$

On GGE (II)

Evidence that

$$\bar{\rho} \to \rho_{GGE}$$
$$L \to \infty$$

However:
identity is exact
for quadratic obervables

$$H = \sum c_x^\dagger M_{x,y} c_y$$
$$A = \sum c_x^\dagger A_{x,y} c_y$$
$$R_{y,x} = \langle c_x^\dagger c_y \rangle$$

Proof:

$$Tr\left(A e^{-itH} \rho e^{itH}\right) = Tr\left(A e^{-itM} R e^{itM}\right)$$

$$\Downarrow$$

$$Tr\left(A\bar{\rho}\right) = Tr\left(A \overline{R(t)}\right) = Tr\left(A \rho_{\bar{R}}\right)$$

Size of fluctuations: general case

$$a(t) = Tr\, A\rho(t)$$

$$\Delta^2 a = \overline{a(t)^2} - \bar{a}^2 \leq \mathrm{Ran}_A^2\, Tr\, \bar{\rho}^2$$

+ non-resonant condition:

$$E_i - E_j = E_l - E_m$$
$$\Rightarrow i=j, l=m \vee i=l,\ j=m$$

Reimann, PRL (2008)

Note that:

$$Tr\, \bar{\rho}^2 = L(t)$$

Clustering initial state

$$L(t) = \left| \langle \Psi | e^{-itH} | \Psi \rangle \right|^2 = \exp 2 \sum_{n=1} \langle H^n \rangle_c \frac{(-t^2)^n}{2n!} = e^{Vf(t)}$$

$$Tr\, \bar{\rho}^2 \leq e^{-\alpha V}$$

What about free systems?

Fluctuations: integrable case

$$a(t) = \langle A(t) \rangle \quad \rightarrow \quad P_A(a) = \overline{\delta(a - a(t))}$$

Generally, for *A* extensive:

$$H = \sum c_x^\dagger M_{x,y} c_y$$
$$A = \sum c_x^\dagger A_{x,y} c_y$$
$$R_{y,x} = \langle c_x^\dagger c_y \rangle$$

$$\overline{a(t)} \propto V \qquad \Delta^2 a \leq O(e^{-\alpha V})$$

Integrable systems (free Fermions)

$$a(t) = Tr(A e^{-itM} R e^{itM}) = \sum_{k,q} A_{q,k} \underbrace{R_{k,q}}_{F_{k,q}/2} e^{-it(\epsilon_k - \epsilon_q)}$$

Analogous bound:

$$\Delta^2 a \leq \|a\|_\infty^2 \, Tr \overline{R}^2 \leq \|a\|_\infty^2 \nu V$$

All the cumulants: Stat-mech parallel

Rational independence

$$\overline{e^{\lambda a(t)}} = \sum_{\theta' s} e^{\lambda E(\theta' s)} = e^{f(\lambda) V}$$

$$E(\theta' s) = \sum_{k,q} F_{k,q} \cos(\theta_k - \theta_q)$$

$$F_{k,q} = F(|k-q|)$$

Classical *XY* model
on lattice $F_{i,j}$
(infinite temperature)

All cumulants extensive:
CLT

$$Z = \frac{(a(t) - \bar{a})}{\sqrt{V}}$$ Gaussian

Gaussian Equilibration

Gaussian equilibration: Examples

$$H_{XY} = -\sum_{i=1}^{L}\left[\frac{(1+\gamma)}{2}\sigma_i^x\sigma_{i+1}^x + \frac{(1-\gamma)}{2}\sigma_i^y\sigma_j^y + h\sigma_i^z\right]$$

$$H = t\sum_{i=1}^{L}\left[c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right]$$

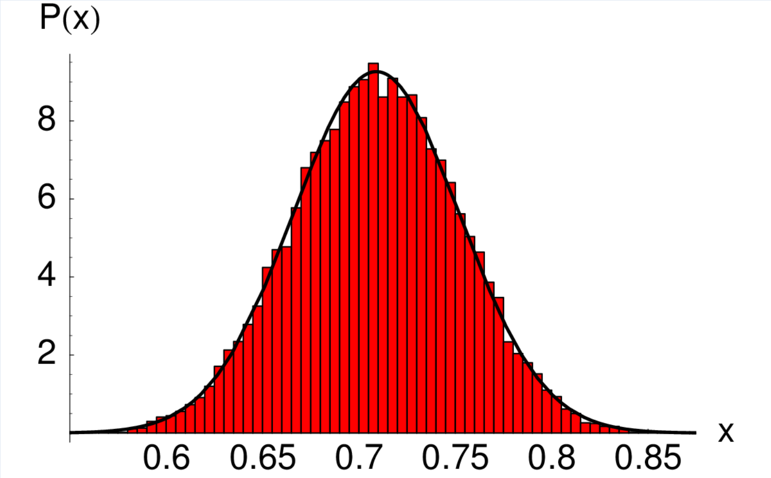
$$c_{L+1} = e^{i\theta L} c_1$$

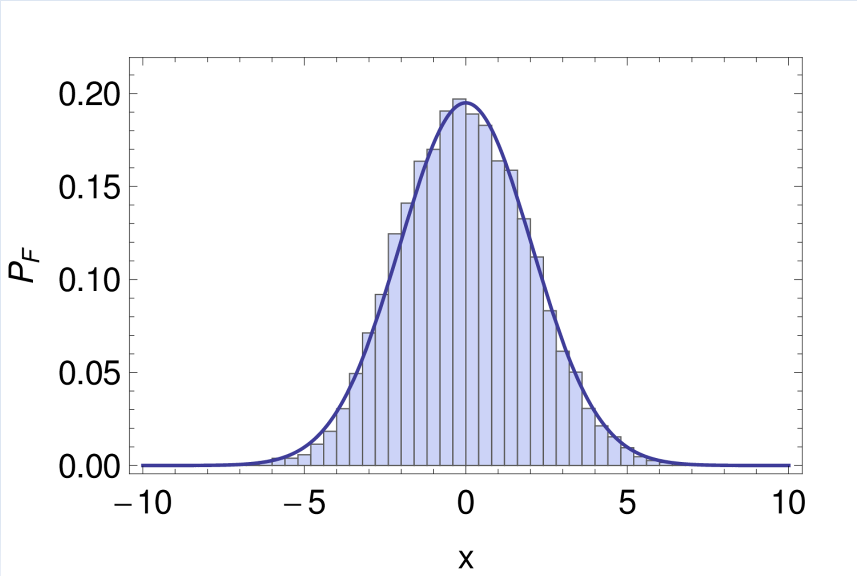
$$m(t) = \langle \sigma_i^z(t)\rangle$$

$$A(t) = \sum_{i=1}^{l}\langle n_i(t)\rangle$$

Theorem: (assuming RI) m(t) Gaussian

Theorem: extensive fluctuations





Equilibration & Integrability

Generally, for *A* extensive:

$$\overline{a(t)} \propto V \qquad \Delta^2 a \leq O\left(e^{-\alpha V}\right)$$

Integrable systems (free Fermions)

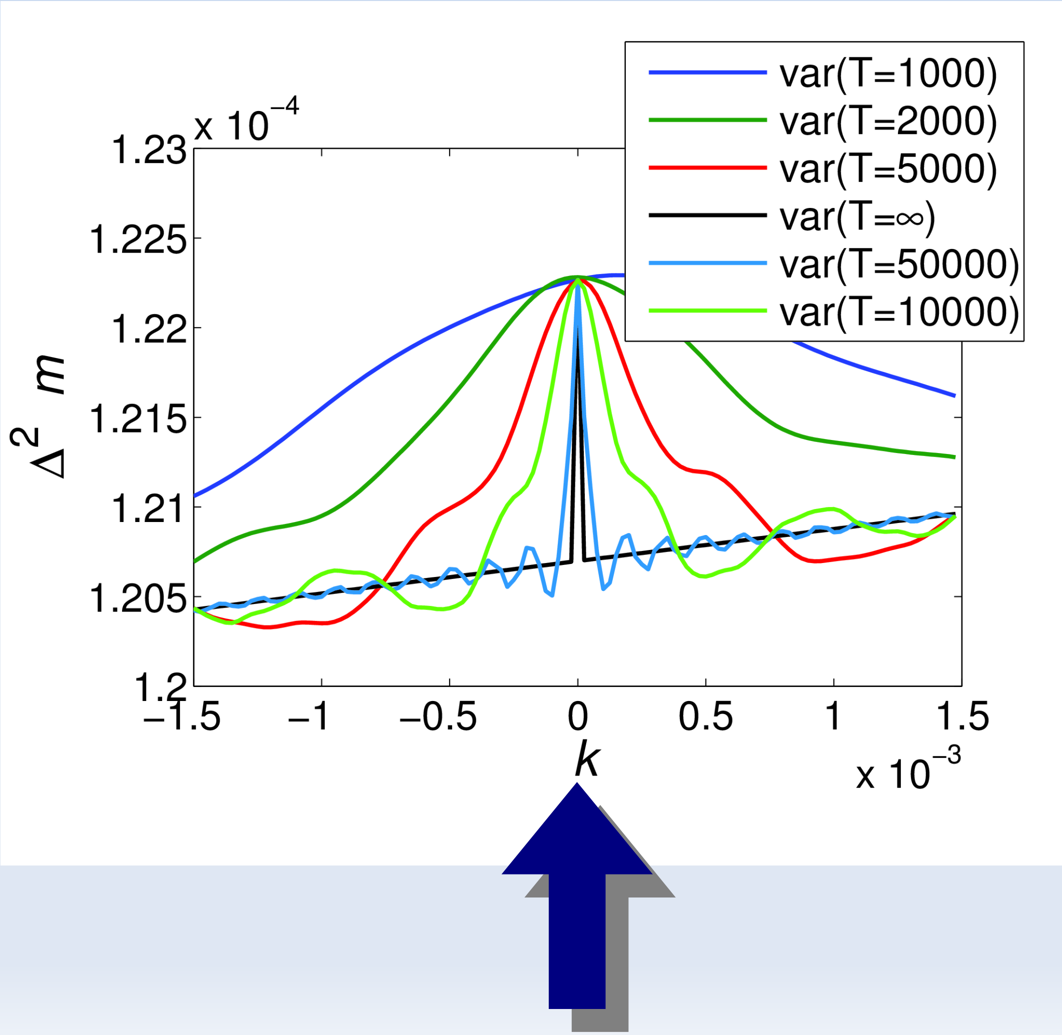
$$\overline{a(t)} \propto V \qquad \Delta^2 a = O(V)$$

Gaussian (poor) equilibration

Detecting integrable points

$$H=-\sum_i \left[\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z - \kappa\sigma_i^x \sigma_{i+2}^x\right]$$

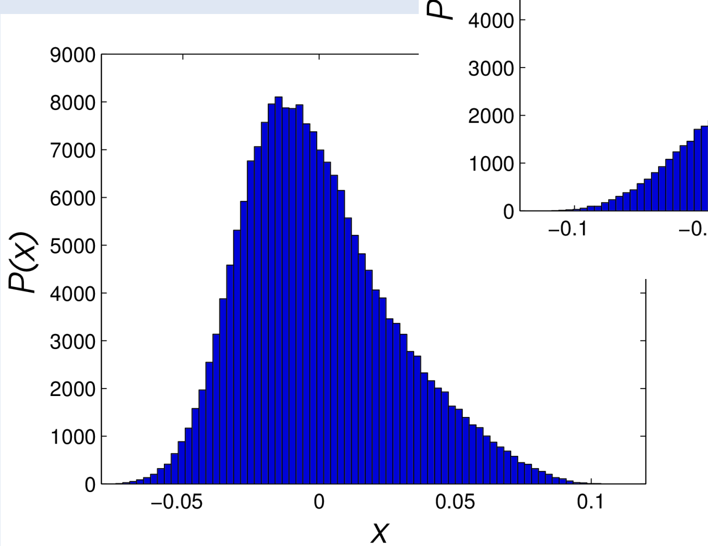
$$m(t)=\langle\sigma_i^z(t)\rangle$$

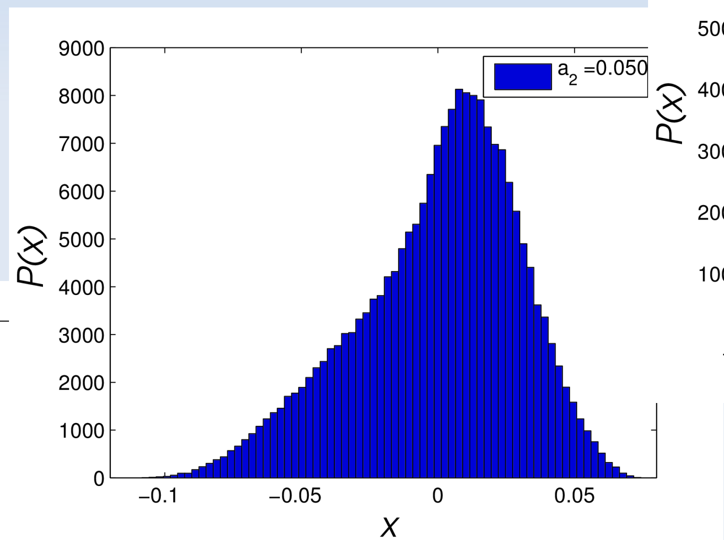


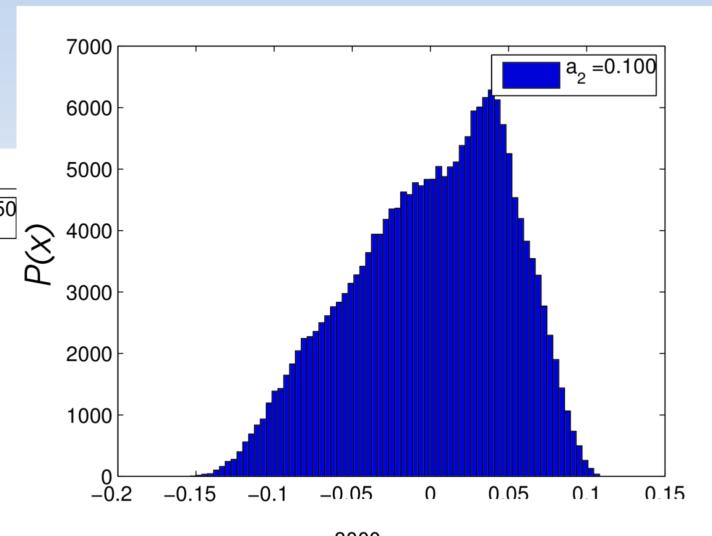
Detecting integrable points (II)

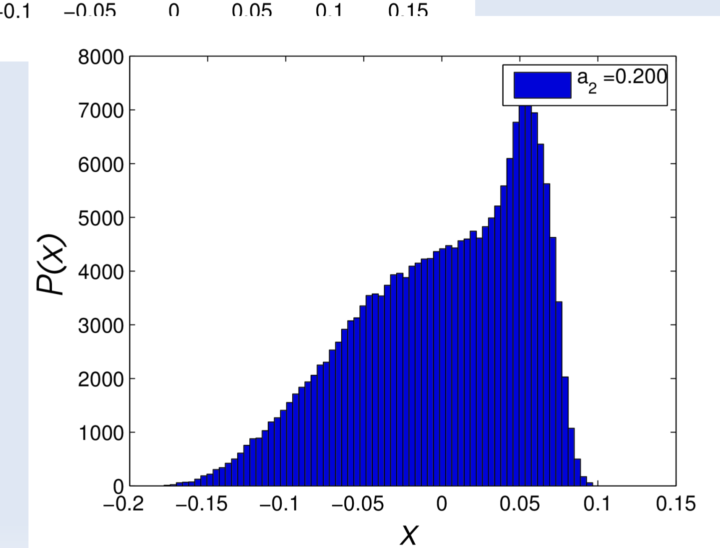
$$H=\sum_i\left[\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\Delta_1\sigma_i^z\sigma_{i+1}^z+\alpha\left(\sigma_i^x\sigma_{i+2}^x+\sigma_i^y\sigma_{i+2}^y+\Delta_2\sigma_i^z\sigma_{i+2}^z\right)\right]$$

$$x(t)=\langle\sigma_i^+\sigma_{i+1}^-(t)\rangle+h.c.$$





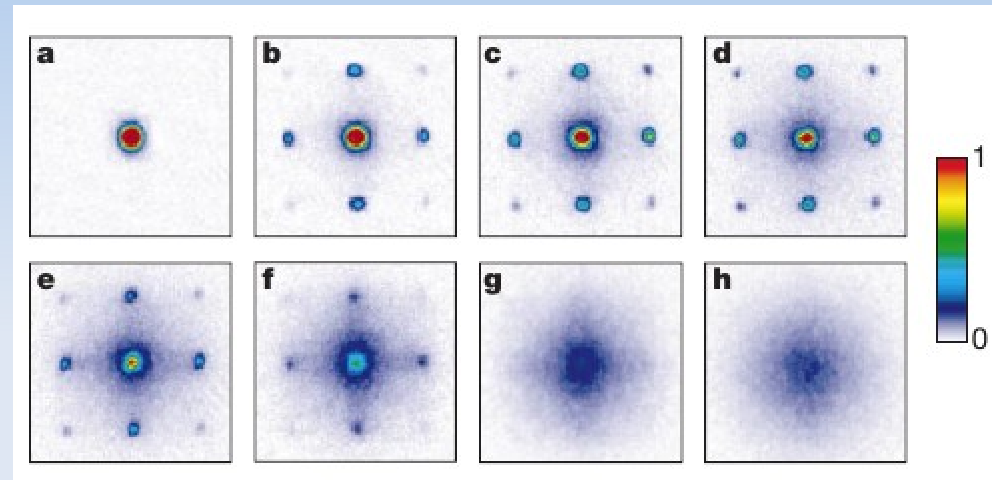




Quench: dynamical detection of QPT's

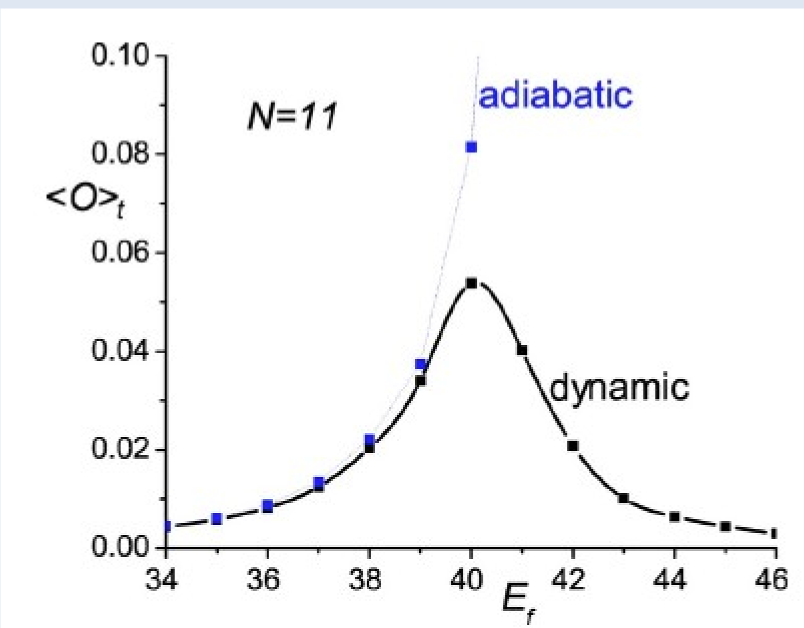
$$H \rightarrow |\Psi_0\rangle$$
$$H' = H + \delta\lambda V$$

superfluid-Mott transition: experiment



Greiner et al., Nature (2002)

Theoretical description



Sengupta, Powell, Sachdev, PRA (2004)

First moment to detect criticality

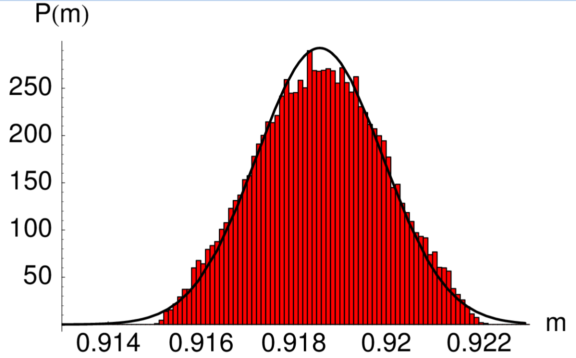
Small* quench: full statistics

$\delta\lambda = 0.04$

- Small quench, off-critical

  $L \gg \xi$

L = 12

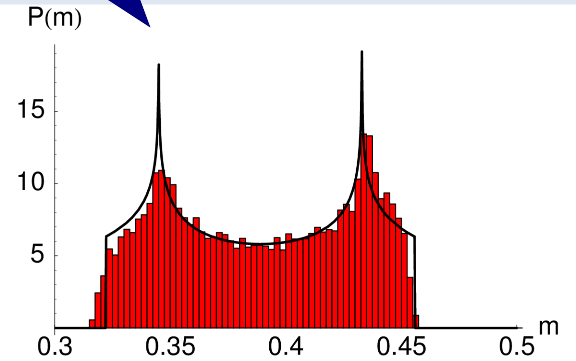


Bi-modal: phase transition

- Small quench, quasi-critical

  $\xi \gg L$

L = 16



(*) Small: $\delta\lambda^2 \chi_F \ll 1 \;\Rightarrow\; \delta\lambda \ll \frac{1}{L^{1/\nu}}$

Small quench: CLT

$$\langle A(t)\rangle = \sum_{n,m} A_{n,m}\overline{c(E_n)}c(E_m)e^{-it(E_m-E_n)}$$

$$c(E_n) = \langle E_n, \Psi_0\rangle$$

$$\approx \overline{A} + \sum_{n>0} F_n \cos(t(E_n - E_0))$$

$$F_n \approx 2A_{n,0}c(E_n)$$

rational independence → *a(t)* sum of independent variables

$$\Delta^2 a = \frac{1}{2}\sum_{n>0} F_n^2 \leqslant 2[\langle 0|A^2|0\rangle - \langle 0|A|0\rangle^2] \propto L$$

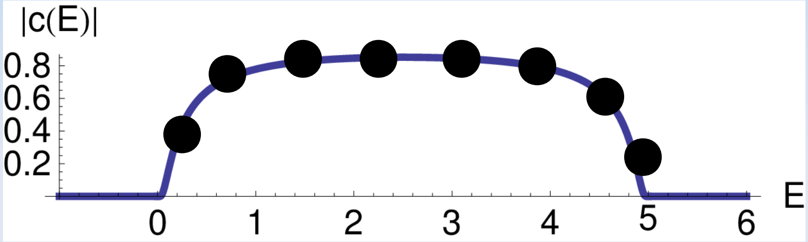
→ *"Generally" CLT*

Look at F_n , $c(E_n)$

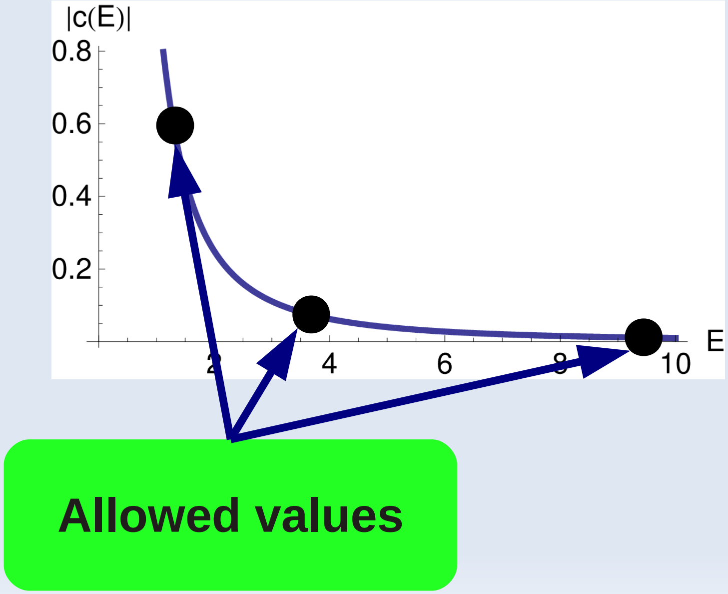
Small quench: explanation

Look at $c(E_n)$

Off-critical distribution



Critical case distribution



Scaling prediction at criticality:

$$c(E) \sim E^{-1/\nu}$$

sum rule

$$\sum_E |c(E)|^2 = 1$$

Only few $|E\rangle$ states contribute: Bi-modal distribution

Small quench: critical case

- Q: How to break CLT?
- A: most $F_n \to 0$

(Quantum) critical points are
more stable against perturbations

Conclusions

- Finite systems
- Look at full time statistics

} Ingredients

- **Integrability & equilibration:**

  Gaussian equilibration: LCV, Paolo Zanardi arXiv:1208.1121

  integrable systems concentrate less

- **Small quench:** a tool to detect criticality, engineer ”*new quantum states of matter*”
  LCV, Jacobson, Santra, Zanardi, PRL (2011)
  LCV, Zanardi, PRA (2010)

Thank you

Loschmidt echo

$$[R, M] = 0$$

$$L(t) = \prod_k \left(1 - \alpha_k \sin^2(t \epsilon_k / 2)\right)$$

- Applies to *XY* model
- Generalizes to thermal quenches
- Generalizes to Ulman Fidelity

$$Z = \frac{\log L(t) - \overline{\log L(t)}}{\sqrt{L}}$$

Gaussian, => *L(t)* Log-Normal

For general models (RI spectrum), work in progress

$$\mu_n(L(t)) = f\left(Tr(\bar{\rho}^{2k})\right)$$

Curiosity: Riemman zeta

$$\zeta(\sigma+it) = Tr\left(e^{-itH}\rho_\sigma\right), \quad \rho_\sigma = e^{-\sigma H}$$

H primon gas:
free bosons

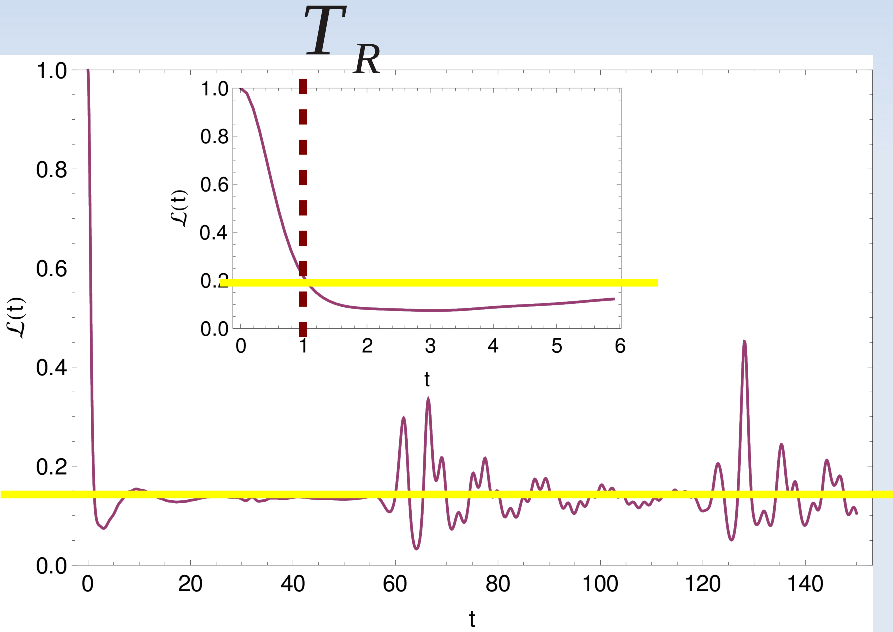
$$Z := \log\left|\zeta(\sigma_0+it)\right|^2$$

Satisfies
CLT

Very similar
to Loschmidt Echo)

Relaxation time

(Talk by Michael Pustilnik)



$$\langle A(T_R) \rangle := \overline{A}$$

Loschmidt echo

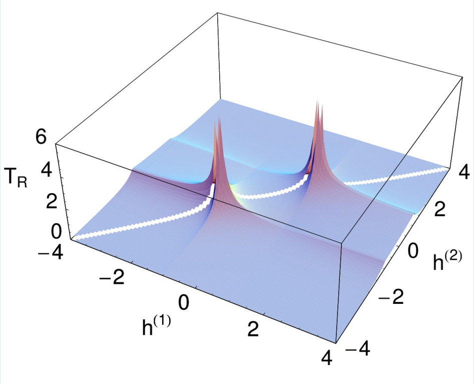
$$L(T_R) := \overline{L}$$

Relaxation time II

Loschmidt echo: short time - cumulant expansion

Generally (+small quench off-critical)

$$L(t) = e^{-\sigma^2 t^2} \qquad \overline{L} = e^{-\alpha L^d}, \ \sigma^2 \sim L^d \ \Rightarrow \ T_R = O(L^0)$$



small quench criticality

$$\overline{L} \simeq e^{-2\delta\lambda^2 \chi_F}, \ \sigma^2 \sim L^{2(d-\Delta)} \ , \chi_F \sim L^{2(d+\zeta-\Delta)} \ \Rightarrow \ T_R = O(L^{\zeta})$$

$$T_R \sim |\lambda - \lambda_c|^{-\zeta\nu}$$

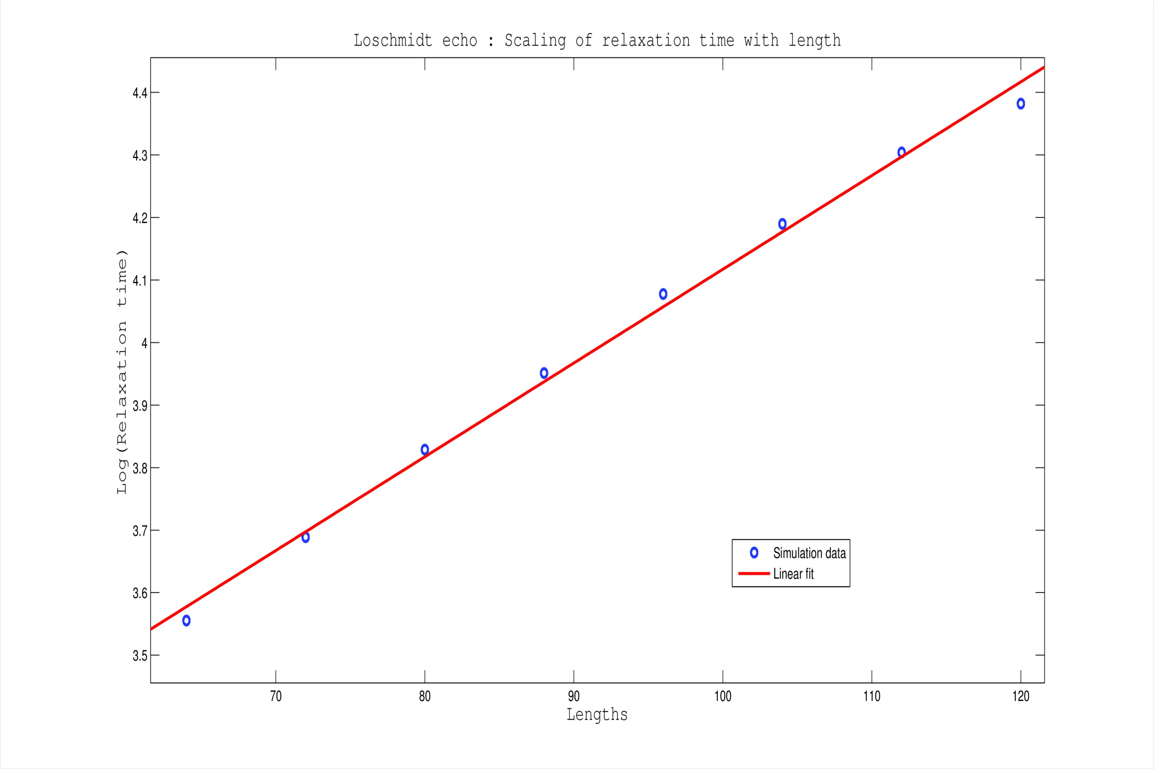
Relaxation time: Random Systems

~ Inguscio, Modugno, LENS

$$H = \sum_x \left(c_x^\dagger c_{x+1} + c_{x+1}^\dagger c_x \right) - \mu_x c_x^\dagger c_x$$

Random field

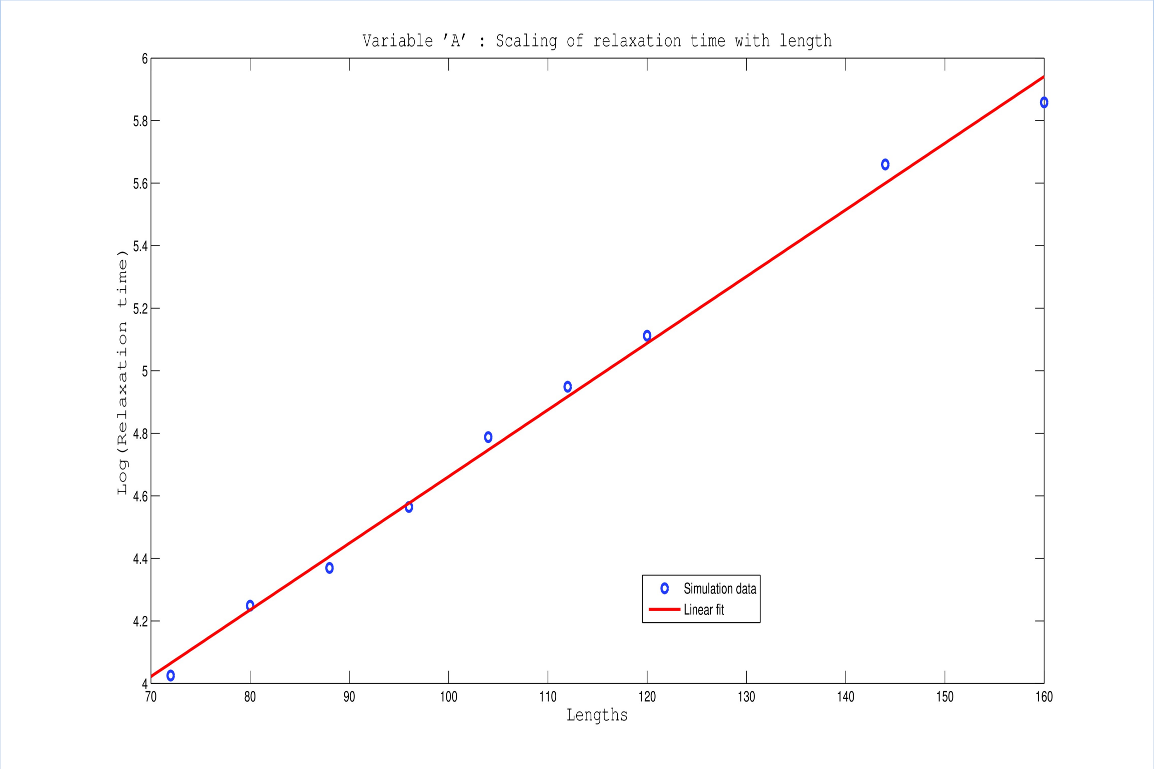
Loschmidt echo

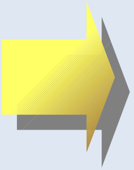


$$\mathrm{E}\left[L(t) \right]$$

Relaxation time: Random Systems

Number operator $\mathrm{E}\left[\left\langle N_l(t)\right\rangle\right]$





$$T_{\mathrm{Relax}} \sim e^{\alpha L}$$