Resurgence and trans-series in QFT: Towards a non-perturbative continuum definition

Mithat Ünsal

In collaboration with
Gerald Dunne (2d QFT, CP(N-1))
Philip Argyres (4d QFT, QCD(adj))

+Aleksey Cherman, Daniele Dorigoni, Gokce Basar

Motivation: Can we make sense out of QFT? When is there a continuum definition of QFT?

Dyson(50s), 't Hooft (77),

Quoting from M. Douglas comments, in Foundations of QFT, talk at String-Math 2011

"A good deal of mathematical work starts with the Euclidean functional integral. There is no essential difficulty in rigorously defining a Gaussian functional integral, in setting up perturbation theory, and in developing the BRST and BV formulations (see e.g. K. Costello's work).

A major difficulty, indeed many mathematicians would say the main reason that QFT is still "not rigorous," is that standard perturbation theory only provides an asymptotic (divergent) expansion. There is a good reason for this, namely exact QFT results are not (often) analytic in a finite neighborhood of zero coupling.

Recently, few people are attempting to answer and reinvigorate this question, whether/when a N.P. continuum definition of QFT may exist.

Argyres, Dunne, MÜ: Resurgence in QFTs, QM, and path integrals
Schiappa, Marino, Aniceto, ... : Resurgence in string theory and matrix models
Kontsevich: recent talk at PI, Resurgence from the path integral perspective
Garoufalidis, Costin: Math and Topological QFTs

The common concept, which all these folks seem to be highly influenced by (and which is virtually unknown in physics community) is a "recent" mathematical progress, called

Resurgence Theory, developed by Jean Ecalle (80s)

and applied to QM by Pham, Delabaere, Voros, Zinn-Justin.
(also relevant Dingle-Berry-Howls)

Ecalle's theory changed (will change?) the overall perspective on asymptotic analysis, for both mathematicians and physicists alike.

Main promise: The P-data and NP-data are tightly knitted together. NP-data (or NP-completion) can be extracted from P-data.

CP(N-1) model on R2 and standard problems

verbatim in 4d QCD.

An asymptotically free non-linear sigma model with a complex projective target space. Large-N, successful. Many problems are still unresolved at finite-N.

1) Pert. theory is an asymptotic (divergent) expansion even after regularization and renormalization. Is there a meaning to pert. theory?

2) Invalidity of the semi-classical dilute instanton gas approximation on R2.
DIG assumes inter-instanton separation is much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption. "Infrared embarrassment" large-instanton contribution to vacuum energy is IR-divergent, see Coleman's lectures.

3) A resolution of 2) was put forward by considering the theory in a small thermal box. But in the weak coupling regime, the theory always lands on the deconfined regime. (Affleck, 80) No semi-classical approx. for the confined regime.

4) Incompatibility of large-N with DIG. It better be so, we trust former and not the latter. (Witten, Jevicki, 79)

5) The renormalon ambiguity (technical, but deeper, to be explained), ('t Hooft,79)

PCM on R_2 (trivial homotopy group)

An asymptotically free matrix field theory with U(N) target space.

1**)** Pert. theory is an asymptotic (divergent) expansion even after regularization and renormalization. Is there a meaning to pert. theory?

2**)**No Instantons! Trivial hom. group. Hence no DIG.

3**)** No semi-classical approx. for the confined regime. The only known stable saddle is P-saddle.

4**)** Large-N, not much success. (despite heroic efforts from Polyakov)

5**)** The renormalon ambiguity is present. (Fateev, Frolov, Kazakov,94)

Recall **Main promise:** If P-data and NP-data are so tightly knitted together, why do we not see any NP-saddles in this problem? Why is this matrix model so "different" from QCD and CP(N-1)? Are we somehow missing the whole picture here?

CP(N-1) model on R2 and standard problems

Despite these problems, the subject is viewed to be "mature". In my opinion, our inheritance from the few generation earlier does not seem so good.

If we are going to make progress in some foundational aspects of QFT, it is, of course, preferable to have a formalism of *practical utility,* whose results can be compared with *numerical experiments,* i.e., lattice field theory. Lattice simulations, of course are useful, but also a black box.

So far, there had been no such useful continuum formulation of general QFTs. "Constructive QFT" (Glimm, Jaffe, Spencer, Bridges) only attempts the first problem listed above.

This talk: Report progress in this direction, and argue a useful N.P. definition may underly the resolution of these problems at least in some regime of QFT.

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ, Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)} \, (\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n} \, e^{-n\, A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)} \, (\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence terminology.

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

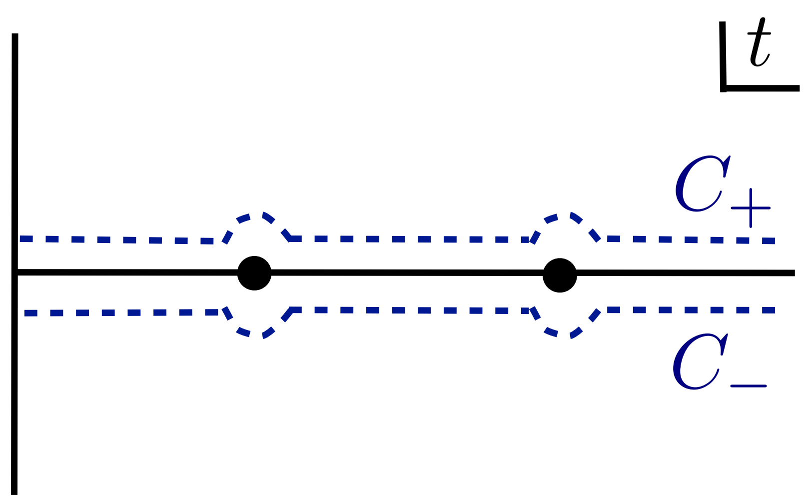
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2} dt \, .$$

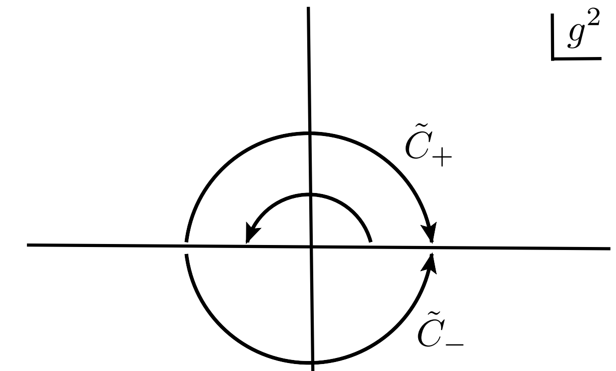
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Perturbation theory: Borel plane, lateral Borel sums, ambiguity

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



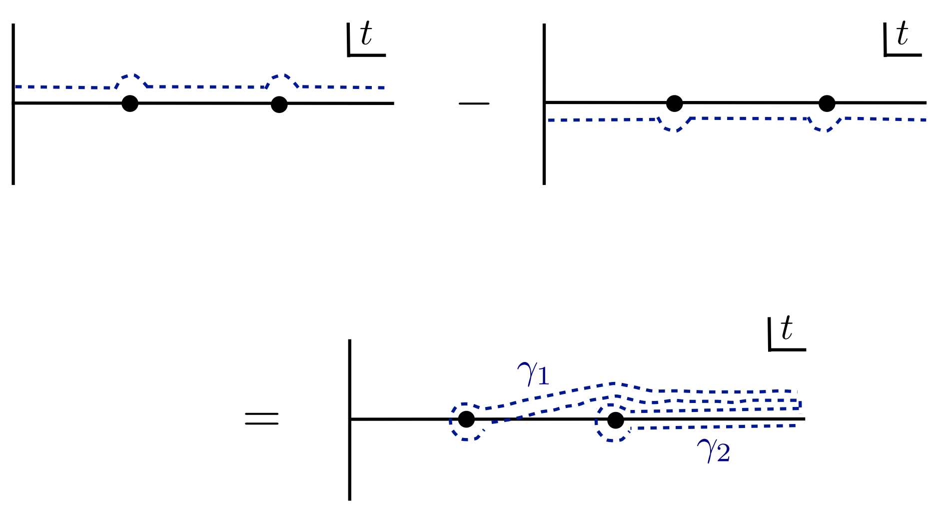
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0^\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor (not 1). The measure of ambiguity (Stokes automorphism/jump in g-space interpretation):



$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta^-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta^-}),$$

$$\mathrm{Disc}_{\theta^-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \dots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

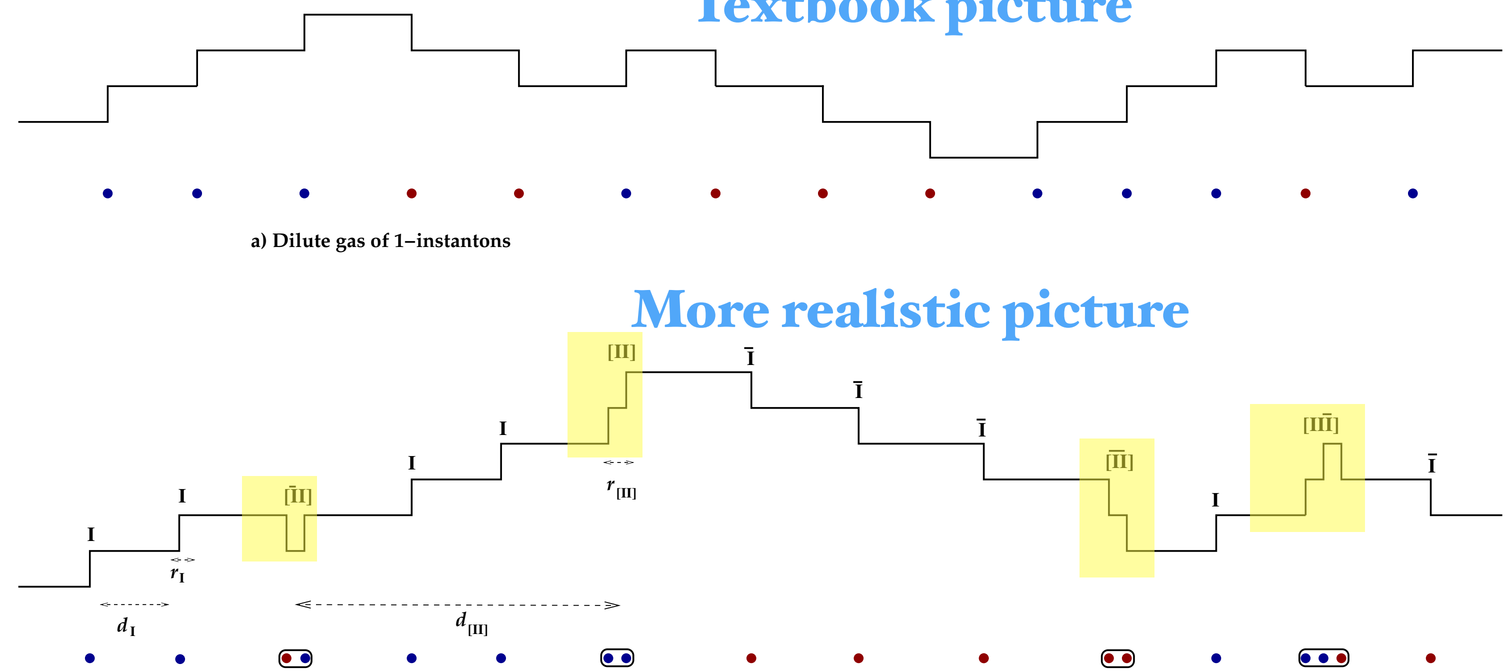
Jean Ecalle, 80s

Instantons and Bogomolny--Zinn-Justin (BZJ) prescription

BZJ, QM (80s): for double well potential,
Here, we work with a periodic potential.
Dilute instanton, molecular instanton gas.

Textbook picture

More realistic picture



a) Dilute gas of 1−instantons

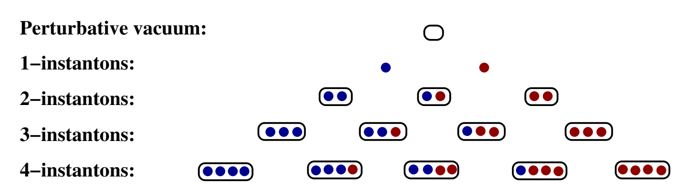
b) Dilute gas of 1−instantons, 2−instantons, and other molecular−events

$$
\begin{array}{ccccccc}
r_{\rm I} & \ll & r_{\rm [II]} \sim \ell_{\rm qzm} & \ll & d_{\rm I} & \ll & d_{\rm [II]}, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \ll & L \log\left(\frac{1}{g^2}\right) & \ll & L\, e^{S_0} & \ll & L\, e^{2S_0}.
\end{array}
$$

How to make sense out of correlated events?

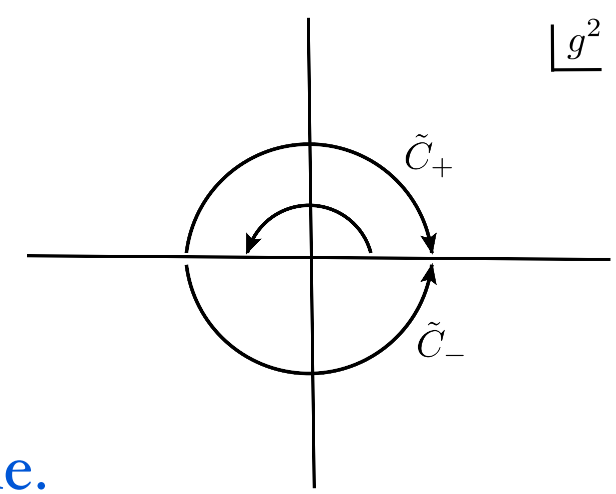
$[\mathcal{I}\mathcal{I}]$ Evaluate quasi-zero mode integral.Easy.

$[\mathcal{I}\bar{\mathcal{I}}]$ Naive calculation meaningless* at $g^2 > 0$.
The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. Continue to $g^2 < 0$, evaluate there, and continue back to $g^2 > 0$: two fold-ambiguous!



c) Representatives of n−instanton events, sketched according to the resurgence triangle.

$$[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm} = \mathrm{Re}\,[\mathcal{I}\bar{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm}$$



*: Retrospectively, it better be so, because we are on a Stokes line.

Remarkable fact: Leading ambiguities cancel.

Non-Borel summable. But a generalized notion of summability exits.
An elementary incidence of Borel-Ecalle summability.

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

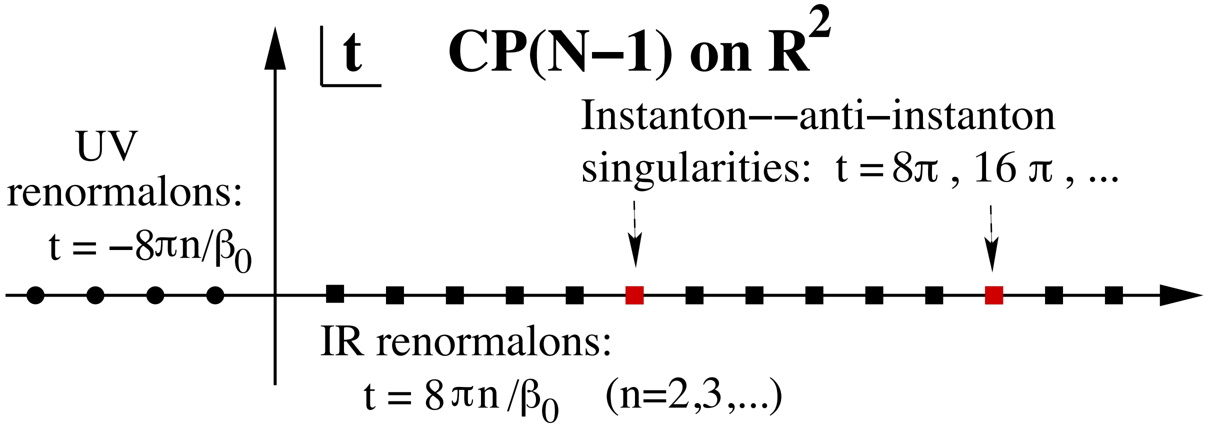
Data from P.T. Data from N.P. sector

Can this work in QFT? QCD on R4 or CP(N-1) on R2?

't Hooft(79) :***No,*** *on R4,* Argyres, MÜ: ***Yes,*** *on R3 x S1,*
F. David(84), Beneke(93) : ***No,*** *on R2.* Dunne, MÜ: ***Yes,*** *on R1 x S1*

Why doesn't it work, say for CP(N-1) on R2?

$[\mathcal{I}\bar{\mathcal{I}}]$ contribution, calculated in some way, gives an $\pm i \exp[-2S_I]$.
Lipatov(77): Borel-transform BP(t) has singularities at $t_n = 2n\, g^2 S_I$.

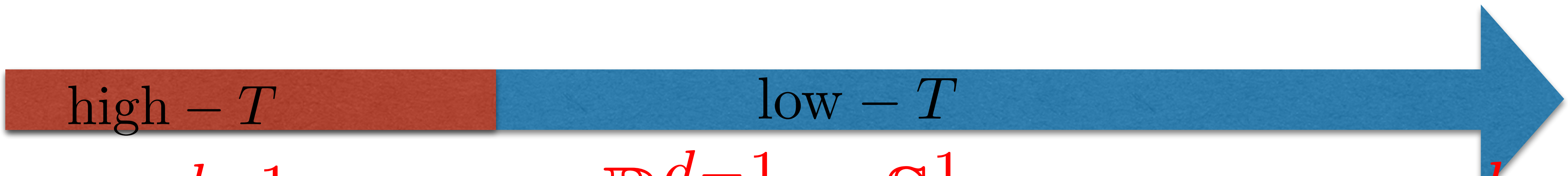


BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams.)

't Hooft called these IR-renormalon singularities with the hope that they would be associated with a saddle point like instantons. **No such configuration is known!**

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from micro-dynamics?

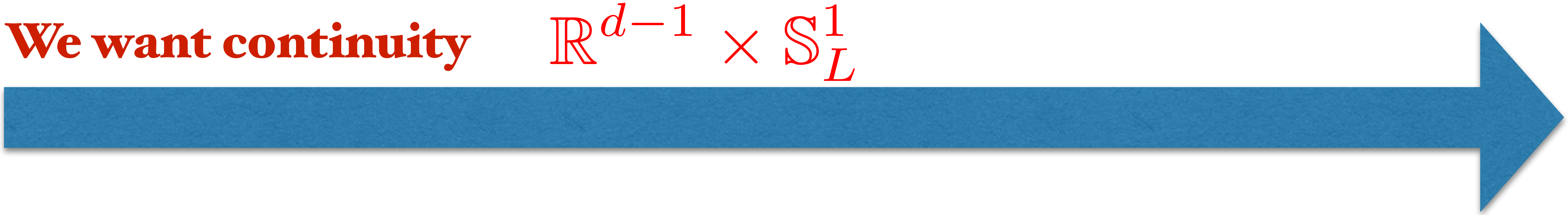
$\mathbb{CP}^{N-1}$ on $\mathbb{R}^1 \times \mathbb{S}^1_L$ and Continuity



$\mathbb{R}^{d-1}$ $\mathbb{R}^{d-1} \times \mathbb{S}^1_\beta$ $\mathbb{R}^d$

Thermal: Rapid crossover at finite-N, phase transition at large-N $Z(\beta) = \mathrm{tr} e^{-\beta H}$

We want continuity $\mathbb{R}^{d-1} \times \mathbb{S}^1_L$



Prevent phase transition by using circle compactification, judicious matter choice and boundary conditions, or by deformations.

Supersymmetric theories: Continuity and analyticity is used since supersymmetric index calculation, (Witten,80). $Z(L) = \mathrm{tr}[e^{-LH}(-1)^F]$

Non-supersymmetric theories, including QCD-like theories:
The possibility and utility of continuity is realized in 2007 (M.Ü.).
This is a semi-classical avatar to the Eguchi-Kawai reduction proposal (82) in lattice field theory, whose first working examples are found in 07, by using Z(L)!!

Sigma-connection holonomy (a new line operator)

Point-wise modulus and phase splitting, derivative of each phase transform as "gauge" connection.

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos\frac{\theta_1}{2} \\ e^{i\varphi_2} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \\ e^{i\varphi_3} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \cos\frac{\theta_3}{2} \\ \vdots \\ e^{i\varphi_N} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \sin\frac{\theta_3}{2} \ldots \sin\frac{\theta_{N-1}}{2} \end{pmatrix}$$

$$\theta_i \in [0, \pi], \qquad \varphi_i \in [0, 2\pi).$$

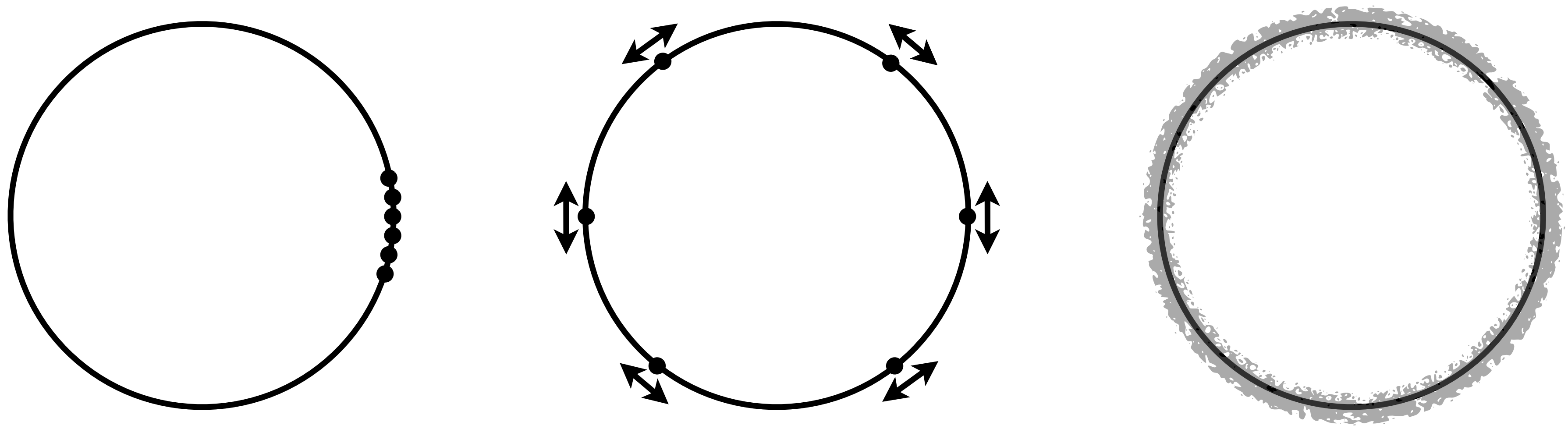
Build a new line operator, counter-part of the Wilson line, the sigma holonomy:

$$({}^L\Omega)_j(x_1) = \exp\left[i \int_0^L dx_2 \, \mathcal{A}_{2,j}\right] = \exp\left[i(\varphi_j(x_1, 0) - \varphi_j(x_1, L))\right]$$

$${}^L\Omega(x_1) = \begin{pmatrix} e^{i[\varphi_1(x_1,0)-\varphi_1(x_1,L)]} & 0 & \ldots & 0 \\ 0 & e^{i[\varphi_2(x_1,0)-\varphi_2(x_1,L)]} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & e^{i[\varphi_N(x_1,0)-\varphi_N(x_1,L)]} \end{pmatrix}$$

One-loop potential for Sigma holonomy

Three types of holonomy, in CP(N-1) with n_f fermions



(a) Thermal:
Eigenvalue attraction

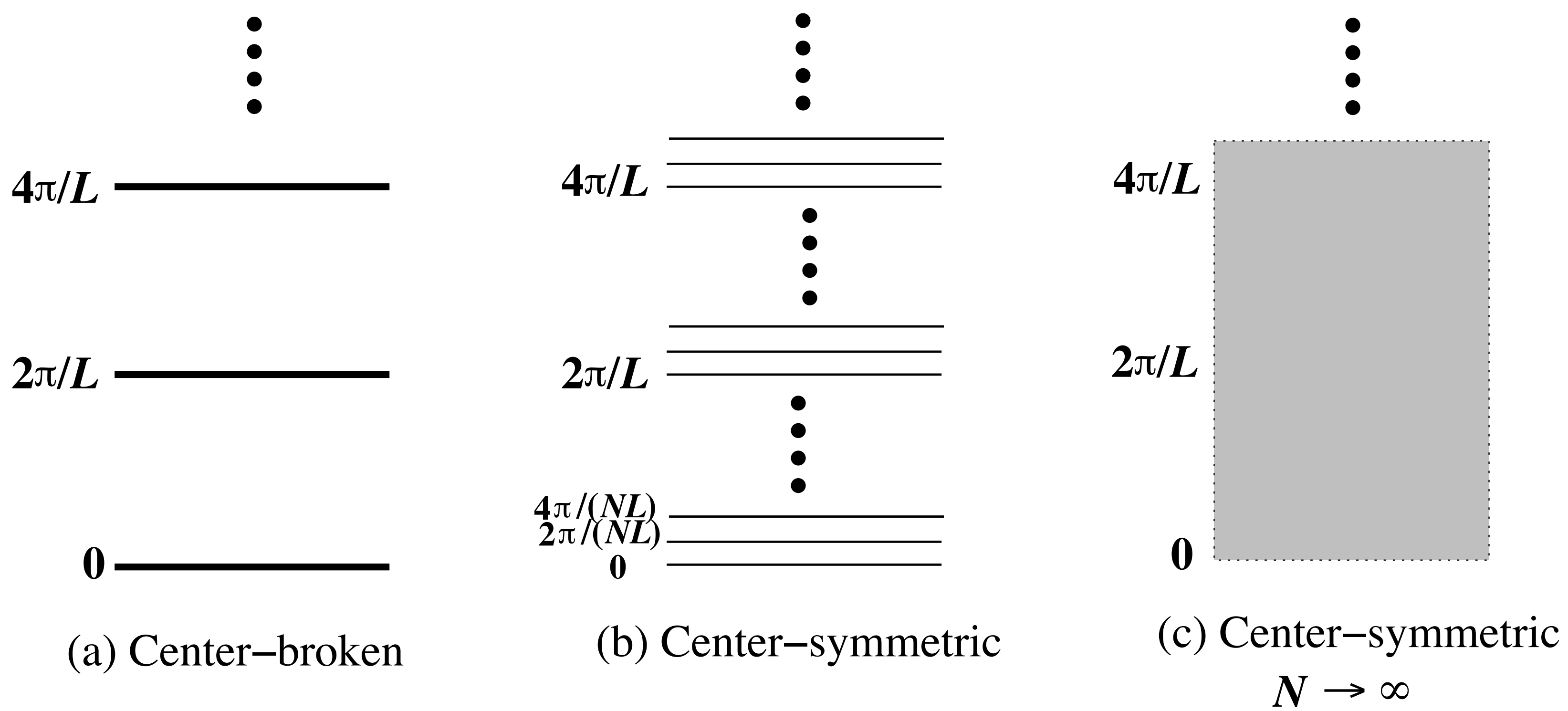
(b) Spatial:
Eigenvalue repulsion

(c) Strong coupling
Randomization

Crucial difference of (a) and (b): van Baal, Kraan, Lee, Yi (97/98) in gauge th. on R_3 x S_1

To achieve (b) in the $n_f = 0$ case require a deformation of the action.
(b) is weak coupling realization of the center-symmetric background.

The dependence of perturbative spectrum to the sigma holonomy background



Same as gauge theory on R3 x S1, the fact that spectrum become dense in the L=fixed, and N-large case is an imprint of the large-N volume independence (Eguchi-Kawai reduction).

Here, we will study non-pert. effects in the long-distance effective theory within Born-Oppenheimer approx. in case (b) for finite-N.

Topological configurations, 1-defects

Kink-instantons: (1d-instanton and twisted instantons) Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The twisted-instanton is present only because the theory is locally 2d! Also see Bruckmann et.al.(07, 09)

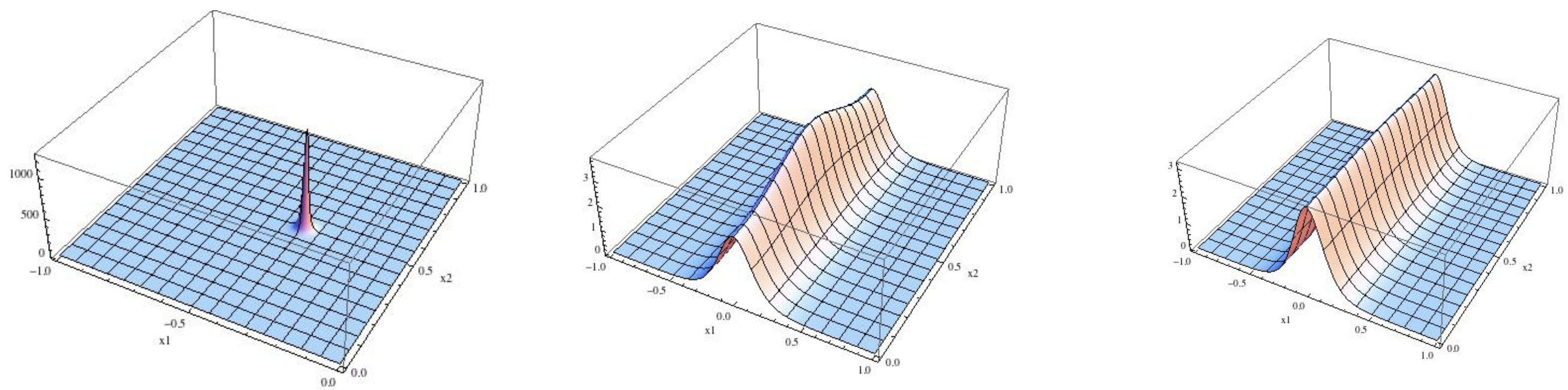
$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i, \qquad \alpha_i \in \Gamma_r^\vee$$

$$\mathcal{K}_k : \qquad S_k = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N} \qquad , \qquad k = 1, \ldots, N$$

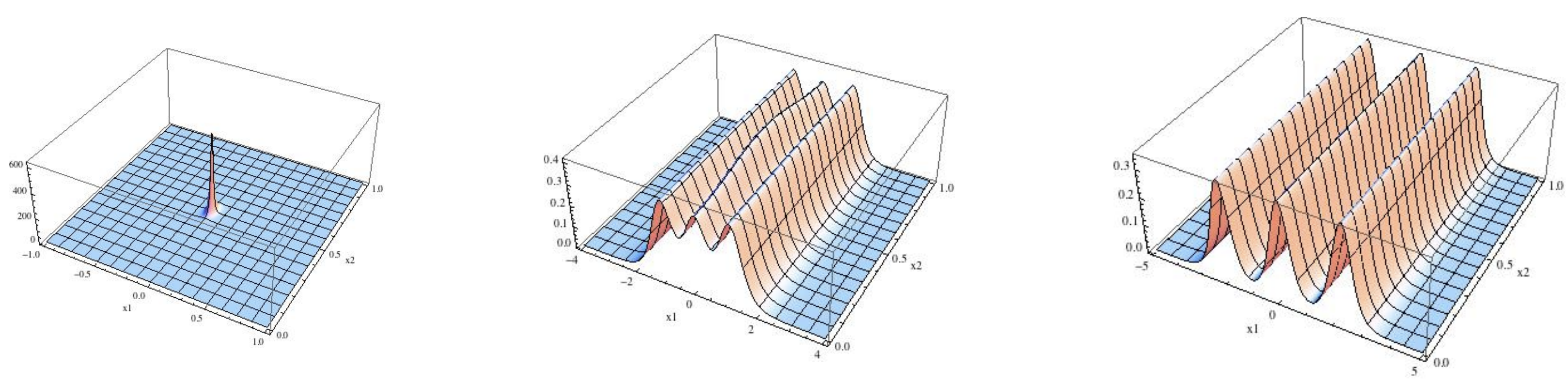
Gauge theory counter-part on R_3 x S_1 :
Monopole-instantons or 3d-instanton and twisted instanton.
(also called caloron constituents) : van Baal, Kraan, (97/98), Lee-Yi (97)

Topological configurations, 1-defects

In thermal box, and high T, associated with trivial holonomy, the fractionalization does not occur (Affleck, 80s). Plot is for CP(2)



In spatial box, and small-L, associated with non-trivial holonomy, the fractionalization does occur. Large-2d BPST instanton in CP(2) fractionates into 3-types of kink-instantons.

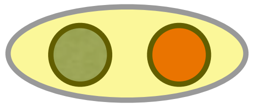


Topological molecules: 2-defects

2-defects are also universal, dictated by Cartan matrix of Lie algebra: We call them charged and neutral bions

Charged bions: For each negative entry of the extended Cartan matrix $\widehat{A}_{ij} < 0$, there exists a bion $\mathcal{B}_{ij} = [\mathcal{K}_i \overline{\mathcal{K}}_j]$, associated with the correlated tunneling-anti-tunneling event

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_j \qquad \alpha_i \in \Gamma_r^\vee$$



Neutral bions: For each positive entry of the extended Cartan matrix $\widehat{A}_{ii} > 0$, there exists a neutral bion $\mathcal{B}_{ii} = [\mathcal{K}_i \overline{\mathcal{K}}_i]$, associated with the correlated tunneling-anti-tunneling event

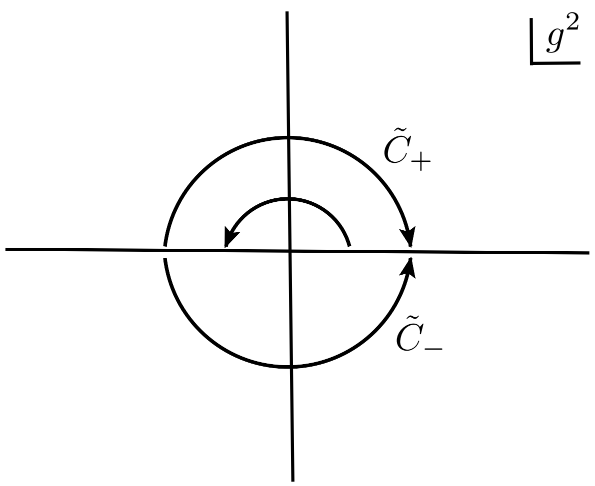
$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_i \qquad \alpha_i \in \Gamma_r^\vee$$

Charged bion: Counter-part of magnetic bion in gauge theory on R3 x S1 (generates mass gap for gauge fluctuations), MÜ 2007

Neutral bion: Same as in the gauge theory on R3 x S1 (generates a center-stabilizing potential), Poppitz-MÜ 2011, Poppitz-Schäfer-MÜ, Argyres-MÜ 2012

Neutral bion and non-perturbative ambiguity in semi-classical expansion

Naive calculation of neutral bion amplitude, as you may guess following QM example, meaningless at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular event is meaningless. Apply BZJ. Result is two fold-ambiguous!



$$
\begin{aligned}
[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} &= \mathrm{Re}\,[\mathcal{K}_i\overline{\mathcal{K}}_i] + i\,\mathrm{Im}\,[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} \\
&= \left(\log\left(\frac{\lambda}{8\pi}\right) - \gamma\right)\frac{16}{\lambda}e^{-2S_0} \pm i\frac{16\pi}{\lambda}e^{-\frac{8\pi}{\lambda}}
\end{aligned}
$$

As it stands, this looks terrible. Is semi-classical expansion at second order void of meaning? This is a general statement valid for many QFTs admitting semi-classical approximation, including the Polyakov model (77)! And it has not been addressed in literature until recently.

In QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects. The truth is far more subtler!

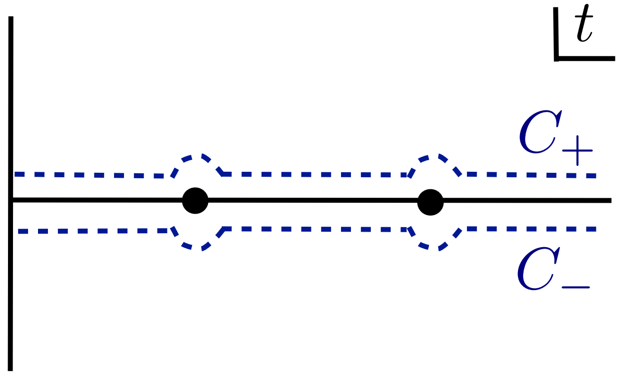
Disaster or blessing in disguise?

Go back to pert. theory, for the compactified center-symmetric CP(N-1) theory. We reduce the long-distance effective theory to simple QM with periodic potentials. Thankfully, the large-order behavior of pert. theory in such QM problems is studied by M. Stone and J. Reeve (78), by using the classic Bender-Wu analysis (69-73).

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \qquad a_q \sim -\frac{2}{\pi}\left(\frac{1}{4\xi}\right)^q q!\left(1 - \frac{5}{2q} + O(q^{-2})\right)$$

Divergent non-alternating series, non-Borel summable, but right and left Borel resummable, with a result:

$$\mathcal{S}_{0\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\ B\mathcal{E}(t)\ e^{-t/g^2} = \mathrm{Re}\mathcal{S}\mathcal{E}(g^2) \mp i\frac{8\xi}{g^2}e^{-\frac{4\xi}{g^2}}$$
$$= \mathrm{Re}\mathbb{B}_0 \mp i\frac{16\pi}{g^2 N}e^{-\frac{8\pi}{g^2 N}}$$



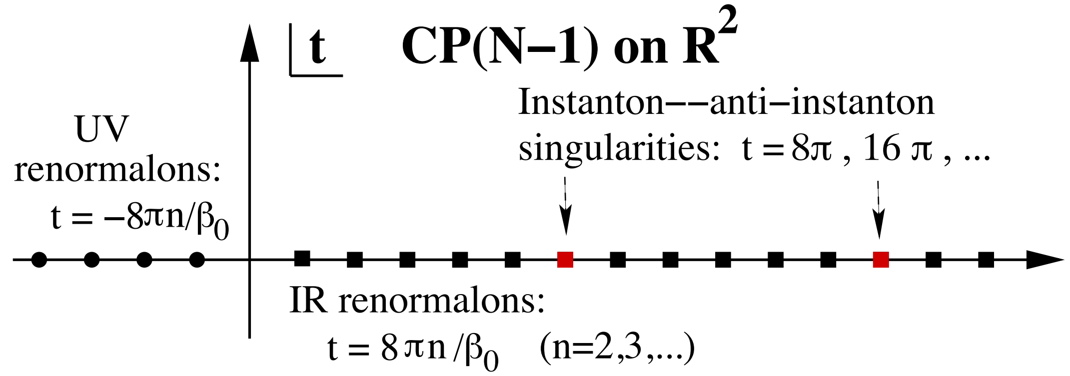
Remarkably,

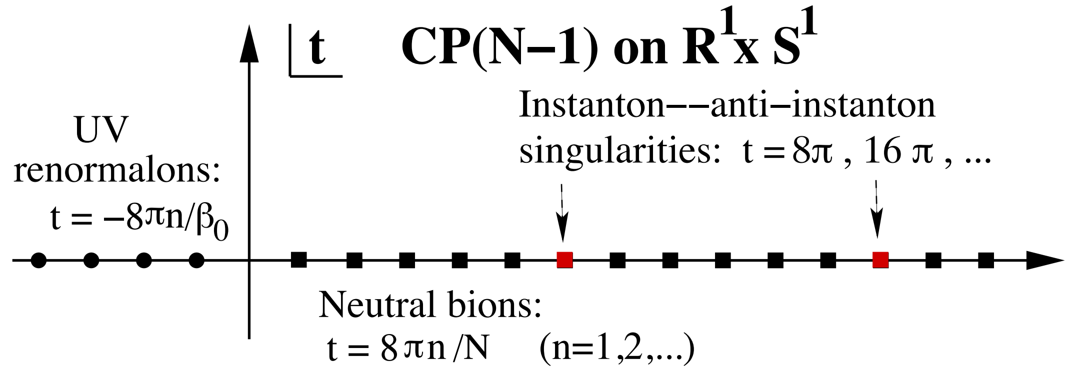
$$\mathrm{Im}\left[\mathcal{S}_\pm\mathcal{E}(g^2) + [\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm}\right] = 0 \qquad \text{up to } e^{-4S_0} = e^{-4S_I/\beta_0}$$

The ambiguities at order $\exp[-2S_I/N]$ cancel and
QFT is well-defined up to the ambiguities of order $\exp[-4S_I/N]$! Ambiguities exactly in the IR-renormalon territory as per ‘t Hooft, David.

Semi-classical renormalons as neutral bions

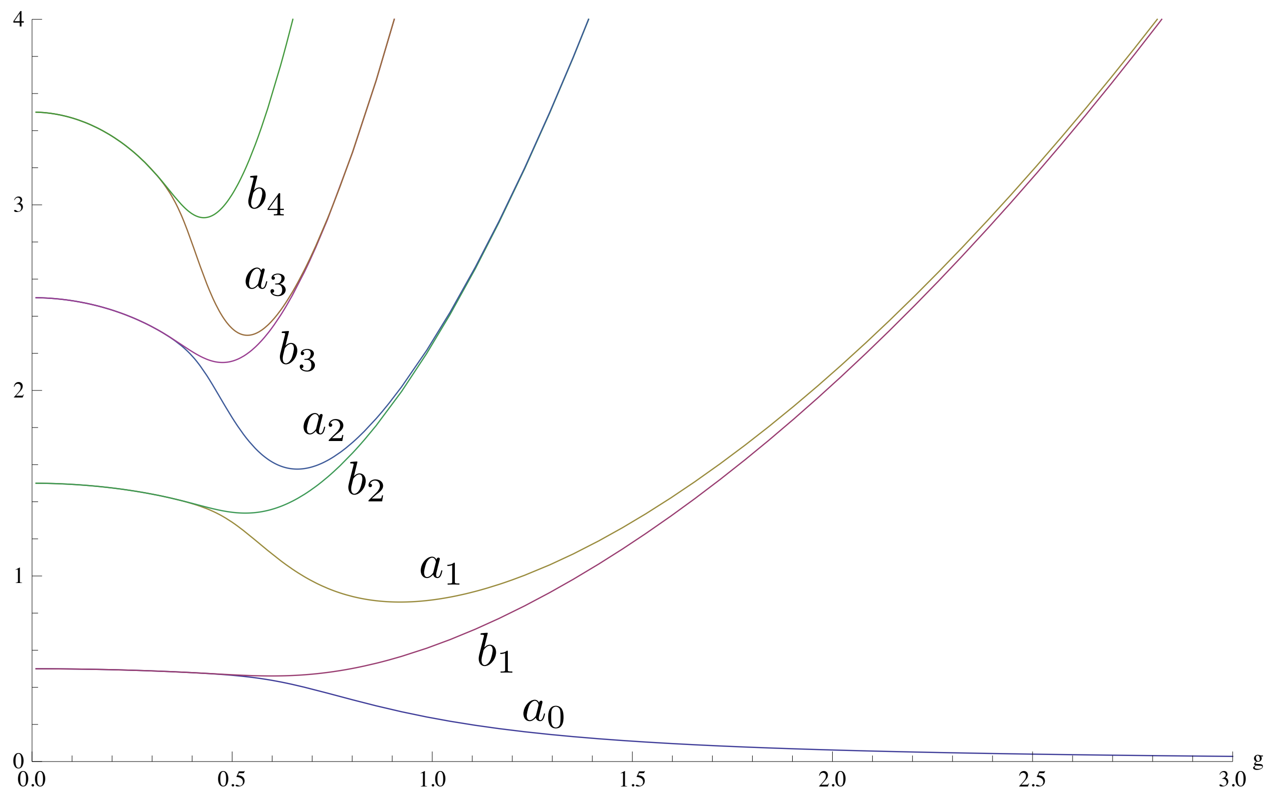
Claim (with Argyres in 4d) and (with Dunne in 2d): Neutral bions and neutral topological molecules are semi-classical realization of 't Hooft's elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion.
Shown only at leading order so far.





More than three decades ago, 't Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that, we can do so, at least, in the semi-classical regime of QFT.

Mass gap in the small-S1 regime



Eigen-energies for CP(1) in reduced QM. In Born-Oppenheimer approx., the zero mode Hamiltonian at small g reduce to Mathieu ODE.

$$H^{\rm zero}_{\alpha_k} = -\frac{1}{2}\frac{d^2}{d\theta^2} + \frac{\xi^2}{4g^2}[1-\cos(2g\theta)]$$

$$m_g = \frac{C}{\sqrt{\lambda}}\left(1 - \frac{7\lambda}{32\pi} + O(\lambda^2)\right) e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N} \qquad \text{for } \mathbb{CP}^{N-1}$$

The functional form of the small-S1 result for CP(N-1) is same as large-N result on R2! This is the first derivation of the factor exp[-SI/N] from microscopic considerations at finite N.

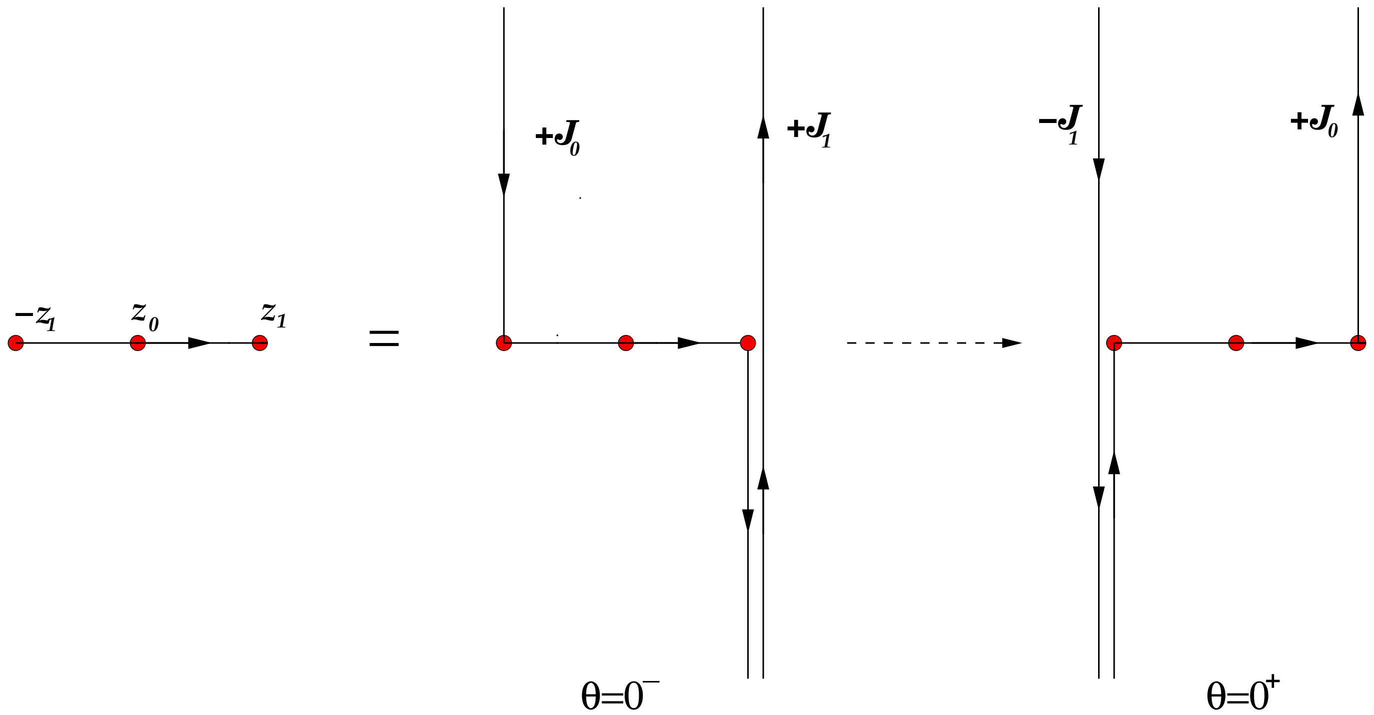
At least in the small-S1 regime, this solves the large-N vs. instanton puzzle. BPST instantons are unimportant, kink-instantons survive large-N limit.

Why is this happening? Stokes line and Stokes phenomenon

Zero dimensional toy example in steepest descent (semi-classical) approx.

$$Z^{0\mathrm{d}}(\lambda) = \frac{1}{\sqrt{\lambda}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, e^{-\frac{1}{2\lambda}\sin^2(x)}$$

To each saddle, there is a unique steepest descent path (generalization to multi-dimension is Lefschetz thimble). These thimbles form natural basis for analytic continuation.
P-saddle and NP-saddle. Original cycle = linear combination of these thimbles.
But on the Stokes line, this is two-fold ambiguous.



$$I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \Sigma = \begin{cases} \mathcal{J}_0(0^-) + \mathcal{J}_1(0^-) \\ \mathcal{J}_0(0^+) - \mathcal{J}_1(0^+) \end{cases}$$

This does not do justice to QFT case, but at least helps visualization of the cancellation.

Resurgence Theory and Transseries

Ecalle (1980s) formalized asymptotic expansion with exponentially small terms (called trans-series) & generalized Borel resummation for them by incorporating the Stokes phenomenon. Generalization of Borel-summability, a way to deal with non-Borel summable series...

Basic idea: Start with a formal power series, e.g. an asymptotic (divergent) expansion of Gevrey-1 type: $\Sigma\, a_n g^{2n}$ where $a_n \leq A\, n!\, c^n$ (generic in QFT).

Borel transformation maps this formal series to a series (called germ of an analytic function) with a finite radius of divergence around the origin in Borel plane.

Analytic continuation of the series to a holomorphic function except a set of singularities (pole or branch points) in the complex Borel-plane.

Directional Laplace transforms to find sectorial sums by invoking Stokes phenomenon.

Resurgence theory of Ecalle

Main result: Borel-Ecalle resummation of a transseries exists and is unique, if the Borel transforms of all perturbative series are all "endlessly continuable" =Set of all singularities on all Riemann sheets on Borel plane do not form any natural boundaries.

Such transseries are called "resurgent functions": Example of transseries:

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

Formal: perturbative + (non-perturbative) x (perturbative)

Resurgence theory in path integrals

Pham, Delabaere,....(1990s): Using the theory of resurgent functions, they claim to prove that (in Hamiltonian formulation) the semi-classical (perturbative+ non-perturbative) transseries expansion in Quantum mechanics with double-well and periodic potentials are summable to finite, exact results.

We want to understand this better in path integrals, which more easily generalize to QFT.

Key step is in the analytic continuation of paths in field space (cf. Pham, and recent papers by Witten), to make sense of steepest descent and Stokes phenomenon in path integrals.
A recent talk by Kontsevich "Resurgence from the path integral perspective", Perimeter Institute, August, 2012.

Graded Resurgence triangle

The semi-classical exp. of CP(N-1) and many QFTs is encoded into the following structure.

$f_{(0,0)}$ → pert. theory around pert. vacuum

kink x (pert. fluctuations) ← $e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)}$

$e^{-\frac{2A}{\lambda}} f_{(2,0)}$ → bions x (pert. fluctuations)

$$f_{(0,0)}$$

$$e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)} \qquad e^{-\frac{A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(1,-1)}$$

$$e^{-\frac{2A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,2)} \qquad e^{-\frac{2A}{\lambda}} f_{(2,0)} \qquad e^{-\frac{2A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,-2)}$$

$$e^{-\frac{3A}{\lambda}+3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,3)} \qquad e^{-\frac{3A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(3,1)} \qquad e^{-\frac{3A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-1)} \qquad e^{-\frac{3A}{\lambda}-3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-3)}$$

$$e^{-\frac{4A}{\lambda}+4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,4)} \qquad e^{-\frac{4A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,2)} \qquad e^{-\frac{4A}{\lambda}} f_{(4,0)} \qquad e^{-\frac{4A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-2)} \qquad e^{-\frac{4A}{\lambda}-4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-4)}$$

$$\iddots \qquad \vdots \qquad \ddots$$

No two column can mix with each other in the sense of cancellation of ambiguities.

N.P. confluence equations

In order QFT to have a meaningful semi-classical continuum definition, a set of perturbative--non-perturbative confluence equations must hold. Examples are

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm} + \mathbb{B}_{[2,0],\theta=0^\pm}[\mathcal{B}_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0],\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0]\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0^\pm} + \dots\Big)$$

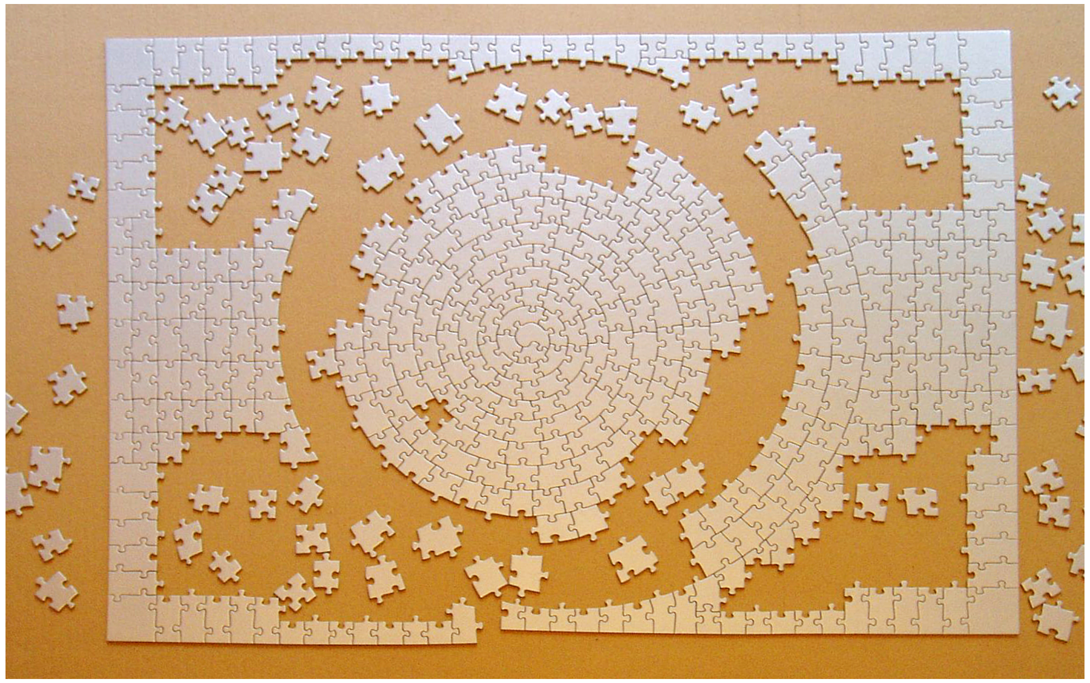
Meaning, order by order hierarchical confluence equations:

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm \ , \qquad (\text{up to } e^{-4S_0})$$

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm + \mathrm{Im}\mathbb{B}_{[2,0]\pm}\mathrm{Re}[\mathcal{B}_{ii}] + \mathrm{Re}\mathbb{B}_{[4,0]}\mathrm{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_\pm \qquad (\text{up to } e^{-6S_0})$$

$$0 = \dots$$

These equations are recently shown to be consequence of the median resummation in resurgence theory, shown in a beautiful paper by Ines Aniceto and Ricardo Schiappa (13).



Decoding late terms in pert. theory.

$$\mathrm{Disc}\,\mathbb{B}_{[0,0]} = -2\pi i \lambda^{-r_2} P_{[2,0]} e^{-2A/\lambda} + \mathcal{O}(e^{-4A/\lambda}), \tag{1}$$

Using dispersion relation, we obtain

$$\begin{aligned} a_{[0,0],q} &= \sum_{q'=0}^{\infty} a_{[2,0],q'} \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2-q'}} + O\left(\left(\frac{1}{4A}\right)^q\right) \\ &= \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2}} \left[a_{[2,0],0} + \frac{2A}{(q+r_2-1)} a_{[2,0],1} + \frac{(2A)^2}{(q+r_2-1)(q+r_2-2)} a_{[2,0],2} + \ldots \right] \\ &+ O\left(\left(\frac{1}{4A}\right)^q\right) \end{aligned} \tag{2}$$

Late terms in pert.exp. around the pert. vac.

Early terms in pert.exp. around neutral bion= 1/q corrections:

Neutral bion action, or two-kink instanton action

Exponentially suppressed corrections: Bion-bion etc. terms.

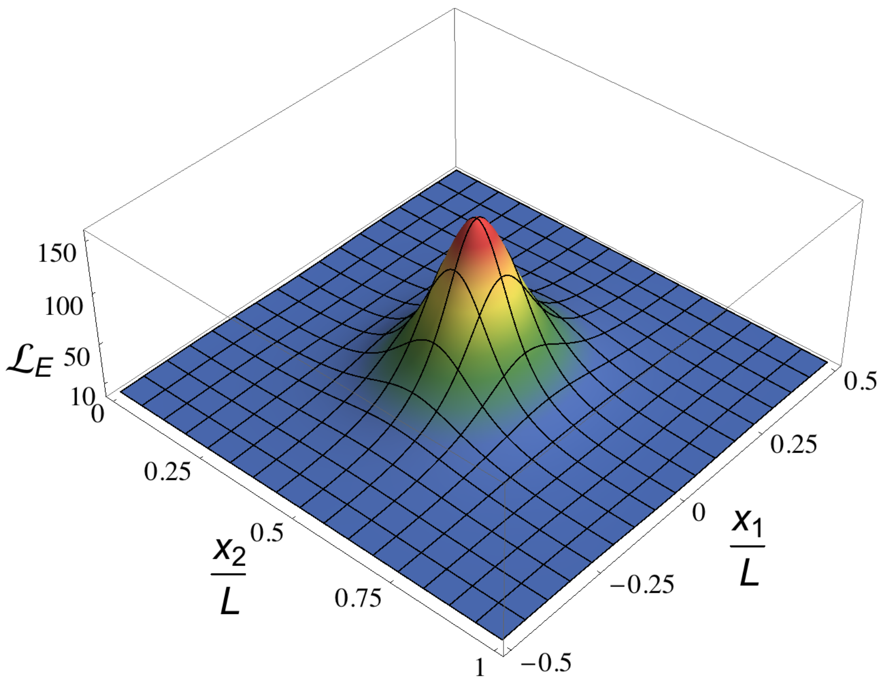
Is all NP data is somehow already present in PT?

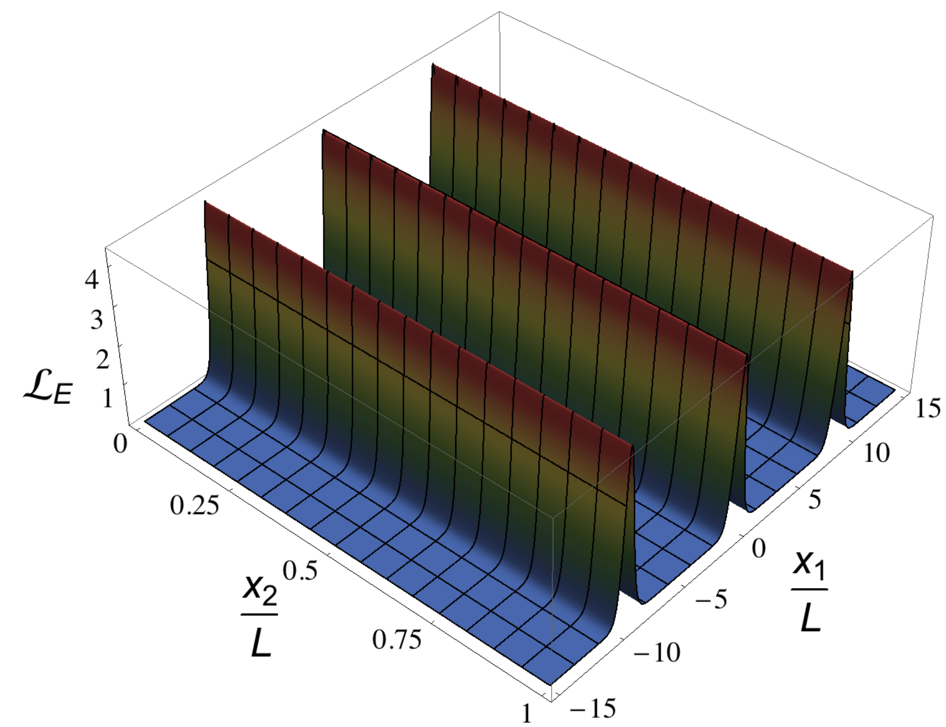
PCM (trivial homotopy group)

Uniton saddle: Harmonic map from S2 to SU(N) found by Uhlenbeck(89).
Later, shown to have a negative mode, and did not receive much attention in physics.
Quantum interpretation: Singularity in Borel plane (but far from the origin).

Fracton saddles:(1-d instanton and twisted -instanton),
with action 1/N of uniton. See below for SU(3)...

Correlated events:....





Conclusions and prospects

It seems plausible that continuity and resurgence theory can be used in combination to provide a non-perturbative continuum definition of asymptotically free theories, and more general QFTs.

The construction may have practical utility and region of overlap with lattice field theory. One can check predictions of the formalism numerically.

Resurgence provides a more refined classification of non-perturbative saddles wrt topological classification, e.g., as shown in resurgence triangle.

Even in theories without a homotopy group, it yields a classification of NP-saddles. There are actually infinitely many NP-saddles, similar to CP(N-1). But not enough time to put them in this talk.

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