The 3d Ising Spectrum Minimizes c [1]

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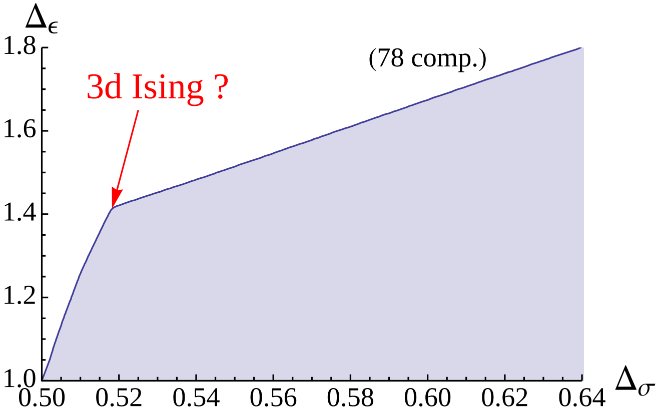
[1] (A conjecture)

Bound on Lowest Dimension Scalar in $\sigma \times \sigma$ OPE



- From studying $\langle \sigma\sigma\sigma\sigma \rangle$
- Assuming only conformal invariance, unitarity, crossing symmetry

A Conjecture

- Let's take seriously the idea: **the 3d Ising Model lies on the boundary of the allowed space of 3d CFTs.**
- For this talk, we'll explore a stronger conjecture: $\langle \sigma\sigma\sigma\sigma \rangle$ **lies on boundary of space of unitary, crossing symmetric 4-pt functions.**

Outline

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The Space of 4-pt Functions

Define $\mathcal{C}_{\Delta_\sigma}$ to be the space of maps

$$(\Delta, \ell) \quad \mapsto \quad p_{\Delta,\ell} \in \mathbb{R}$$

such that

1. $p_{0,0} = 1$ (the unit operator is present)
2. $p_{\Delta,\ell} \geq 0$ (unitarity)
3. $p_{\Delta,\ell}$ gives a crossing-symmetric conformal block expansion:

$$G(u,v) \equiv \sum_{\Delta,\ell} p_{\Delta,\ell} g_{\Delta,\ell}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\sigma} G(v,u)$$

(Think of $p_{\Delta,\ell}$ as a squared OPE coefficient if Δ, ℓ is in the spectrum, 0 otherwise.)

Some Properties of $\mathcal{C}_{\Delta_\sigma}$

- $\mathcal{C}_{\Delta_\sigma}$ is $\mathcal{C}$onvex:

$$t p_{\Delta,\ell} + (1-t) p'_{\Delta,\ell} \quad \text{with} \quad t \in [0,1]$$

  also gives a unitary crossing symmetric 4-pt function.
- $\mathcal{C}_{\Delta_\sigma}$ is nonempty
  - Contains 4pt function for any CFT with scalar of dimension Δ_σ
  - Contains 4pt function for Mean Field Theory (aka Generalized Free Fields)
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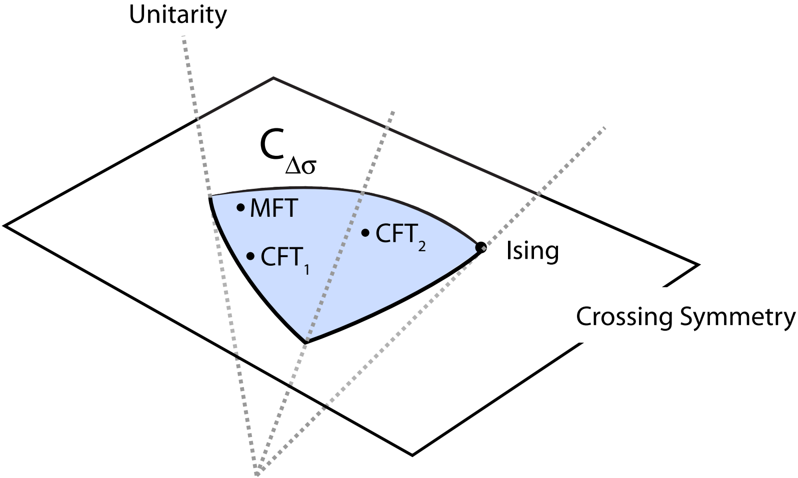
$$\begin{aligned} \dim \mathcal{C}_{\Delta_\sigma} &= \#(\text{dimensions and spins } (\Delta,\ell)) \\ &\quad - \#(\text{constraints from crossing symmetry}) \\ &= \infty - \infty = \infty \end{aligned}$$

A Picture of $\mathcal{C}_{\Delta_\sigma}$



Getting To the Boundary of $\mathcal{C}_{\Delta_\sigma}$

Points on the boundary of a convex space are extrema of some linear function. So...

- The 3d Ising Spectrum Maximizes *something*.

Candidates:

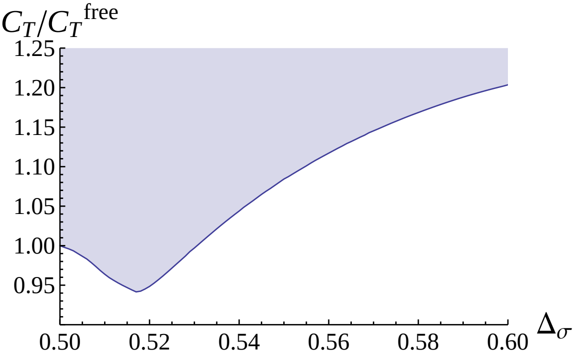
- The 3d Ising Spectrum Maximizes Δ_ϵ (dimension of lowest-dimension scalar in $\sigma \times \sigma$)
- The 3d Ising Spectrum Maximizes $p_T = p_{3,2}$ (coefficient of stress-tensor conformal block)

p_T Maximization $= c$ Minimization

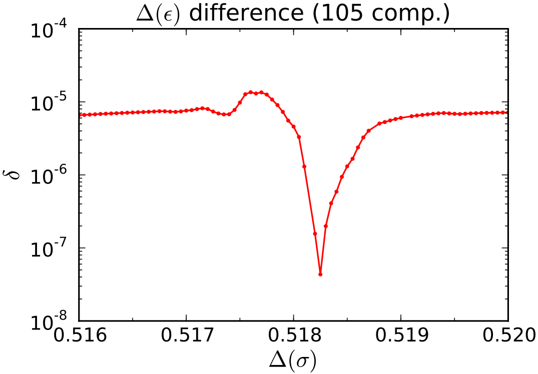
The coefficient p_T is fixed by Ward identities

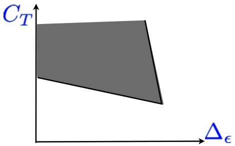
$$\begin{aligned} \langle T_{\mu\nu}\sigma\sigma\rangle &\propto \Delta_\sigma \\ \langle T_{\mu\nu}T_{\rho\sigma}\rangle &\propto c \end{aligned} \implies p_T \propto \frac{\Delta_\sigma^2}{c}$$

Bounds support idea that Ising Model minimizes c



Equivalence of c-minimization and Δ_ϵ-maximization





Precise Conjecture

$$\Delta_\sigma, p_{\Delta,\ell} \text{ in 3d Ising} \quad = \quad \text{argmax}_{\Delta_\sigma, p_{\Delta,\ell} \in \mathcal{C}_{\Delta_\sigma}} [p_T]$$

- Conceptually nice
  - Conjecture is in terms of $T_{\mu\nu}$, which is present in every CFT
  - Ising is as far as possible from MFT ($c_{\text{MFT}} = \infty$)
  - Smallest $c \approx$ "simplest" theory
- Computationally nice
  - p_T is a linear function on $\mathcal{C}_{\Delta_\sigma}$, so we have a linear program for each Δ_σ
  - Solve with Dantzig's simplex method ('47)

Outline

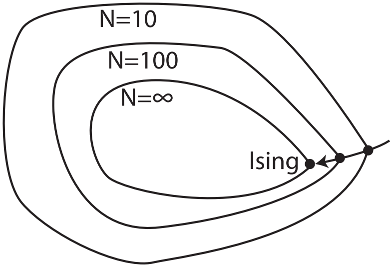
1. An Optimization Problem For the Spectrum
2. Simplex Algorithm
3. Results

Making the Problem Finite

We first relax the crossing constraint to a finite set of constraints

$$\partial_u^m \partial_v^n \left(G(u,v) - \left(\frac{u}{v}\right)^{\Delta_\sigma} G(v,u) \right) \Bigg|_{u=v=1/4} = 0$$

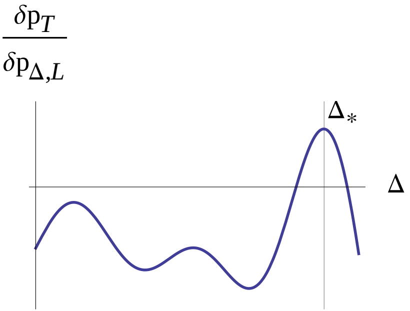
for N pairs of derivatives (m,n). Recover $\mathcal{C}_{\Delta_\sigma}$ as $N \to \infty$.



- Optimum is achieved with N nonzero $p_{\Delta,\ell} \implies N$ operators.
- Take $N \to \infty$ to recover spectrum.

The Simplex Method

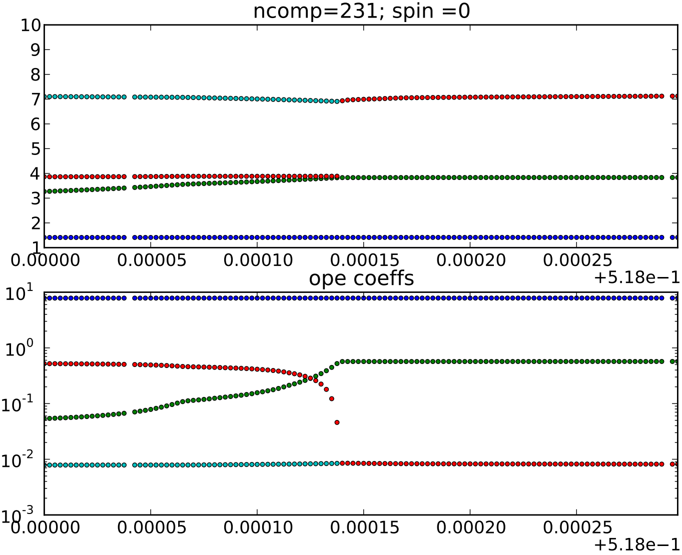
1. Start with N positive coefficients $\{p_{\Delta_1,\ell_1},\ldots,p_{\Delta_N,\ell_N}\}$ satisfying the N crossing constraints.
2. Consider turning on some new p_{Δ_*,ℓ_*}, adjusting the p_{Δ_i,ℓ_i} to preserve crossing symmetry. Choose Δ_*,ℓ_* to maximize $\frac{\delta p_T}{\delta p_{\Delta,\ell}}$.



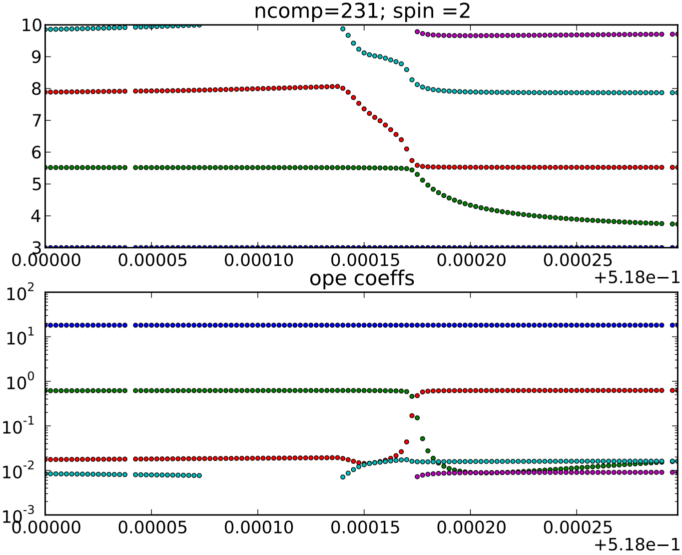
3. Turn on p_{Δ_*,ℓ_*} as much as possible until some p_{Δ_k,ℓ_k} goes to zero, leaving N nonzero coefficients again.
4. Repeat.

Outline

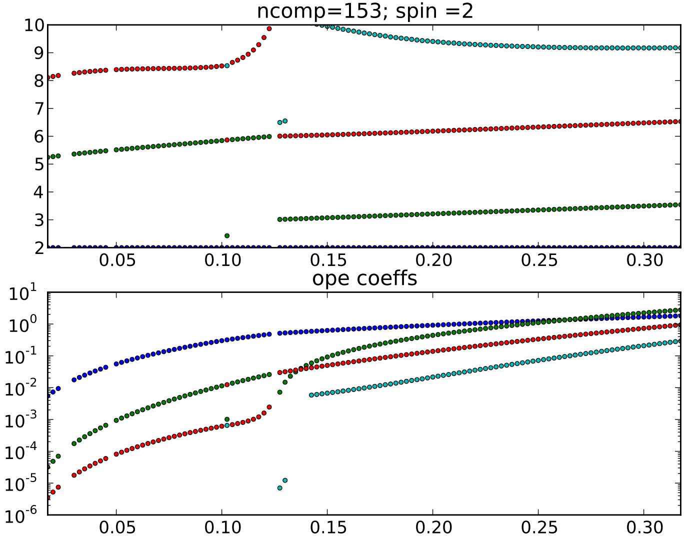
Spin-0 Spectrum



Spin-2 Spectrum



Spin-2 Spectrum in 2d (For Comparison)



Conclusions

Results:

- Special value of Δ_σ emerges as $N \to \infty$
- Extremely precise determinations of critical exponents and OPE coefficients
  - $\Delta_\sigma = 0.518155(15)$
  - $\Delta_\epsilon = 1.41268(12)$
  - $c/c^{\text{free}} = 0.946533(10)$
- Certain operators predicted by Exact RG methods not actually present in spectrum.

Future Directions:

- Improve algorithm/precision
- Study optimization analytically
- Investigate other CFT constraints