Entanglement Negativity in Quantum Field Theory

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Outline

- Bipartite entanglement in pure states
- Path integral approach and correlators of twist operators in QFT
- Entanglement in mixed states and negativity
- Results in 1+1-dimensional CFT
- Higher dimensions

Work largely carried out with Pasquale Calabrese (Pisa)
and Erik Tonni (Trieste)

Bipartite Entanglement in Pure States

Quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Schmidt decomposition:

$$|\Psi\rangle = \sum_j c_j \, |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

with $c_j \geq 0$, $\sum_j c_j^2 = 1$, and $|\psi_j\rangle_A$, $|\psi_j\rangle_B$ orthonormal.

One quantifier of the amount of entanglement is the entropy

$$S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$$

Equivalently, in terms of A's reduced density matrix $\rho_A \equiv \text{Tr}_B \, |\Psi\rangle\langle\Psi|$

$$S_A = -\text{Tr}_A \, \rho_A \log \rho_A = S_B$$

Similar information is contained in the Rényi entropies

$$S_A{}^{(n)} = (1 - n)^{-1} \log \text{Tr}_A \, \rho_A{}^n$$

$$S_A = \lim_{n \to 1} S_A{}^{(n)}$$

Other measures of bipartite entanglement exist, but entropy has several nice properties: additivity, convexity, . . .

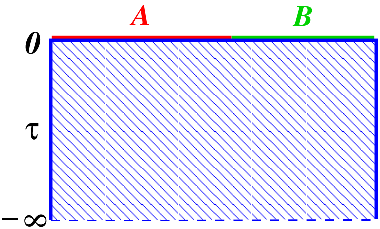
It is monotonic under Local Operations and Classical Communication (LOCC)

It gives the amount of classical information required to specify ρ_A (important for numerical computations)

It gives a basis-independent way of identifying and characterising quantum phase transitions

In a relativistic QFT the entanglement in the vacuum encodes all the data of the theory (spectrum, anomalous dimensions, . . .)

Rényi entropies from the path integral



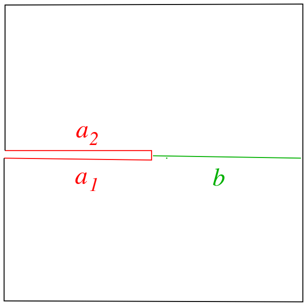
$$\Psi(\{a\},\{b\}) = Z_1^{-1/2} \int_{a(0)=a,b(0)=b} [da(\tau)][db(\tau)]\, e^{-(1/\hbar)S[\{a(\tau)\},\{b(\tau)\}]}$$

where $S = \int_{-\infty}^{0} L\big(a(\tau), b(\tau)\big)\, d\tau$

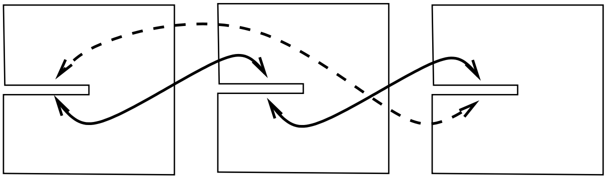
Similarly $\Psi^*(\{a\},\{b\})$ is given by the path integral from $\tau = 0$ to $+\infty$

$$\rho_A(a_1, a_2) = \int db\, \Psi(a_1, b)\Psi^*(a_2, b)$$

This is given by the path integral over $\mathbb{R}^2$ cut open along $A \cap \{\tau = 0\}$, divided by Z_1:



Rényi entropies

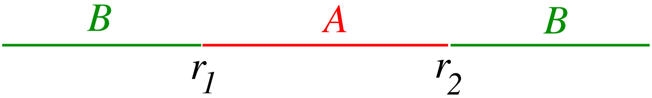


$\mathrm{Tr}_A\, \rho_A^n$ is given by the partition function on n sheets sewn together cyclically along $A \cap \{\tau = 0\}$, forming a conifold $\mathcal{R}_n$, with opening angles $2\pi n$ at each conical singularity.

$$\mathrm{Tr}_A\, {\rho_A}^n = Z(\mathcal{R}_n)/Z_1^n$$

– equivalently, n copies of the CFT within the fields cyclically identified across $A \cap \{\tau = 0\}$:

$$a_j(0-) = a_{j+1}(0+) \qquad \text{mod } n$$



If space is 1d and A is an interval (r_1, r_2) (and B is the complement) then $Z(\mathcal{R}_n)$ can be thought of as the insertion of twist operators into n copies of the CFT:

$$Z(\mathcal{R}_n)/Z_1^n = \langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\rangle_{(CFT)^n}$$

These have similar properties to other local operators e.g. in a massless QFT (a CFT)

$$\langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\rangle \sim |r_1 - r_2|^{-2\Delta_n}$$

Main result for $d = 1$ [Holzhey et al., CC]:

$$\Delta_n = (c/12)(n - 1/n)$$

where c is the central charge of the UV CFT

Two intervals

$$Z(\mathcal{R}_n)/Z^n = \langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\mathcal{P}_n^{-1}(r_3)\mathcal{P}_n(r_4)\rangle$$

In general there is no simple result but for $r_{12}, r_{34} \ll r_{23}, r_{14}$ we can use an operator product expansion [Headrick, CCT]

$$\mathcal{P}_n^{-1}(r_1)\cdot\mathcal{P}_n(r_2) = \sum_{\{k_j\}} C_{\{k_j\}}(r_1 - r_2)\prod_{j=1}^{n}\Phi_{k_j}\big(\tfrac{1}{2}(r_1 + r_2)_j\big)$$

in terms of a complete set of local operators Φ_{k_j}.

This shows that the mutual information $S_{A_1\cup A_2} - S_{A_1} - S_{A_1}$ is more related to correlations between A_1 and A_2 and not their quantum entanglement.

Twist operators correspond to a cyclic permutation P_n of the replicas as we go around the conical singularity.

More generally we could consider

$$\langle \mathcal{P}_n^{(1)}(r_1)\mathcal{P}_n^{(2)}(r_2)\mathcal{P}_n^{(3)}(r_3)\mathcal{P}_n^{(4)}(r_4)\rangle$$

where the $\mathcal{P}_n^{(k)}$ are more general permutations of n objects (with $\prod_k \mathcal{P}_n^{(k)} = 1$.)

These are related to new measures of the *mixed state* entanglement between A_1 and A_2.

In particular

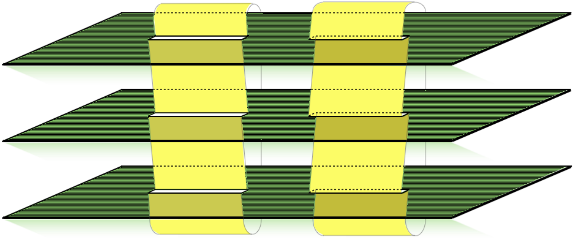
$$\langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\mathcal{P}_n(r_3)\mathcal{P}_n^{-1}(r_4)\rangle$$

gives

$$\mathrm{Tr}\,\left(\rho_{A_1\cup A_2}^{T_2}\right)^n$$

where $\rho_{A_1\cup A_2}^{T_2}$ is the partial transpose

$$\rho_{A_1\cup A_2}^{T_2}(a_1,a_2;a_1',a_2') = \rho_{A_1\cup A_2}(a_1,a_2';a_1',a_2)$$



This is related to *negativity* [Vidal-Werner 2002].

Although $\mathrm{Tr}\,\rho^{T_2} = 1$, it may have negative eigenvalues λ_j, and this will happen if

$$\mathcal{E} \equiv \log \mathrm{Tr}\,|\rho^{T_2}| = \log \sum_j |\lambda_j| > 0$$

$\mathcal{E}$ has nice quantum information properties, e.g. monotonicity under LOCC.

Negativity in 1+1 dimensional CFT

Note that

$$\mathrm{Tr}\left(\rho^{T_2}\right)^n = \sum_j \lambda_j^n = \sum_j |\lambda_j|^n \qquad \text{for } n \text{ even}$$

so if the continuations to $n = 1$ from even and odd n are different, we can have negativity.

$$\langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\mathcal{P}_n(r_3)\mathcal{P}_n^{-1}(r_4)\rangle$$

This can happen if $r_{23} \ll r_{12}, r_{34}$, because of the OPE

$$\begin{aligned}\mathcal{P}_n \cdot \mathcal{P}_n &\cong \mathcal{P}_n \qquad n \text{ odd} \\ &\cong \mathcal{P}_{n/2} \otimes \mathcal{P}_{n/2} \qquad n \text{ even}\end{aligned}$$

This has scaling dimension $2(c/12)(n/2 - 2/n) \to -c/4$ as $n \to 1$.

In this limit we get

$$\mathcal{E} \sim (c/4)\log(r_{12}r_{34}/r_{23}r_{14})$$

This has been confirmed numerically for uncompactified free boson and for the Ising model.

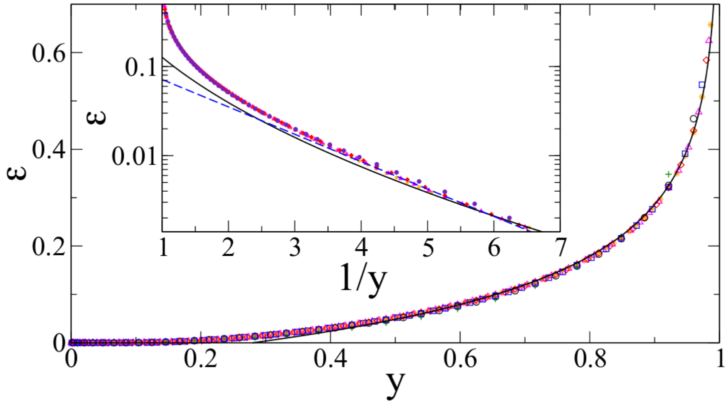
$$\langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\mathcal{P}_n(r_3)\mathcal{P}_n^{-1}(r_4)\rangle$$

In the opposite limit $r_{12}, r_{34} \ll r_{23}, r_{14}$ we can use the short interval expansion, analytically continued from the usual ordering of the arguments: as $n \to 1$ every term in the OPE vanishes!

But numerically we find [Markovitch et al., CCT]

$$\mathcal{E} \propto \exp\big(-C\, r_{23}r_{14}/r_{12}r_{34}\big)$$

Non-perturbative terms in the OPE!



$$y = r_{12}r_{34}/r_{13}r_{24}$$

Other results

In general $\mathrm{Tr}\left(\rho_{A_1\cup A_2}^{T_2}\right)^n$ is given by the partition function on a surface of the same genus as that for $\mathrm{Tr}\left(\rho_{A_1\cup A_2}\right)^n$, but on a different section of the moduli space.

However for $n=2$

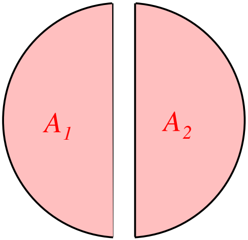
$$\mathrm{Tr}\left(\rho_{A_1\cup A_2}^{T_2}\right)^2 = \mathrm{Tr}\left(\rho_{A_1\cup A_2}\right)^2 \propto Z_{\text{torus}}$$

but

$$\mathrm{Tr}\left(\rho_{A_1\cup A_2}^{T_2}\cdot\rho_{A_1\cup A_2}\right) \propto Z_{\text{Klein bottle}}$$

Correlators of products of twist operators corresponding to general permutations are given by CFT partition functions on non-orientable surfaces.

Higher dimensions



For $d > 1$ for 2 large regions a finite distance apart

$$\mathcal{N}(A_1, A_2) \propto \text{Area of common boundary between } A_1 \text{ and } A_2$$

- universal corrections to this 'area law'?
- if A_1 and A_2 are far apart, a generalisation of small interval expansion again gives vanishing negativity to all order vanishes to all orders – non-perturbative corrections??

Summary

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...when gravity fails and negativity won't pull you through...