Overview: Entanglement Entropy

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Quantum Fields beyond Perturbation Theory
KITP

0 Intro & disclaimer

Over past 10 years, explosion of activity in entanglement entropy in QFT:

- many conceptual ramifications
- enormous number of applications.

It would be impossible for me to cover even just the most important developments in 50 minutes. Instead, I will focus on a small & idiosyncratic selection of topics, emphasizing some important open problems.

Citations will generally be limited to the paper that (as far as I know) initiated the given subject.

1 Basic definitions

Divide a quantum system into subsystems A, A^c, such that $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$.

Entangled pure state: $|\psi\rangle = \sum_i \lambda_i |i\rangle_A |i\rangle_{A^c}$.

Entanglement leads to mixedness of reduced density matrix:

$$\rho_A := \mathrm{tr}_{\mathcal{H}_{A^c}} |0\rangle\langle 0| = \sum_i |\lambda_i|^2 \, |i\rangle_A \langle i|_A$$

Best way to quantify amount of entanglement is by entropy of ρ_A—*entanglement entropy*:

$$S(A) := -\,\mathrm{tr}\,\rho_A \ln \rho_A = -\sum_i |\lambda_i|^2 \ln |\lambda_i|^2$$

Entanglement is a ubiquitous phenomenon in quantum systems.
EE is a central concept in quantum statistical mechanics & quantum information theory.

For a mixed state ρ, we similarly define

$$\rho_A := \mathrm{tr}_{\mathcal{H}_{A^c}} \rho\,, \qquad S(A) := -\,\mathrm{tr}\,\rho_A \ln \rho_A\,.$$

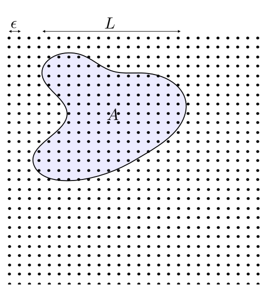
Also called EE (although it doesn't measure just entanglement).

2 Quantum field theories

Consider a lattice system in $D-1$ spatial dimensions with Hilbert space $\mathcal{H} = \bigotimes_{\text{sites } i} \mathcal{H}_i$, local Hamiltonian.

Let A be a region of size $L \gg \epsilon =$ lattice spacing. Two patterns are observed:

- In a generic state,
$$S(A) \sim \ln \dim \mathcal{H}_A \sim (\#\text{ lattice sites in } A) \sim (L/\epsilon)^{D-1}$$
- In the ground state (or other low-lying pure state),
$$S(A) \sim (\#\text{ links cut by } \partial A) \sim (L/\epsilon)^{D-2} \quad (\sim \ln\epsilon \text{ in } D=2)$$
(Bombelli, Koul, Lee, Sorkin '86, . . .)

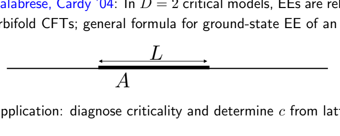


Typical physical states have very *little* entanglement, & most of it is local.
(Basis for efficient numerical simulation methods: build lack of entanglement into variational ansatz—DMRG, MPS, PEPS, MERA, etc.; see review Vidal '09. Can it be applied to lattice gauge theories?)

In $\epsilon \to 0$ limit, system may be describable by a QFT. Holding A fixed, $S(A) \to \infty$. By removing UV-divergent part, can we extract "universal" quantities that characterize the QFT (are independent of lattice realization)?

Following some early work (Bombelli, Koul, Lee, Sorkin '86, Srednicki '93, Callan & Wilczek '94, Holzhey, Larsen, Wilczek '94, . . .), starting ~ 10 years ago, 3 key advances convinced people that the answer is *yes*, and that EE is a useful tool for studying QFTs:

- Calabrese, Cardy '04: In $D=2$ critical models, EEs are related to twist-field correlation functions in cyclic orbifold CFTs; general formula for ground-state EE of an interval:



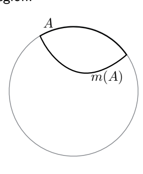
$$S(A) = \frac{c}{3} \ln \frac{L}{\epsilon} + (\text{non-universal constant})$$

  Application: diagnose criticality and determine c from lattice simulations.

- Kitaev, Preskill '05, Levin, Wen '05: In massive $D=3$ theories, for a simply-connected region much larger than correlation length,
$$S(A) = \frac{L}{\epsilon} \times (\text{non-universal constant}) - \gamma\,,$$
  where *topological EE* γ characterizes topological QFT that controls IR.

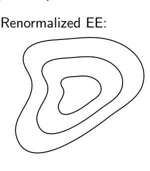
  Application: diagnosing topological order & phase transitions.

- Ryu, Takayanagi '06: In holographic CFT, conjecture for EE of arbitrary region:
$$S(A) = \frac{1}{4G_N} \mathrm{area}(m(A))$$
$$m(A) = \text{minimal surface in bulk anchored on } \partial A$$
  UV divergence arises from part of $m(A)$ near boundary.
  Application: many (see below).



Examples of quantities that are (believed to be) UV-finite & universal, characterize physics at scale L:

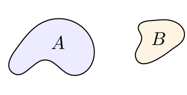
- Renormalized EE:



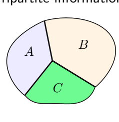
$$D=2: \qquad \mathcal{F}(L) = L\frac{d}{dL} S(A_L)$$
$$D=3: \qquad \mathcal{F}(L) = \left(L\frac{d}{dL} - 1\right) S(A_L)$$
$$D=4: \qquad \mathcal{F}(L) = \frac{1}{2} L\frac{d}{dL}\left(L\frac{d}{dL} - 2\right) S(A_L)$$
$$\vdots$$

  where A_L is a family of regions related by uniform dilatation.
  (Casini, Huerta '04, . . . , Liu, Mezei '12, . . .)

- Mutual information:
$$I(A:B) := S(A) + S(B) - S(AB)$$
  where A, B do not share a boundary.
  Measures entanglement + classical correlation.
  (Calabrese, Cardy '04, . . .)



- Tripartite information:



$$I_3(A:B:C) := S(ABC) + S(A) + S(B) + S(C) - S(AB) - S(AC) - S(BC)$$
$$= I(A:B) + I(A:C) - I(A:BC)$$

  Measure correlations of correlations.
  (Kitaev, Preskill '05, . . .)

Since 2004, an explosion of activity in the study of EEs in QFTs, addressing old problems and posing new ones. Examples (almost all are studied in both holographic & non-holographic theories):

- Dependence of $S(A)$ on state, & on geometry & topology of A (e.g. divergences from singularities in geometry of ∂A).
- Characterizing fixed points and constraining RG flows (see talk by Myers).
- Probe of confinement (Kutasov, Klebanov, Murugan '07, Velytsky '08, . . .).
- Condensed-matter applications (e.g. probe of Fermi liquid vs. non-Fermi-liquid behavior).
- Probe of quenches & thermalization processes, "propagation" of entanglement (Calabrese, Cardy '05, Abajo-Arrastia, Aparício, López '10, . . .)
- Probe of correlations of fields on cosmological backgrounds (Maldacena, Pimentel '12, . . .).
- Bekenstein-Hawking entropy as EE of fields on black-hole background: $S_{\rm BH} \stackrel{?}{=} S(\text{exterior})$ (Bombelli, Koul, Lee, Sorkin '86, . . .).
- Structure of reduced density matrix ρ_A (Casini, Huerta '09, Hung, Myers, Smolkin, Yale '11,. . .).

There remain outstanding problems of practice & principle in definition & calculation of EEs in QFTs, e.g.:

- It is awkard that to define EE one has to fall back on a lattice regularization (e.g. breaks Lorentz invariance). Also, in (lattice or continuum) theory with (discrete or continuous) gauge symmetry, $\mathcal{H} \neq \mathcal{H}_A \otimes \mathcal{H}_{A^c}$, because of Gauss-law constraint (Buividovich, Polikarpov '08, . . .); to define EE one must enlarge the Hilbert space in an ad hoc way.

  The *entanglement Rényi entropy*
$$S_\alpha(A) := \frac{1}{1-\alpha} \ln \mathrm{tr}\, \rho_A^\alpha$$
  is also a measure of entanglement, and can be calculated for $\alpha = 2, 3, \ldots$ in terms of a Euclidean path integral on a certain branched-cover manifold. This can be regulated in any convenient way and automatically takes proper account of gauge symmetries (Headrick, Lawrence, Roberts '12, . . .). However, the ERE does not have most of the special properties of EE.

  Challenge 1: Give a good general field-theoretic definition of EE (or of universal quantities derived from it).

- The main tool for calculating EEs analytically in (non-holographic) QFTs is the replica trick (Holzhey, Larsen, Wilczek '94, . . .):
  - calculate $S_\alpha(A)$ for $\alpha = 2, 3, \ldots$
  - fit to analytic function of α
  - evaluate at $\alpha = 1$.

  This method is roundabout & often impossible to carry out. E.g. for free compact boson CFT, EE of 2 separated intervals remains unknown: $S_\alpha(A)$ is known explicitly for $\alpha = 2, 3, \ldots$, but it is not known how to fit the values to an analytic function (Calabrese, Cardy, Tonni '09, . . .).

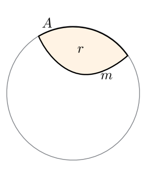
  Challenge 2: Find a more direct way to analytically calculate EEs.

Hilbert space of a QFT can also be factorized in other ways (e.g., given a perturbative realization, by regions of momentum space: Balasubramanian, McDermott, Van Raamsdonk '11, . . .). Is it possible to extract universal quantities from such EEs?

3 Ryu-Takayanagi formula

Consider a theory holographically dual to classical Einstein gravity, in a state described by a static spacetime. Let A lie on a constant-time slice Σ. Ryu-Takayanagi formula:

$$S(A) = \frac{1}{4G_N} \min_m (\mathrm{area}(m))$$
$$m = \text{codimension-2 surface in bulk s.t. } \exists r \subset \Sigma \text{ with } \partial r = A \cup m$$



There is a large amount of evidence that RT formula is correct:

- agrees with EEs computed from first principles (usually using replica trick) in many specific cases (both divergent & finite parts); general argument (Lewkowycz, Maldacena '13)
- obeys all applicable properties of EEs: strong subadditivity & many more (Headrick, Takayanagi '07, . . .)
- has been applied to a wide variety of holographic systems, always giving physically reasonable results

In addition to specific applications, RT formula also implies special properties of EEs in holographic theories, e.g.

- if A is varied continuously, $S(A)$ is continuous but has phase transitions due to competing minimal surfaces
- monogamy of mutual information:
$$I_3(A:B:C) \le 0\,, \quad \text{i.e.} \quad I(A:BC) \ge I(A:B) + I(A:C)$$
  (Hayden, Headrick, Maloney '11, . . .)

Effects of higher-derivative & perturbative quantum $(1/N)$ corrections in bulk have been investigated (Fursaev '06, Faulkner, Lewkowycz, Maldacena '13, . . .).

There are also important non-perturbative quantum $(1/N)$ corrections, e.g. to smooth out phase transitions, which are not understood. E.g. in presence of competing minimal surfaces m_1, m_2, replica-trick calculation following Lewkowycz, Maldacena '13 gives

$$S(A) = \frac{1}{4G_N} (\mathrm{area}(m_1) + \mathrm{area}(m_2))$$

(Myers, personal comm.), but RT formula says

$$S(A) = \frac{1}{4G_N} \min(\mathrm{area}(m_1), \mathrm{area}(m_2))$$

Challenge 3: Understand non-perturbative quantum corrections to RT formula.

EREs have also been calculated in holographic theories (Headrick '10, . . .). Involves finding bulk solutions with complicated boundary conditions (vs. applying RT to given solution for EE).
Challenge 4: Understand the structure of ρ_A in holographic theories.

Evidence from several directions suggests that a large class of large-N theories shares special properties of holographic EEs, e.g.

- $S(A)$ given by minimization
- phase transitions in $S(A)$ as A is varied
- strict vanishing (at order $1/N^2$) of $I(A:B)$ when A, B are sufficiently far apart
- generic saturation (at order $1/N^2$) of Araki-Lieb inequality: $S(AB) = S(A) - S(B)$
- monogamy of mutual information

(Headrick '10, . . .).
Challenge 5: Understand EEs and reduced density matrices in general large-N theories.
Hint: Large-N limits can be thought of as thermodynamic limits. In a thermodynamic limit, some observables are good thermodynamic variables: respect ensemble equivalence (e.g. same for microcanonical & canonical ensembles), are functions of the thermodynamic state. E.g.

- $\langle E \rangle$ (not ΔE)
- entropy S (not Rényi entropy S_α)
- in large-N theories: single-trace operators (not multi-trace)

In holographic theories

complete thermodynamic description = bulk classical field configuration

RT formula shows that EE is a good thermodynamic variable (ERE is not). True for general large-N theories?

4 Hubeny-Rangamani-Takayanagi formula

To avoid RT formula's static condition, Hubeny, Rangamani, Takayanagi ('07) proposed a covariant generalization:

$$S(A) = \frac{1}{4G_N} \min_m (\mathrm{area}(m))\,,$$

where now m is an extremal spacelike codimension-2 surface s.t. $\exists$ (spacelike? achronal?) codimension-1 surface r with $\partial r = A \cup m$.

Reduces to RT in static case.

HRT formula has been applied in many cases, always giving physically reasonable results.
Also, there is evidence that it obeys strong subadditivity (subject to null energy condition; Allais, Tonni '11,. . .).

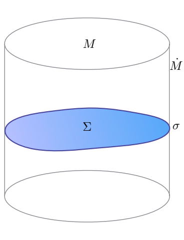
There remain many fundamental questions.
Challenge 6: Show that, under physically reasonable assumptions, HRT formula has (or doesn't have) the following properties:

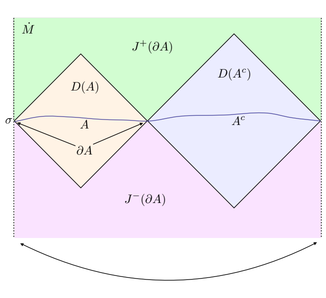
- a surface m exists
- strong subadditivity is obeyed
- $S(A)$ is continuous under continuous variations in A
- monogamy of mutual information is obeyed
- it can be derived!

By reformulating HRT in terms of a maximin construction, Wall ('12) has made significant progress on several of these problems.

Finally: Another strong check on consistency of HRT (Headrick, Hubeny, Lawrence, Rangamani '14):

M = bulk
$\dot M$ = boundary
Σ = Cauchy slice of M
$\sigma := \Sigma \cap \dot M$ = Cauchy slice of $\dot M$





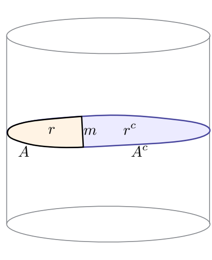
Given a subset $A \subset \sigma$, $A^c := \sigma \setminus A$, there is a natural decomposition of $\dot M$ into 4 spacetime regions:

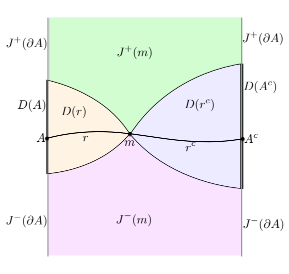
$$D(A)\,, \quad D(A^c)\,, \quad J^+(\partial A)\,, \quad J^-(\partial A)$$

separated by Cauchy horizons of A, A^c.

m = HRT extremal codimension-2 surface for A, $m \cap \dot M = \partial A$
$r \subset \Sigma$ is codimension-1 surface interpolating between m & A: $\partial r = m \cup A$
$r^c := \Sigma \setminus r$
Decomposition of M into 4 regions:

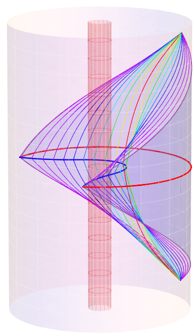
$$D(r)\,, \quad D(r^c)\,, \quad J^+(m)\,, \quad J^-(m)$$



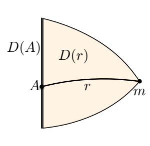


Theorem: If bulk obeys NEC, then boundary & boundary decompositions are compatible:

$$D(A) = D(r) \cap \dot M$$
$$D(A^c) = D(r^c) \cap \dot M$$
$$J^+(\partial A) = J^+(m) \cap \dot M$$
$$J^-(\partial A) = J^-(m) \cap \dot M$$

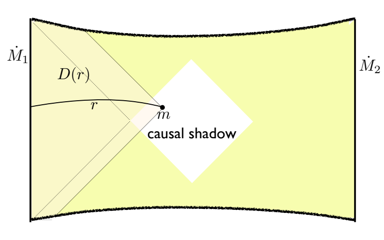


Suggests that $D(r)$ is dual of field theory on $D(A)$ in state ρ_A (cf. Czech, Karczmarek, Nogueira, Van Raamsdonk '12).



Protects $S(A)$ from being influenced by perturbations in $D(A)$ or $D(A^c)$, which would violate causality in boundary theory.

For example, suppose M has two boundaries $\dot M_1$, $\dot M_2$: entangled but non-interacting CFTs. Theorem prevents $S(A)$ for $A \subset \dot M_1$ from being influenced by $\dot M_2$.
For $A = \dot M_1$, theorem implies m lies inside "causal shadow", the set of points not causally connected to either boundary. $D(r)$ contains (but is generically larger than) exterior region of black hole.



5 Advertisement

KITP Program: Entanglement in Strongly-Correlated Quantum Matter

April-July 2015

Organizers & advisors:
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