Supersymmetry in Curved Space

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Sources and Background Fields

A standard tool in QFT is to turn on sources and study the response of the system.

Example: In a theory with a global flavor symmetry we can turn on a gauge field A_μ that couples to the conserved flavor current j_μ,

$$\mathscr{L}' = \mathscr{L} + A^\mu j_\mu + \mathcal{O}\left(A^2\right)$$

- The higher-order seagull terms ensure invariance under gauge transformations of A_μ (equivalently $\partial^\mu j_\mu = 0$).
- A_μ is a non-dynamical background field (no e.o.m.)
- Small variations of A_μ around $A_\mu = 0$ are captured, order by order in a power series, by correlation functions of j_μ in the original theory (linear response).

QFT in Curved Space

Every Poincaré invariant QFT has a conserved, symmetric stress tensor $T_{\mu\nu}$. The appropriate source is a background spacetime metric $g_{\mu\nu}$. In Euclidean theories, $g_{\mu\nu}$ is a Riemannian metric. Around flat space:

$$g_{\mu\nu} = \delta_{\mu\nu} + \Delta g_{\mu\nu}\ , \qquad \mathscr{L}' = \mathscr{L} - \frac{1}{2}\Delta g^{\mu\nu} T_{\mu\nu} + \mathcal{O}\left(\Delta g^2\right)$$

- The effect of $\Delta g_{\mu\nu}$ is captured by correlation functions of $T_{\mu\nu}$.
- Some higher-order terms are fixed by diff-invariance (seagull terms). The fully invariant theory can then be studied on any Riemannian manifold $\mathcal{M}$. In particular, $\mathcal{M}$ may be compact.
- The curved space Lagrangian is not unique: we can add curvature couplings, which correspond to a choice of $T_{\mu\nu}$,

$$T'_{\mu\nu} = T_{\mu\nu} + \left(\partial_\mu \partial_\nu - \delta_{\mu\nu}\partial^2\right) u \quad \Leftrightarrow \quad \mathscr{L}' = \mathscr{L} - \frac{1}{2} R[g] u$$

The Partition Function

$$Z_{\mathcal{M}}[g_{\mu\nu}, A_\mu, \cdots] = \int \mathcal{D}\Phi \; e^{-\int \mathscr{L}_{\mathcal{M}}[\Phi, g_{\mu\nu}, A_\mu, \cdots]}$$

IR finite if $\mathcal{M}$ is compact, but possible UV ambiguities (scheme dependence). The physical part of $Z_{\mathcal{M}}$ is a rich observable:

- Dependence on $g_{\mu\nu}, A_\mu$ encodes correlators of $T_{\mu\nu}, j_\mu$ on $\mathcal{M}$.
- It can detect global degrees of freedom, which are activated by the topology of $\mathcal{M}$ (e.g. Chern-Simons theory).
- In a CFT, the theory on $\mathcal{M}$ is sometimes related to flat space by a conformal transformation (fixes curvature couplings):
  - States on $S^{d-1} \times \mathbb{R} \Leftrightarrow$ Local operators on $\mathbb{R}^d$. They are counted by the partition function on $S^{d-1} \times S^1$.
  - Correlation functions on $S^d \Leftrightarrow$ Correlation functions on $\mathbb{R}^d$.
  - Partition function on $S^d \Leftrightarrow$ Entanglement entropy across a sphere in $\mathbb{R}^d$. [Casini, Huerta, Myers] + Talks by Headrick and Myers.

In general, computing $Z_{\mathcal{M}}$ is very challenging.

Supersymmetric Theories

Flat Space: SUSY provides a powerful handle on the dynamics of QFT. This is especially useful for BPS observables that preserve some of the supercharges: their dependence on the parameters of the theory is tightly constrained and can sometimes be determined exactly, e.g. superpotential $W(\Phi)$ in 4d $\mathcal{N}=1$ theories.

Curved Space: Generic choices of $\mathcal{M}$ and $g_{\mu\nu}, A_\mu$ break SUSY,

$$[Q, T_{\mu\nu}] \neq 0 \ , \qquad [Q, j_\mu] \neq 0 \qquad \text{(not BPS)} \ .$$

Preserving some of the supercharges requires additional background fields and/or additional geometric structures on $\mathcal{M}$.

- When and how can we preserve SUSY on $\mathcal{M}$?
- What extra data does the Lagrangian $\mathscr{L}_\mathcal{M}$ depend on?
- The partition function $Z_\mathcal{M} = \langle 1 \rangle$ is a BPS observable. How does it depend on the data in the Lagrangian?

Selected Examples

- Twisting [Witten]: consider a theory with R-symmetry G_R, a metric on $\mathcal{M}$ with holonomy G_{hol}, and a Q that is a singlet under $G_R \times G_{hol}|_{\text{diagonal}}$. Then Q can be preserved on $\mathcal{M}$.

More recently, other SUSY backgrounds (not twisting). Some backgrounds are highly symmetric and preserve all supercharges:

- $\mathcal{N} = 2$ on S^4 [Pestun], $\mathcal{N} = 2$ on S^3 [Kapustin, Willett, Yaakov; Jafferis; Hama, Hosomichi, Lee], $\mathcal{N} = (2,2)$ on S^2 [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee], ...
- $\mathcal{N} = 1$ on $S^3 \times S^1$ [D. Sen; Römelsberger], $\mathcal{N} = 2$ on $S^2 \times S^1$ [Imamura, Yokoyama; Kapustin, Willett], $\mathcal{N} = (2,0)$ on T^2 [Witten; Benini, Eager, Hori, Tachikawa; Gadde, Gukov], ... (Supersymmetric Indices)

Others are less symmetric and preserve fewer supercharges. They often come in continuous families labeled by some parameters:

- $\mathcal{N} = 2$ in $\Omega(\varepsilon_1, \varepsilon_2)$ background [Nekrasov; Nekrasov, Okounkov], $\mathcal{N} = 2$ on squashed S_b^3, $\mathcal{N} = (2,2)$ on squashed S_b^2 [Gomis, Lee], ...

Case Study: Squashed S^3_b

In $\mathcal{N} = 2$ theories with an R-symmetry, the partition function on a round S^3 can be computed using supersymmetric localization techniques [Kapustin, Willett, Yaakov; Jafferis; Hama, Hosomichi, Lee] + Talk by Willett. Many authors have generalized this to squashed spheres [Hama, Hosomichi, Lee; Gadde, Yan; Imamura; Imamura, Yokoyama; Martelli, Passias, Sparks; Nishioka, Yaakov; Alday, Martelli, Richmond, Sparks; ...]. The metric can contain arbitrary functions, in addition to continuous parameters.

Explicit localization computations suggest:

- $Z_{S^3_b}$ only depends on the geometry of the background through a single complex parameter b (squashing parameter).
- Some deformations of the background geometry do not affect $Z_{S^3_b}$, even though the metric changes.

More generally, examples suggest that $Z_{\mathcal{M}}$ only depends on a finite number of continuous parameters, rather than all the data used to define $\mathscr{L}_{\mathcal{M}}$ (several arbitrary functions).

Goal

In the remainder of the talk I will review a unified approach to supersymmetric theories on curved manifolds $\mathcal{M}$, describe the data that enters the Lagrangian $\mathscr{L}_{\mathcal{M}}$, and explain how this data affects the partition function $Z_{\mathcal{M}}$.

Note: I will have to omit many topics and references. I will restrict myself to R-symmetric theories with four supercharges in $d = 3, 4$ and focus on the partition function $Z_{\mathcal{M}}$. The methods are general and can be applied in many other examples in the literature.

- 4d $\mathcal{N} = 1$ theories on curved manifolds
- Constraints on $Z_{\mathcal{M}}$
- 3d $\mathcal{N} = 2$ theories, squashed S_b^3 revisited

4d $\mathcal{N} = 1$ Theories on Curved Manifolds

Now $T_{\mu\nu}$ resides in a supermultiplet with other currents. For theories with a $U(1)_R$-symmetry, we can use the $\mathcal{R}$-multiplet:

$$\mathcal{R} = \left(j_\mu^{(R)}, S_{\mu\alpha}, T_{\mu\nu}, C_{[\mu\nu]}\right)$$

It controls the coupling of the field theory to background fields, which reside in an off-shell supergravity multiplet:

$$\mathcal{H} = \left(A_\mu^{(R)}, \Psi_{\mu\alpha}, \Delta g_{\mu\nu}, B_{\mu\nu}\right) \ , \qquad V^\mu = \frac{i}{2}\varepsilon^{\mu\nu\rho\lambda}\partial_\mu B_{\nu\lambda}$$

A bosonic background preserves a supercharge Q if $\delta_Q \Psi_{\mu\alpha} = 0$ (independent of the field theory). Given a Lagrangian in flat space, it is very convenient to infer the curved space $\mathscr{L}_\mathcal{M}$ and SUSY transformation rules for the matter fields from the corresponding off-shell supergravity formulas. [Festuccia, Seiberg] + Talk by Festuccia (online)

Example: Free Chiral Multiplet

Consider a free chiral multiplet $\Phi = (\phi, \psi_\alpha, F)$ with R-charge r.
To obtain $\mathscr{L}_\mathcal{M}$ in a bosonic background satisfying $\delta_Q \Psi_{\mu\alpha} = 0$, we can take the linearized coupling to the $\mathcal{R}$-multiplet operators

$$j_\mu^{(R)} = ir\widetilde{\phi}\overleftrightarrow{\partial}_\mu \phi + r\widetilde{\psi}\widetilde{\sigma}_\mu\psi \,,\ \ldots\,,\ T_{\mu\nu} = (\cdots) + \frac{r}{2}\left(\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\right)\widetilde{\phi}\phi\,,$$

and find the non-linear completion using the Noether procedure:

$$\mathscr{L}_\mathcal{M} = \mathscr{L}_{\mathbb{R}^4}|_{\text{covariant}} + V^\mu\left(i\widetilde{\phi}\overleftrightarrow{D}_\mu \phi + \widetilde{\psi}\widetilde{\sigma}_\mu\psi\right) - r\left(\frac{1}{4}R - 3V^\mu V_\mu\right)\widetilde{\phi}\phi$$

The SUSY transformations of Φ are also modified (covariant). To show that $\mathscr{L}_\mathcal{M}$ is supersymmetric we must use $\delta_Q \Psi_{\mu\alpha} = 0$. It is much more convenient to take a rigid limit of the corresponding off-shell supergravity formulas (if available) [Sohnius, West; Festuccia, Seiberg].

Note the explicit dependence of $\mathscr{L}_\mathcal{M}$ on the choice of R-charge r, through covariant derivatives and curvature couplings.

Supersymmetry on Complex Manifolds

The condition $\delta_Q \Psi_{\mu\alpha} = 0$ leads to a generalized Killing spinor equation for the spinor ζ corresponding to Q:

$$\left(\nabla_\mu - i A_\mu^{(R)}\right)\zeta = \frac{i}{2} V_\mu \zeta - i V^\nu \sigma_{\mu\nu}\zeta \ , \quad V = *dB$$

This PDE has a solution $\Leftrightarrow$ $\mathcal{M}$ is a Hermitian manifold: it has an integrable complex structure $J^\mu{}_\nu$ and $g_{\mu\nu}$ is a compatible Hermitian metric. [TD, Festuccia, Seiberg; Klare, Tomasiello, Zaffaroni]

- Relation to twisting: If $\mathcal{M}$ is Kähler ($U(2)$ holonomy) we can find solutions with $V_\mu = 0$ [Johansen, Witten, Vyas].
- In general $\mathcal{M}$ need not be Kähler ($S^3 \times S^1$), $V_\mu \sim \nabla_\nu J^\nu{}_\mu$.
- The supercharge Q transforms as a scalar under holomorphic coordinate changes (crucial) and satisfies $Q^2 = 0$.
- $A_\mu^{(R)}, V_\mu$ are (partially) determined in terms of $J^\mu{}_\nu, g_{\mu\nu}$.
- More supercharges impose further constraints on $\mathcal{M}$.

Background Gauge Fields

If the field theory has continuous flavor symmetries, we can couple a background gauge field A_μ to the flavor current j_μ. (Focus on Abelian case.) With SUSY:

$$\mathcal{J} = (J, j_\alpha, j_\mu) \;, \qquad \mathcal{V} = (D, \lambda_\alpha, A_\mu)$$

As before, a bosonic configuration A_μ, D with $\lambda_\alpha = 0$ preserves Q if $\delta_Q \lambda_\alpha = 0$:

$$\big(F^{0,2}\big)_{\overline{ij}} = 0 \;, \quad F = dA \;, \quad D = -\frac{1}{2} J^{\mu\nu} F_{\mu\nu}$$

Thus SUSY background gauge fields $\Leftrightarrow$ holomorphic line bundles over the complex manifold $\mathcal{M}$.

Ingredients for $\mathscr{L}_{\mathcal{M}}$

The supersymmetric curved-space Lagrangian $\mathscr{L}_{\mathcal{M}}$ depends on:

- The integrable complex structure $J^{\mu}{}_{\nu}$
- A compatible Hermitian metric $g_{i\bar{j}}$
- Background gauge fields $\Leftrightarrow$ holomorphic line bundles
- Coupling constants, e.g. those of the original flat-space theory
- $\cdots$

Some of this data can be varied continuously, and the space of possible variations is infinite-dimensional (functions on $\mathcal{M}$).

What does $Z_{\mathcal{M}}$ Depend On?

We can constrain $Z_{\mathcal{M}}$ by varying the continuous data in $\mathscr{L}_{\mathcal{M}}$ and checking whether the change is Q-exact:

$$\Delta\mathscr{L}_{\mathcal{M}} = (\Delta\mathcal{M})\{Q, \mathcal{O}\} \quad \Rightarrow \quad \Delta Z_{\mathcal{M}} \sim \langle\{Q, \mathcal{O}\}\rangle = 0$$

In principle need full non-linear background supergravity

Simplification: work around flat space $\mathcal{M} \approx \mathbb{R}^4$. Then $\Delta\mathscr{L}_{\mathcal{M}}$ consists of operators in the stress-tensor supermultiplet with known, universal SUSY transformations. What about general $\mathcal{M}$?

Key Fact: Q is a scalar under holomorphic coordinate changes and this is enough to extend the result to general $\mathcal{M}$.

Compare to topologically twisted theories: Q a scalar under all coordinate changes and $T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}$ in flat space. Then the partition function does not depend on the metric for any $\mathcal{M}$.

Varying $J^\mu{}_\nu$ and $g_{\mu\nu}$

$$J^\mu{}_\nu \to J^\mu{}_\nu + \Delta J^\mu{}_\nu \ , \qquad g_{\mu\nu} \to g_{\mu\nu} + \Delta g_{\mu\nu}$$

Use holomorphic coordinates z^i adapted to $J^\mu{}_\nu$. The deformation must lead to another Hermitian structure:

$$\Delta J^i{}_j = \Delta J^{\bar{i}}{}_{\bar{j}} = 0 \ , \qquad \partial_{\bar{j}} \Delta J^i{}_{\bar{k}} - \partial_{\bar{k}} \Delta J^i{}_{\bar{j}} = 0 \ ,$$

$$\Delta g_{i\bar{j}} = \text{arbitrary} \ , \qquad \Delta g_{ij} = \frac{i}{2} \left(\Delta J_{ij} + \Delta J_{ji} \right) \ .$$

An infinitesimal diffeomorphism parametrized by ε^μ leads to $\Delta J^i{}_{\bar{j}} = 2i\partial_{\bar{j}}\varepsilon^i$. Non-trivial deformations correspond to cohomology classes in $H^{0,1}\left(\mathcal{M}, T^{1,0}\mathcal{M}\right)$. If $\mathcal{M}$ is compact, there are finitely many complex structure moduli.

Obtaining $\Delta\mathscr{L}_{\mathcal{M}}$

Recall the coupling of the supercurrent multiplet to the bosonic supergravity background fields:

$$-\frac{1}{2}\Delta g^{\mu\nu}T_{\mu\nu} + A^{(R)\mu}j^{(R)}_{\mu} + B^{\mu\nu}C_{\mu\nu}$$

Since $A^{(R)}_{\mu}$, $B_{\mu\nu}$ are expressed in terms of $J^{\mu}{}_{\nu}$, $g_{\mu\nu}$, we can perform the infinitesimal deformations $\Delta J^{\mu}{}_{\nu}, \Delta g_{\mu\nu}$ to obtain:

$$\Delta\mathscr{L}_{\mathcal{M}} = -\Delta g^{i\bar{j}}\mathcal{T}_{i\bar{j}} - i\sum_{j}\Delta J^{\bar{i}}{}_{j}\mathcal{T}_{\overline{j}\overline{i}} + i\sum_{j}\Delta J^{i}{}_{\bar{j}}\Big(\mathcal{T}_{ij} + i\partial_{j}j^{(R)}_{i}\Big)$$

$$\mathcal{T}_{\mu\nu} = T_{\mu\nu} + \frac{1}{4}C_{\mu\nu} - \frac{i}{4}\varepsilon_{\mu\nu\rho\lambda}\partial^{\rho}j^{(R)\lambda} - \frac{i}{2}\partial_{\nu}j^{(R)}_{\mu}$$

Q-Exactness of Deformations

$$\Delta\mathscr{L}_{\mathcal{M}} = -\Delta g^{i\bar{j}}\mathcal{T}_{i\bar{j}} - i\sum_j \Delta J^{\bar{i}}{}_j \mathcal{T}_{\bar{j}\bar{i}} + i\sum_j \Delta J^i{}_{\bar{j}}\Big(\mathcal{T}_{ij} + i\partial_j j_i^{(R)}\Big)$$

Are any of these operators Q-exact? The only fermionic operator in the same multiplet as $\mathcal{T}_{\mu\nu}, j_\mu^{(R)}$ is the supersymmetry current:

$$\{Q, S_{\mu\alpha}\} = 0\ , \qquad \{Q, \widetilde{S}_{\mu\dot{\alpha}}\} \sim \mathcal{T}_{\mu\bar{i}}$$

We conclude:

- $Z_{\mathcal{M}}$ does not depend on the Hermitian metric $g_{i\bar{j}}$.
- $Z_{\mathcal{M}}$ only depends on $\Delta J^i{}_{\bar{j}}$ but not on $\Delta J^{\bar{i}}{}_j$, i.e. it is a holomorphic function of the complex structure moduli.

Comments

- Independence of $g_{i\bar{j}}$ means invariance of $Z_{\mathcal{M}}$ under scale changes $x^\mu \to \lambda x^\mu$. Thus $Z_{\mathcal{M}}$ is a renormalization group invariant: it can be computed in the UV or the IR, and it must be invariant under (IR) duality.
- The argument only shows that $Z_{\mathcal{M}}$ is locally holomorphic in the complex structure moduli. Sometimes there are singularities (they should be understood better).
- If $\mathcal{M}$ is compact, there is a finite number of complex structure moduli (infinite $\Rightarrow$ finite).
- Applying the same arguments to flavor current multiplets, it follows that $Z_{\mathcal{M}}$ only depends on background gauge fields through the corresponding holomorphic line bundles. It is a locally holomorphic function of the bundle moduli (finitely many, if $\mathcal{M}$ is compact).

Example: $S^3 \times S^1$

Kodaira: Complex Manifolds diffeomorphic to $S^3 \times S^1$ are primary Hopf surfaces,

$$\mathcal{M}^{p,q} = \mathbb{C}^2 - (0,0)/(w,z) \sim (pw, qz) \ , \qquad 0 < |p| \leq |q| < 1 \ .$$

The partition function must be a locally holomorphic function of the complex structure moduli p, q. If there is an Abelian background gauge field, $Z_{\mathcal{M}}$ must be locally holomorphic in the corresponding holomorphic line bundle modulus u.

One can show that $Z_{\mathcal{M}}(p,q,u)$ is nothing but the supersymmetric index $\mathcal{I}(p,q,u)$ for states on $S^3 \times \mathbb{R}$ [Römelsberger; Dolan, Osborn; . . .] at general complex fugacities. In an SCFT, it counts BPS operators in the flat-space theory [Kinney, Maldacena, Minwalla, Raju; . . .].

3d $\mathcal{N} = 2$ Theories on Curved Manifolds

Closely related to 4d $\mathcal{N} = 1$ theories by (twisted) dimensional reduction. The $\mathcal{R}$-multiplet now contains the operators [TD, Seiberg]

$$\mathcal{R} = \left(j_\mu^{(R)}, S_{\mu\alpha}, T_{\mu\nu}, j_\mu^{(Z)}, J \right)$$

and the corresponding background supergravity fields are

$$\mathcal{H} = \left(A_\mu^{(R)}, \Psi_{\mu\alpha}, \Delta g_{\mu\nu}, C_\mu, H \right)$$

Now the condition $\delta_Q \Psi_{\mu\alpha} = 0$ for supersymmetric backgrounds leads to the Killing spinor equation

$$\left(\nabla_\mu - A_\mu^{(R)} \right) \zeta = -\frac{1}{2} H \gamma_\mu \zeta + \frac{i}{2} V_\mu \zeta - \frac{1}{2} \varepsilon_{\mu\nu\rho} V^\nu \gamma^\rho \zeta$$

Transversely Holomorphic Foliations

A Killing spinor ζ exists $\Leftrightarrow$ $\mathcal{M}$ admits a transversely holomorphic foliation (THF) and the metric is transversely Hermitian [Closset, TD, Festuccia, Komargodski]:

- A nowhere vanishing unit vector field ξ^μ, which provides a local $2+1$ decomposition.
- An integrable complex structure J on the $2d$ transverse space, such that J is invariant under flows of ξ, i.e. $\mathcal{L}_\xi J = 0$.
- In the compact case, they are completely classified [Brunella, Ghys]. Topologically, Seifert manifolds or T^2 bundles over S^1.
- Many similarities to complex manifolds:
  - (p, q)-forms, $\overline{\partial}$-operator, Dolbeault cohomology $H^{p,q}(\mathcal{M})$
  - Holomorphic line bundles $\Leftrightarrow$ SUSY background gauge fields.
  - Both structures are parametrized by finitely many complex moduli corresponding to certain $\overline{\partial}$-cohomology classes.

Q-Exactnes in 3d

The supersymmetric Lagrangian $\mathscr{L}_{\mathcal{M}}$ depends on:

- The transversely holomorphic foliation (THF) on $\mathcal{M}$
- A choice of transversely Hermitian metric
- Background gauge fields $\Leftrightarrow$ holomorphic line bundles

Applying the same logic as in $4d$, we obtain the following constraints on the parameter dependence of $Z_{\mathcal{M}}$:

- It does not depend on the transversely Hermitian metric.
- It is a locally holomorphic function of the complex moduli parametrizing the possible THFs on $\mathcal{M}$.
- It only depends on background gauge fields (including real masses) through the corresponding holomorphic vector bundles. It is locally holomorphic in the bundle moduli.

Squashed S_b^3 Revisited

We can use this understanding to explain the observed behavior of partition functions on squashed three-spheres:

The moduli space of THFs on manifolds diffeomorphic to S^3 is well understood [Brunella, Ghys]. The component that contains the usual supersymmetric round sphere of [Kapustin, Willett, Yaakov.; Jafferis; Hama, Hosomichi, Lee] is one-dimensional. Therefore, all squashed S_b^3 partition functions should only depend on a single complex parameter – the squashing parameter b – regardless of how complicated the squashing is. This also shows that no interesting new squashings exist on this branch.

Distinct squashings that give the same value of b correspond to the same THF, but different transversely Hermitian metrics.

The Superconformal R-Symmetry

The SUSY theories on $S^3 \times S^1$ or S^3 depend on a choice of R-symmetry, which affects the curvature couplings. In an SCFT, they are fixed by conformal invariance. Agreement requires choosing the correct superconformal R-symmetry. In $4d$, it can be determined in flat space using a-maximization [Intriligator, Wecht].

In $3d$, the analogous principle is F-maximization: consider Z_{S^3} with a background gauge field for a global flavor current j_μ. It only depends on one holomorphic line bundle modulus u:

$$Z_{S^3} = e^{-F_{S^3}(u)} , \qquad F_{S^3}(u) = F_{S^3}(m + it)$$

Here t controls the mixing of j_μ with the R-symmetry [Jafferis; Festuccia, Seiberg]. Derivatives with respect to t compute integrated correlation functions of j_μ or its superpartners. In the SCFT:

$$\langle j_\mu \rangle = 0 \quad \Rightarrow \quad \partial_t \, \mathrm{Re}\, F_{S^3}|_{\text{SCFT}} = 0 \qquad \text{[Jafferis]}$$

SCFT Correlation Functions

$$\langle j_\mu j_\nu \rangle \sim \tau > 0 \quad \Rightarrow \quad \partial_t^2 \operatorname{Re} F_{S^3}|_{\text{SCFT}} = -\frac{\pi^2}{2}\tau < 0$$

[Closset, TD, Festuccia, Komargodski, Seiberg]. Subtleties due to $\operatorname{Im} F_{S^3}|_{\text{SCFT}}$.

- Once the superconformal point has been found, the second t-derivative can be used to compute $\langle j_\mu j_\nu \rangle \sim \tau$ in the SCFT.
- Higher-order t-derivatives compute integrated higher-point correlators of j_μ in the SCFT.
- Similarly, we can squash the sphere slightly at the SCFT point and compute derivatives with respect to the squashing parameter b. They compute integrated correlation functions of the stress tensor $T_{\mu\nu}$ [Closset, TD, Festuccia, Komargodski], such as

$$\langle T_{\mu\nu} T_{\rho\lambda} \rangle \sim C_T \sim \partial_b^2 \operatorname{Re} F_{S_b^3}|_{b=1}$$

Can these protected correlation functions be computed directly in flat space?

Conclusions

- Supersymmetric QFT on curved manifolds can be described using background supergravity. Around flat space, the coupling proceeds via the stress-tensor supermultiplet. It is a powerful tool for analyzing the curved-space theory.
- 4d $\mathcal{N} = 1$ theories with an R-symmetry require $\mathcal{M}$ to be a Hermitian manifold.
  - $Z_{\mathcal{M}}$ does not depend on the Hermitian metric
  - $Z_{\mathcal{M}}$ depends holomorphically on complex structure and line bundle moduli
- Similar results for 3d $\mathcal{N} = 2$ theories with an R-symmetry
- This explains many observations in the literature and constrains $Z_{\mathcal{M}}$ in situations where no computations are available (complement to explicit localization computations).
- General method that can be applied to many classes of theories in diverse dimensions.