Quantum Fields Beyond Perturbation Theory, KITP, January 2014

Overview: Conformal Bootstrap

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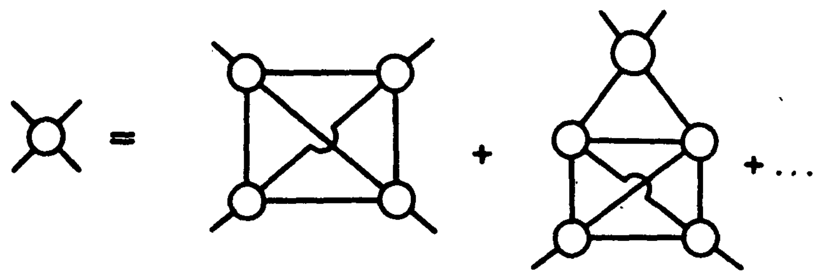
See also David Poland's recorded review talk:
http://online.kitp.ucsb.edu/online/qft14/poland/

Disambiguation

Conformal bootstrap I [Polyakov '70, Migdal '71]



$$\circ\!\!-\!\!\!-\!\!\!-\!\!\circ \;=\; \frac{1}{(x_1 - x_2)^{2\Delta}}$$

(of historical interest only)

Here:

Conformal bootstrap II [Ferrara,Gatto,Grillo '73, Polyakov'74, BPZ'83]

non-perturbative, non-Lagrangian approach to solving/constructing CFTs

to introduce it in some detail...

CFT$_{d\geq3}$ kinematics in flat space

(1) Local operators = primaries + descendants (derivatives)

Quantum numbers: Δ_a - scaling dimension, R_a - irrep of SO(d)
⇒ fix 2-pt function

$$\langle O_a(x) O_b(0)\rangle = \frac{\delta_{ab}}{x^{2\Delta_a}} \times \text{(fixed tensor structure)}$$

(2) 3-pt functions of primaries:

$$\langle O_a(x) O_b(y) O_c(z)\rangle = \sum_{i\leq I(R_a,R_b,R_c)} f^i_{abc} \times (\text{tensor structure})_i$$

(2)⇔(2') OPE

$$O_a(x) O_b(y) = \sum_c \left\{ \sum_i f^i_{abc} P_i(x-y, \partial_y) O_c(y) \right\}$$

Dynamics

whatever fixes Δ_a, R_a and f_{abc}

RG way

view CFT_{IR} as a fixed point of a Lagrangian RG flow

(use CFT kinematics to organize end results)

Bootstrap way

impose associativity condition on the operator algebra:

$$O_a(x)O_b(y)O_c(z) = O_a(x)O_b(y)O_c(z) \qquad \forall a, b, c$$

$$\Updownarrow$$

$$\langle O_a O_b O_c O_d \rangle = \langle O_a O_b O_c O_d \rangle \qquad \forall a, b, c, d$$

Why bother?

practice and principle

- RG method has run out of steam

Example 1: **3d Ising CFT**

	Z2	Δ (RG)
σ	-	0.51675(125)
ε	+	1.4137(33)
ε'	+	3.799(11)

Δ (MC)	
0.51814(5)	factor 25 better
1.41275(25)	factor 13 better
3.832(6)	factor 2 better

Example 2: **3d O(2) model**

Δ_ε =1.5094(2) (He^4 exp)
=1.5112(2) (Lattice) 8σ discrepancy
=1.5081(33) (from RG) inconclusive

- Bootstrap can be used to construct CFTs for which RG interpretation is unknown or does not exist

Bonus points

Bootstrap equations may be our best first-principle definition of CFTs

- convergent, mathematically well-defined

- give results with rigorous error bars

forget about divergences, resummations, asymp. series

O(100) papers since 2008

Red = to do

$2<d\leq 4$ and d=2 SL(2,C)

- bounds (dims, ope coeffs, central charges)
- extremal spectrum studies
- numerical techniques (simplex method, SDPA, dual/ direct) ellipsoid method minor method by Gliozzi
- global syms
- impact of SUSY
- large N ↔ AdS
- lightcone results: large spin, small twist numerical impact?
- several correlators
- basis optimization
- external states with spin (T, J)

Conformal blocks

- -exact expressions
- -power series expansions
- -recursions
- -for ops with spin
- -large d limit

CFT_d with bdry

Conformal defects ↔ d=1 bootstrap d→ 1 limit

[Study of 2d CFT torus partition functions]

d=2 non-rational Virasoro bootstrap

Bootstrap on other geometries (R^{d-1} x S^1)

Why bootstrap is practical- operator decoupling

2d Minimal models

•finitely many primaries, dims known
•bootstrap for ope coeffs = **finite** dimensional linear algebra

CFT($d\geq 3$) & 2d non-rational

- ∞ many primaries, dims unknown
- bootstrap = system of ∞ eqs for ∞ unkowns

Any truncation in Δ space?

High-dim operators decouple exponentially fast (any d):

$$\langle\phi(0)\phi(z)\Big|_{\Delta\geq\Delta_*}\phi(1)\phi(\infty)\rangle \leq C\Delta_*^{2\Delta_\phi}|z|^{\Delta_*} \qquad (\Delta_*\geq\Delta_\phi/(1-z))$$

Pappadopulo, SR, Espin, Rattazzi

Proof:

$$\langle\phi(0)\phi(z)\phi(1)\phi(\infty)\rangle \begin{array}{l} = \sum f_\Delta^2 z^\Delta \\ \sim \dfrac{1}{(1-z)^{2\Delta_\phi}} \qquad (z\to 1)\end{array}$$

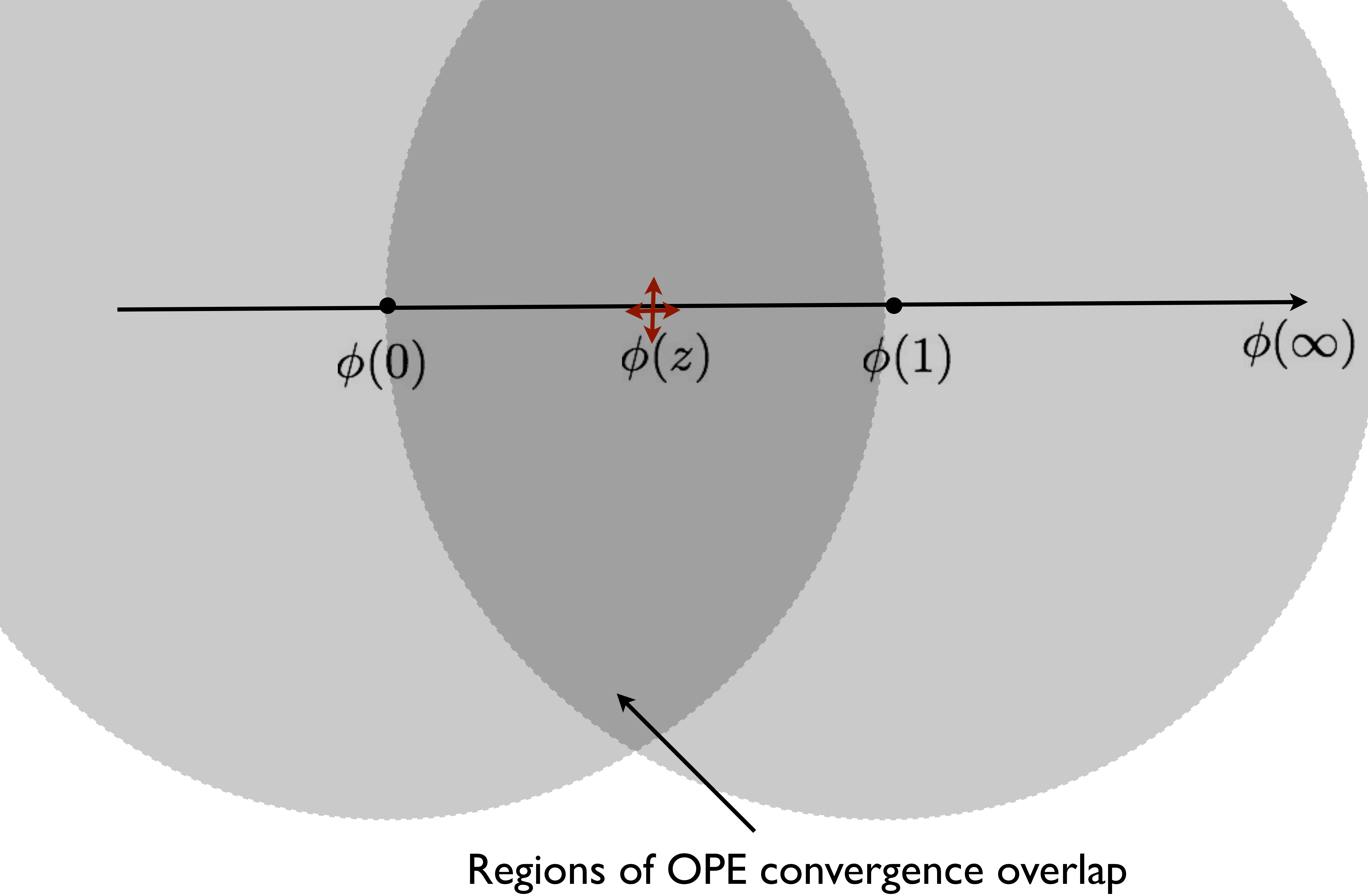
$$\sum_{\Delta\geq\Delta_*} f_\Delta^2 z^\Delta = \sum f_\Delta^2 z_*^\Delta \left(\frac{z}{z_*}\right)^{\Delta} \qquad \text{(rewriting)}$$

$$\leq \left[\sum f_\Delta^2 z_*^\Delta\right]\left(\frac{z}{z_*}\right)^{\Delta_*} \qquad (z<z_*<1)$$

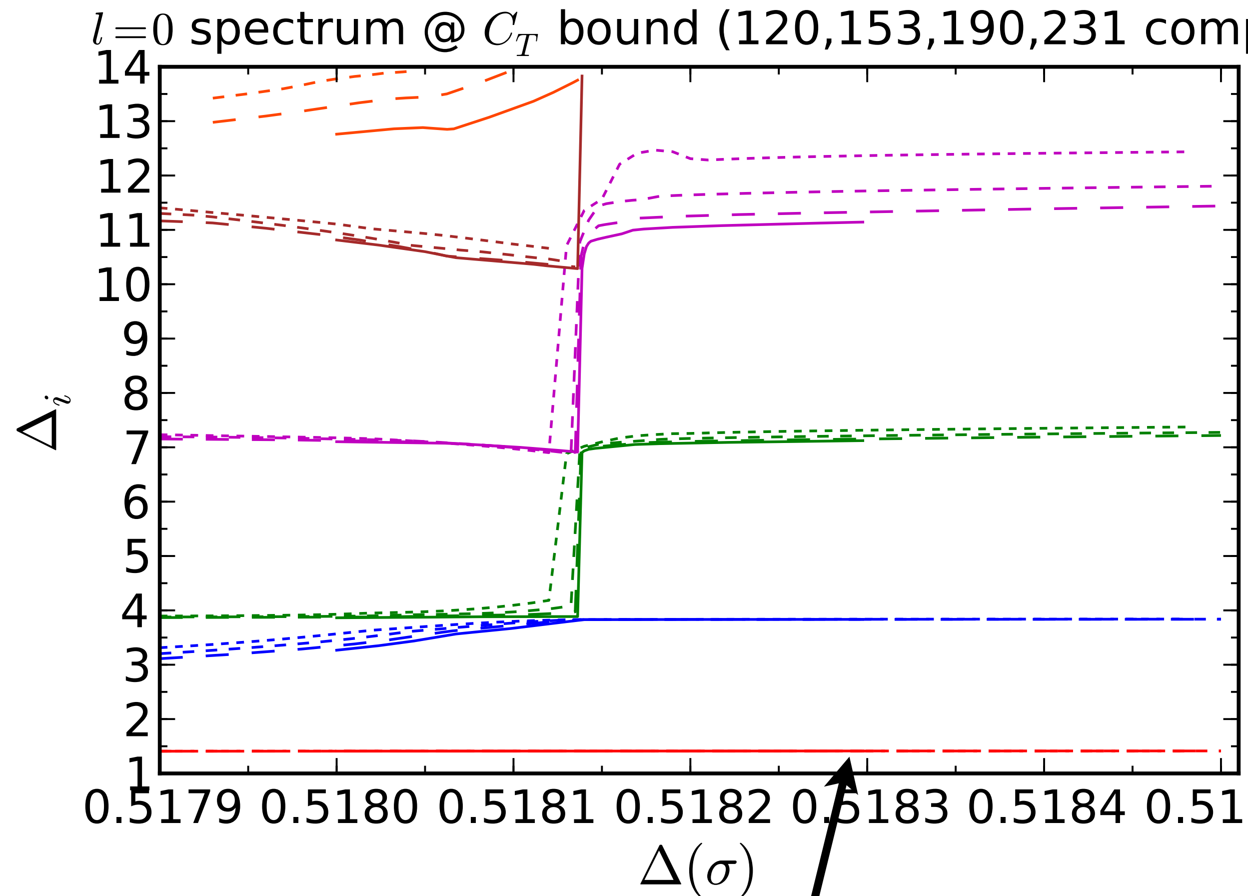
$$\leq \frac{C}{(1-z_*)^{2\Delta_\phi}}\left(\frac{z}{z_*}\right)^{\Delta_*}$$

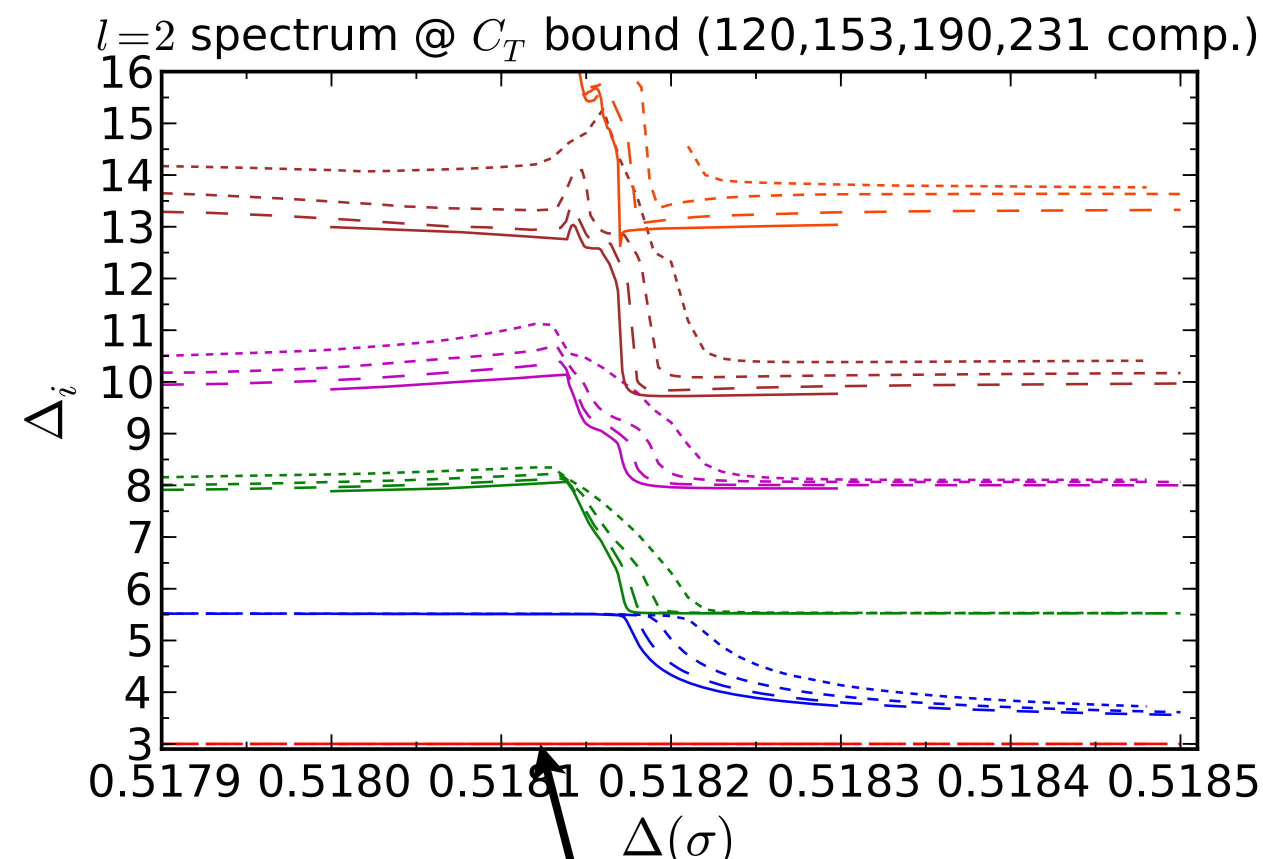
Now pick z* optimally: $z_* = 1 - O(\Delta_\phi/\Delta_*)$

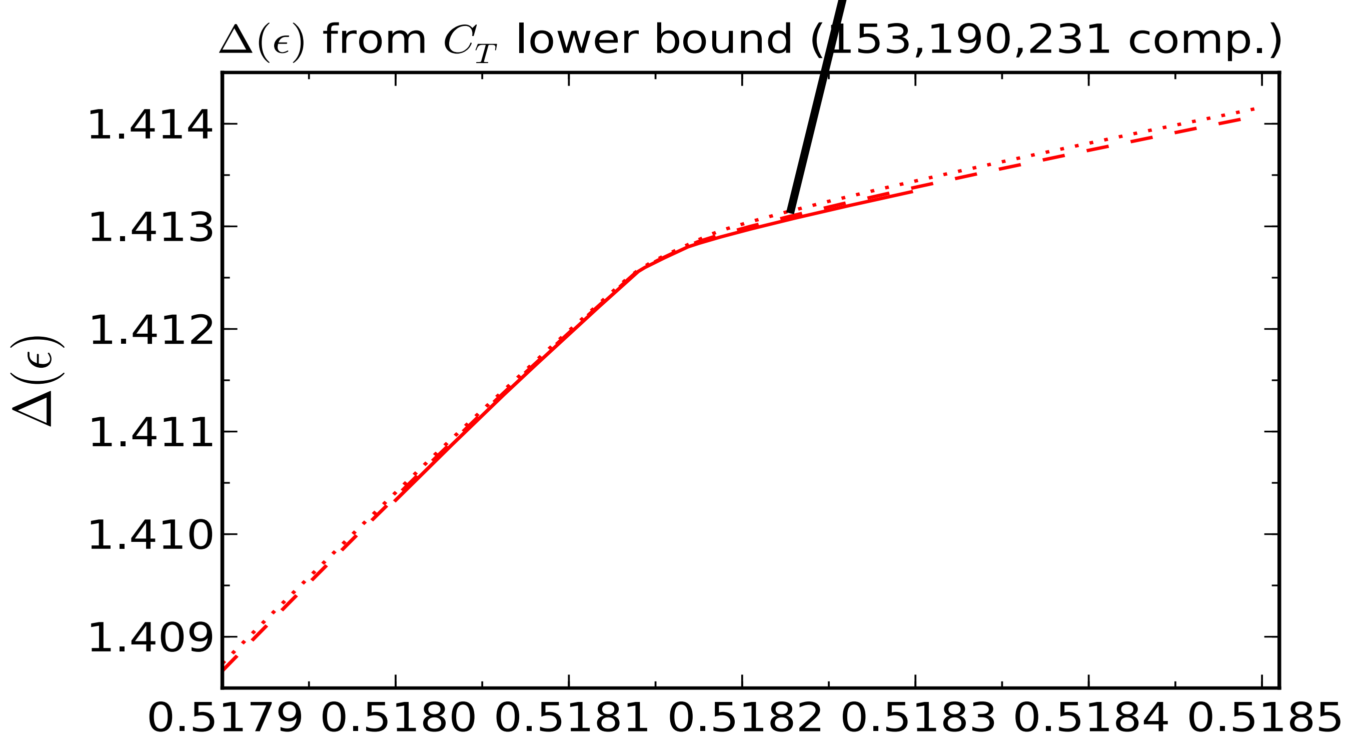
We do numerical bootstrap around z~1/2



A family of spectra solving the bootstrap equation near 3d Ising
(Roughly, z∈[ε,1-ε], ε → 0)







Stress tensor Δ=3

[El-Showk, Paulos, Poland, Simmons-Duffin, SR, Vichi, to appear]

Operator decoupling makes bootstrap practical

- Can imagine "recursive bootstrap":

$\langle O_1 O_1 O_1 O_1 \rangle$ $\Rightarrow$ learn something about O_2's $\Delta_2 \lesssim \text{few} \times \Delta_1$

then

$\langle O_2 O_2 O_2 O_2 \rangle, \; \langle O_1 O_1 O_2 O_2 \rangle$ $\Rightarrow$ learn something about O_3's

ecc.

- Alternatively, can study several correlators together:

$$\langle O_i O_j O_k O_l \rangle, \quad i = 1, 2, \ldots N$$

So far, no systematic study beyond N=1

Improving sensitivity to high-dimension operators

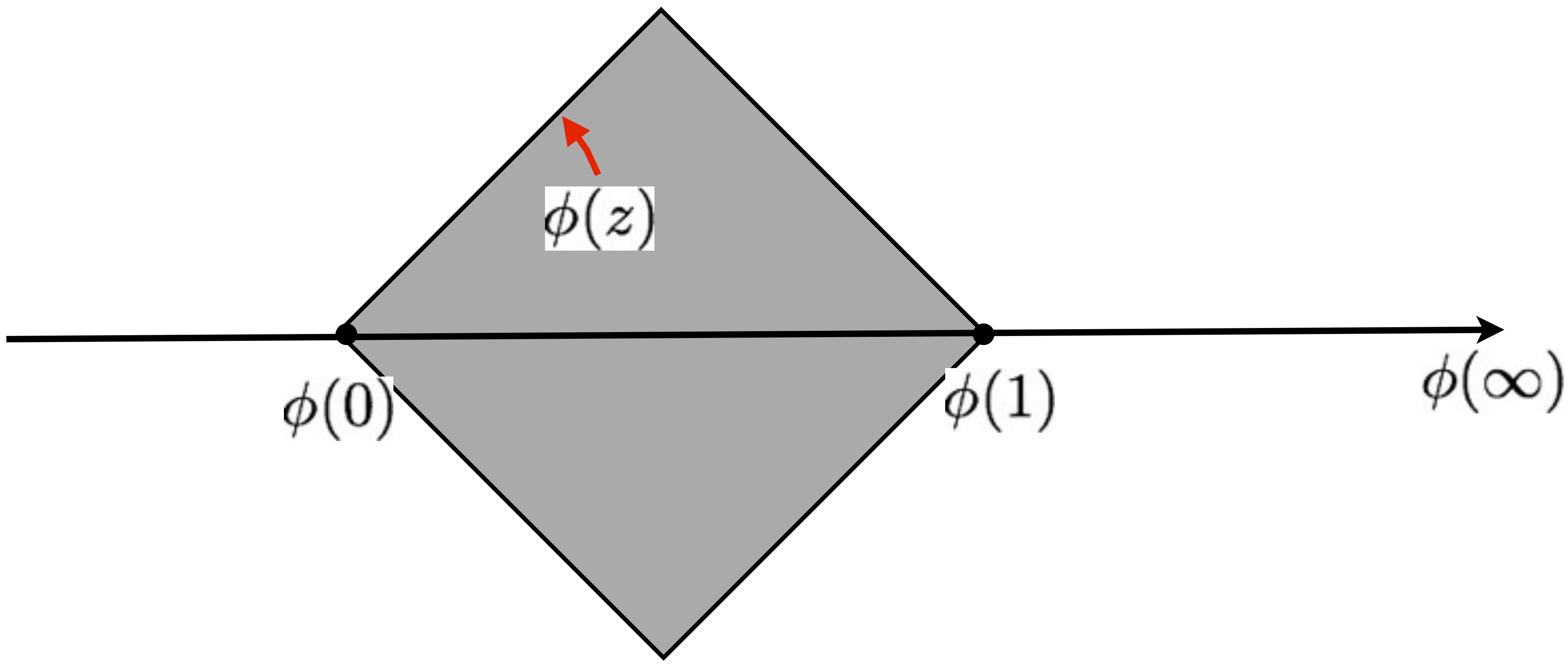
- by $z \to 1$
- external states themselves of high dimension

- by going to Minkowski space

Even mundane, statistical mechanical models like 3d Ising model CFT,
- should be well-defined in Minkowski
- by looking at them in Minkowski one may learn something nontrivial

Example 1: "Callan-Gross"-type relations

[noticed in perturbation theory by Callan-Gross'73]



Sensitive to low-twist operators of arbitrary spin

⇒ Prove existence of large spin operators with low twist:

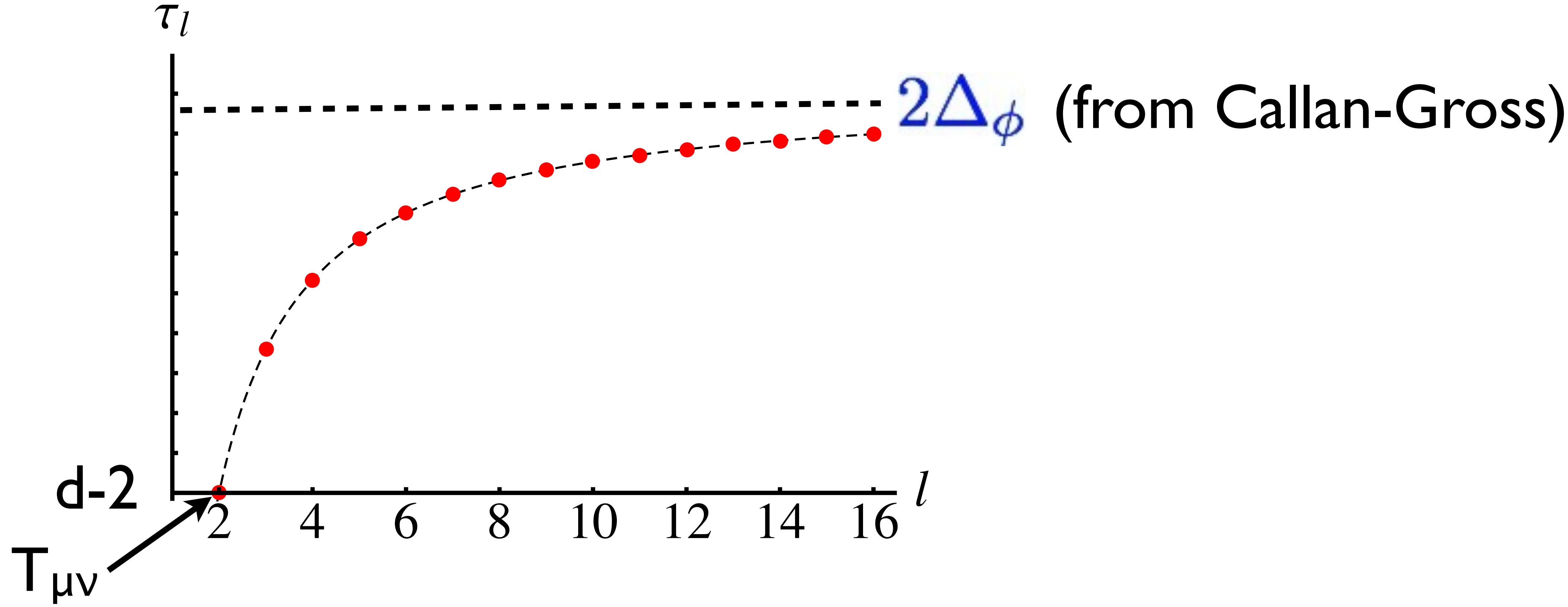
$$\tau_\ell \approx 2\Delta_\phi + O(1/\ell^{d-2})$$

Fitzpatrick, Kaplan, Poland,Simmons-Duffin'12;
Komargodski, Zhiboedov'12
cf. Alday,Maldacena'07

Q: can Minkowski be exploited numerically,
to improve sensitivity say at l=4?

Example 2: Nachtmann's "theorem" [Nachtmann'73]

leading twists form a monotonic, upward convex function

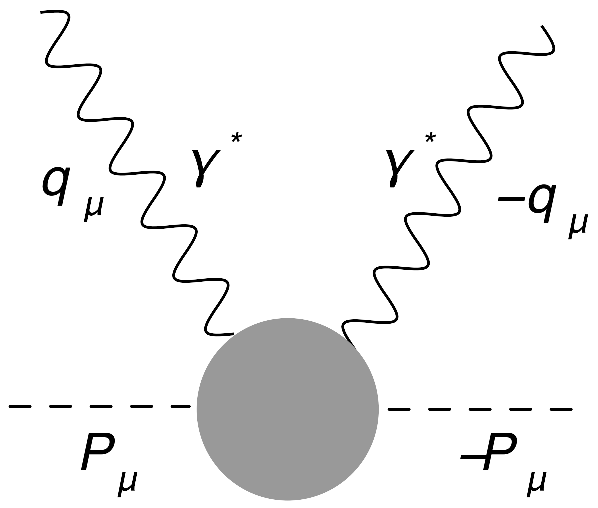


in 3d Ising CFT: $\tau_4 \approx 1.0208(12)$ [Campostrini et al'97]

$\tau_{4,6,8,\ldots}$ can be extracted by numerical bootstrap,
both Nachtmann and Callan-Gross seem OK

Proof - analyze certain "scattering amplitude"

[Nachtmann'73, Komargodski,Zhiboedov'12]



$$A(q_\mu, P_\mu) \equiv \int d^d y e^{iqy} \langle P | T\left(\mathcal{O}(y)\mathcal{O}(0)\right) | P \rangle$$

- a state in a massive deformation
- can be thought of as created by O(∞) ?

Crucial assumption:
Polynomial boundedness in the Regge limit:

$$\lim_{x \to 0} A(x, q^2) \leq x^{-N+1}$$

Monotonic convexity results only for spins ≥ N

However, "experimentally" for 3d Ising CFT holds for spin≥ 2

Can one prove polynomial Regge limit boundedness in CFTs rigorously, using known OPE coefficients asymptotics?

Comment on small anomalous dimensions of spin≥4 currents

In 3d Ising, spin field anomalous dimension is very small, $\gamma=0.01675(125)$

Nachtmann + Callan-Gross & numerical bootstrap
$\Rightarrow$ leading *spin*≥4 currents have small anomalous dimension $\leq 2\gamma$

"Weakly broken higher spin symmetry" (?)

Now that we know this, can we use it to further constrain the 3d Ising CFT?

Future - numerics vs analytics

- physics can be constrained but not fully captured by analytical constraints (simply because not all theories will saturate them)

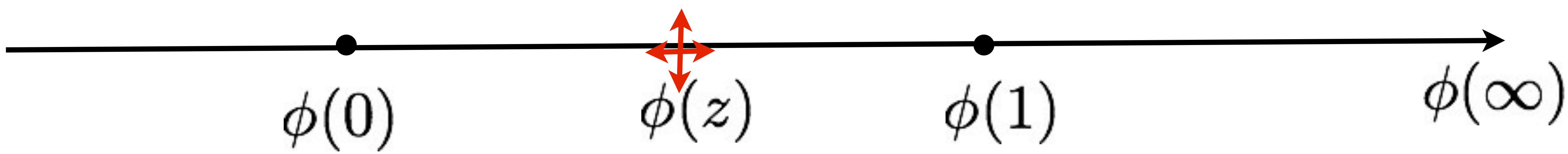
- numerics suggest the existence of some very special "extremal" theories, like 3d Ising CFT ⇒ some new form of "integrability" (?)

- but it would be equally cool if 3d Ising CFT is **not** "integrable", yet we can find a way to solve with with arbitrary accuracy

- in any case we don't expect that all CFTs are "integrable"

⇒ important to keep developing numerical methods, which will be applicable to **any** CFT

Basis problem (Truncation No.2, in z-space)



- Bootstrap eq.: $Q(z,\bar{z}) = 0$ *Q*=crossing deficit

- Functional equation, has to be truncated in some basis

Standard way: $\partial_z^m \partial_{\bar{z}}^n Q \Big|_{z=\bar{z}=1/2} = 0 \qquad (m, n \leq N)$

N as large as possible

Is this analytically most justified/numerically most economical way to truncate?

E.g. why not choose a set of points z_i and study $Q(z_i, \bar{z}_i) = 0$

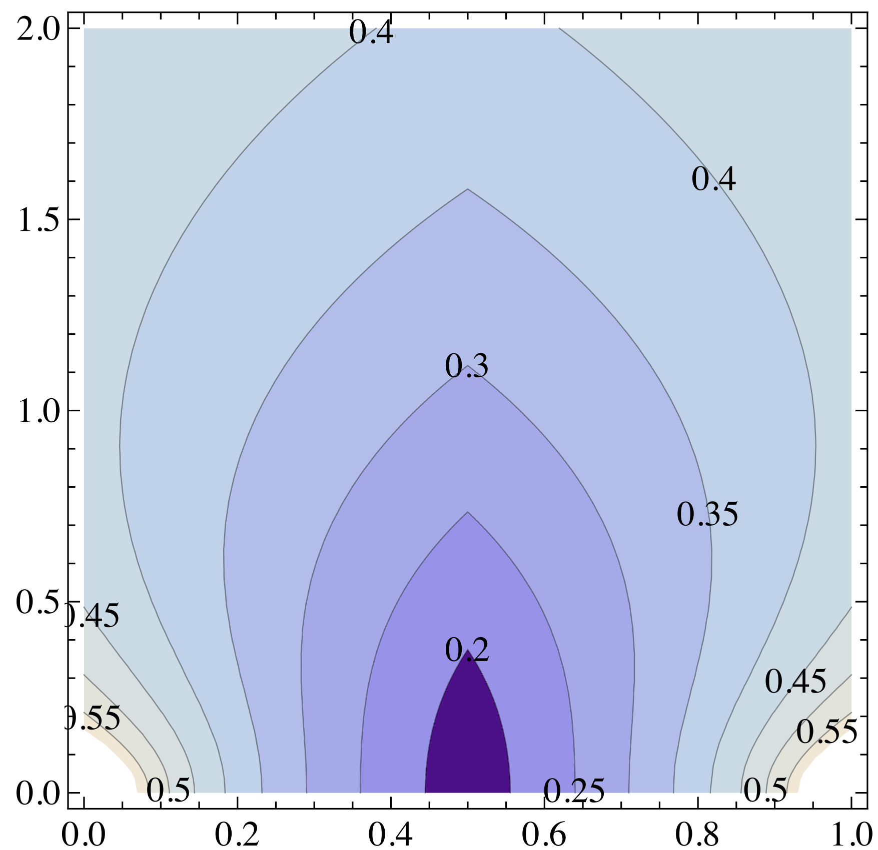
Try to use information from OPE convergence rate?

$$\langle \phi(0)\phi(z)\Big|_{\Delta\geq\Delta_*}\phi(1)\phi(\infty)\rangle \lesssim \frac{\Delta_*^{4\Delta_\phi}}{\Gamma(4\Delta_\phi+1)}|\rho(z)|^{\Delta_*}$$

$$\rho(z) = \frac{z}{(1+\sqrt{1-z})^2}$$

Truncated bootstrap equation: [Hogervorst,SR]

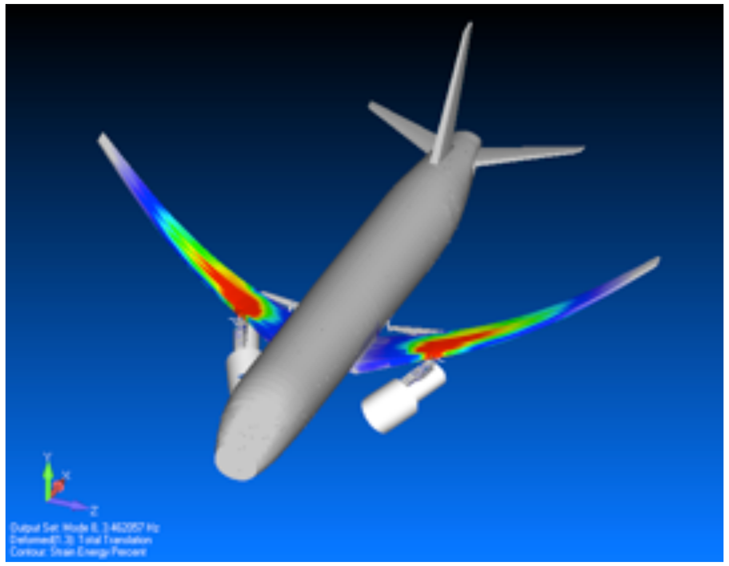
$$Q(z,\bar{z})\Big|_{\Delta\geq\Delta_*} \lesssim \lambda(z)^{\Delta_*} \qquad \lambda(z) = \max(|\rho(z)|, |\rho(1-z)|)$$



How to distribute points efficiently?

Bootstrap turns Conformal Field Theory from an art into a craft

Becomes a linear algebra problem which you can give to a computer



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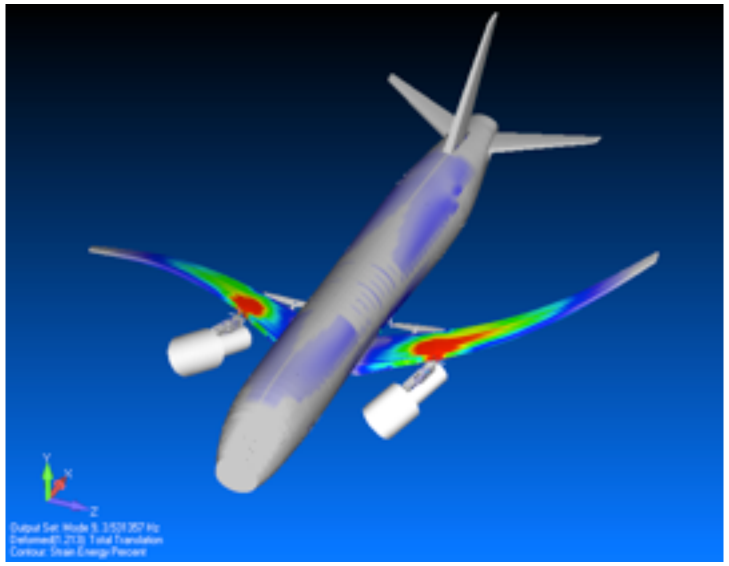

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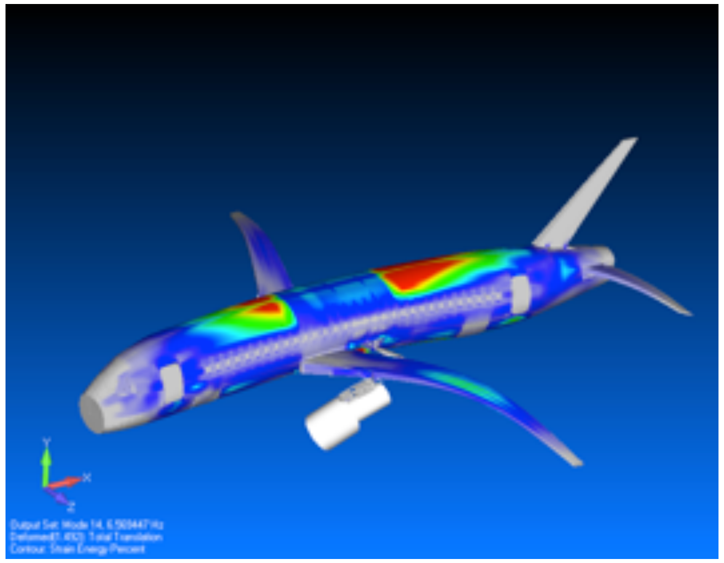

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Like in engineering!