Supersymmetry in Curved Spacetime

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January 7, 2014

Why Susy in curved space

Allows to compute exactly many interesting observables such as:

- Partition function Z on a compact manifold $\mathcal{M}$.
- Expectation value of supersymmetric operators.

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For instance:

- The partition function on $S^3 \times S^1$ of $\mathcal{N}=1$ theories with a $U(1)_R$ symmetry. [Romelsberger; Dolan Osborn ...]
- The partition function on S^4 of $\mathcal{N}=2$ theories. [Pestun;...]
- The partition function on S^3 of $\mathcal{N}=2$ theories with a $U(1)_R$ symmetry.[Kapustin, Willett, Yaakov;...]

Why Susy in curved space

These results can be extended to less symmetric manifolds:

- $\mathcal{N} = 1, 2, 4$ on $S_b^3 \times R$

- $\mathcal{N} = 2$ on S_b^3 [Hama, Hosomichi, Lee; Imamura]
- $\mathcal{N} = 2$ on S_b^4 [Hama, Hosomichi]

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- $\mathcal{N} = 2$ on S_b^3 [Hama, Hosomichi, Lee; Imamura]
- $\mathcal{N} = 2$ on S_b^4 [Hama, Hosomichi]

Different squashings of S^3 have been considered

- Some preserve an $SU(2) \times U(1)$ isometry others just $U(1) \times U(1)$
- The metric can depend on arbitrary functions or on a finite number of parameters

The partition function on all these backgrounds is a certain function of a single complex parameter b.
[Alday Martelli Richmond Sparks]

Questions

- Which manifolds $\mathcal{M}$ allow for Susy?
- What is the structure of supersymmetric theories on $\mathcal{M}$?
- Dependence of susy observables on the geometry of $\mathcal{M}$.

Outline

A general framework to understand Susy on curved manifolds.

Classification of Susy backgrounds. Survey of results in different number of dimensions.

Dependence of partition functions on geometry

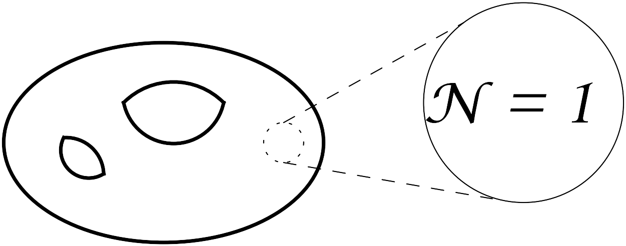
Susy on Curved Manifolds

Consider a supersymmetric theory in flat space.

We want to place it on a manifold $(\mathcal{M}, g)$ so that:

- The short distance limit of the theory is unaffected.
- The theory is invariant under some Supersymmetry.

At short distances the Susy transformations are part of the flat space superalgebra.



The Rigid Limit of SUGRA

Consider an off-shell formulation of Supergravity and give arbitrary background values to the fields in the gravity multiplet:

- The metric $g_{\mu\nu}$
- Various auxiliary fields.
- Set the gravitino $\psi_{\mu\alpha} = 0$

Send $M_p \to \infty$ keeping the background values for the metric and auxiliary fields fixed.

Some supersymmetry is preserved if it is possible to find ζ_α such that the SUSY variation of the gravitino is zero:

$$\delta_\zeta \psi_{\mu\alpha} = 0 \quad \Rightarrow \quad \nabla_\mu \zeta_\alpha = \mathcal{M}_{\mu\alpha}{}^{\beta} \zeta_\beta$$

where $\mathcal{M}_\mu$ depends on the the metric and auxiliary fields.

The Rigid Limit of SUGRA: Comments

- Different backgrounds treated in a unified way.

- Different than Linearized Sugra.

$$\nabla_\mu \zeta_\alpha = \mathcal{M}_{\mu\alpha}{}^{\beta} \zeta_\beta$$

- The "Killing" equation for ζ depends only on the fields in the gravity multiplet

$$\Downarrow$$

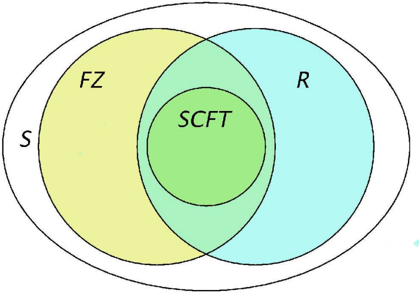
Weak dependence on the matter content.

Generalized treatment of different theories.

- We do not impose e.o.m. for the auxiliary fields. Different off shell formulations of SUGRA can lead to distinct deformations.

The Rigid Limit of SUGRA: Comments

A $\mathcal{N}=1$ theory in 4D possesses an $\mathcal{S}$ multiplet containing the energy momentum tensor as top component. Sometimes it can be shortened; e.g if the theory has a $U(1)$ R-symmetry. SCFT's have an even shorter multiplet. [Komargodski, Seiberg, Dumitrescu]



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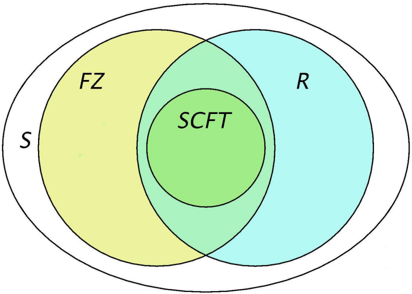
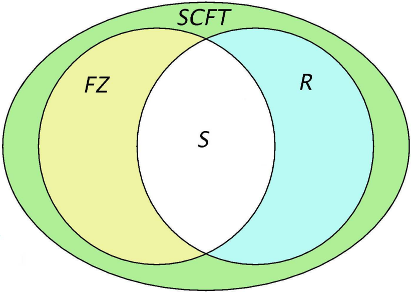




Every multiplet can be coupled to a different off shell formulation of SUGRA. Taking the rigid limit each allows a different set of supersymmetric manifolds. The shorter the multiplet the more manifolds the theory can be placed on preserving susy.

Example: New Minimal Sugra

In a $\mathcal{N} = 1$ theory with a $U(1)_R$ symmetry consider

- The energy momentum tensor $T_{\mu\nu}$
- The conserved R-current j^R_μ
- The supercurrent $S_{\mu\alpha}$

Together with a string current $C_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\rho \mathcal{A}^\sigma$ they form the $\mathcal{R}$-multiplet. It couples to the fields in New Minimal Sugra:

- The metric $g_{\mu\nu}$
- The gravitino ψ^μ_α
- An auxiliary $U(1)_R$ connection $A_\mu \sim A_\mu + \partial_\mu a$
- An auxiliary vector V^μ. It is conserved $\nabla_\mu V^\mu = 0$

In the Rigid Limit we set $\psi^\mu_\alpha = 0$ and freeze the metric and auxiliary fields to arbitrary background values.

New Minimal Sugra, the Rigid Limit

Consider a flat space $\mathcal{N} = 1$ theory with an $U(1)_R$ symmetry. Coupling to Sugra and taking the rigid limit we obtain:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

- $\mathcal{L}_0$ is the flat space theory minimally coupled to the metric.
- $\mathcal{L}_1$ are terms of order $\frac{1}{r}$ which couple the auxiliary fields to definite components of the R-multiplet

$$\mathcal{L}_1 = -j_\mu^{(R)}\left(A^\mu\right) - \mathcal{A}_\mu V^\mu .$$

  At this order the deformation of the flat space theory can be described also when a Lagrangian is not available.
- $\mathcal{L}_2$ are $\frac{1}{r^2}$ terms with curvature or two auxiliary fields.

$$q\left(\frac{1}{4}R + \frac{3}{2}V_\mu V^\mu + 2V_\mu A^\mu\right)(\phi\bar{\phi})$$

Rigid Variations

The Susy transformation are deformed from their flat space counterparts. E.g. for a chiral multiplet of R-charge q:

$$\delta\phi^i = -\sqrt{2}\zeta\psi^i$$
$$\delta\psi^i_\alpha = -\sqrt{2}\zeta_\alpha F^i - i\sqrt{2}(\sigma^\mu\bar{\zeta})_\alpha(\partial_\mu - iq(A_\mu + \frac{3}{2}V_\mu))\phi^i$$
$$\delta F^i = -i\sqrt{2}\bar{\zeta}\bar{\sigma}^\mu\left(\nabla_\mu - i(q-1)(A_\mu + \frac{3}{2}V_\mu) - \frac{i}{2}V_\mu\right)\psi^i$$

Setting to zero the gravitino variation gives the Killing spinor equations:

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{i}{2}V^\nu\sigma_\mu\bar{\sigma}_\nu\zeta \;, \qquad (\nabla_\mu + iA_\mu)\bar{\zeta} = \frac{i}{2}V^\nu\bar{\sigma}_\mu\sigma_\nu\bar{\zeta}$$

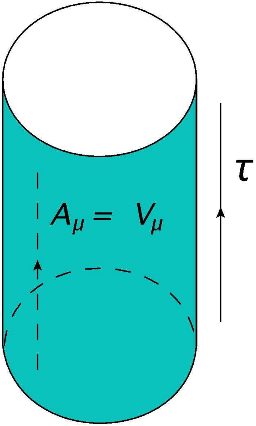
On an Euclidean manifold $\mathcal{M}$ the spinors ζ and $\bar{\zeta}$ are independent and V_μ, A_μ are complex.

Example: $S^3 \times S^1$ [D. Sen; Romelsberger]

Consider the (Euclidean) cylinder $S^3 \times R$

The isometry group is $SU(2)_\ell \times SU(2)_r \times R$

$$A = V = \frac{i}{r} d\tau$$



- The spinors ζ and $\bar{\zeta}$ are τ independent.
- The spinors ζ and $\bar{\zeta}$ are in $(\frac{1}{2}, 0)$.
- No need for superconformal symmetry.

The resulting superalgebra is $SU(2|1)_\ell \times SU(2)_r \times R$
Because the spinors are τ independent we can compactify R to S^1.

If additional $U(1)_f$ flavor symmetries are present we can add complex background gauge fields $A^f = -\frac{i}{r}\mu_f d\tau$ along S^1.

3D New Minimal Sugra

$\mathcal{N} = 1$ theories in 4D reduce to 3D $\mathcal{N} = 2$ theories.

$$\{Q_\alpha, \bar{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\epsilon_{\alpha\beta} Z \ , \quad \{Q_\alpha, Q_\beta\} = 0 \ , \quad \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0$$

The $\mathcal{R}$-multiplet reduces to: [Dumitrescu,Seiberg]

$$T^{\mu\nu}, \quad S^\mu_\alpha, \quad j^R_\mu, \quad j^Z_\mu, \quad J$$

New Minimal Sugra in $3D$ has three auxiliary fields with couplings:

$$j^R_\mu \left(A^\mu - \frac{3}{2} V^\mu \right) - i\, j^Z_\mu C^\mu + JH, \qquad V = *dC$$

For a Superconformal theory j^Z_μ and J are redundant operators.

3D New Minimal Sugra

Matter multiplets and the form of supersymmetric Lagrangians in the rigid limit can be worked out.
[Closset, Dumitrescu, Komargodski, GF]

The Killing Spinor equations corresponding to 3D New Minimal Sugra are:

$$\begin{aligned}
(\nabla_\mu - iA_\mu)\zeta &= -\tfrac{1}{2}H\gamma_\mu\zeta - iV_\mu\zeta - \tfrac{i}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta \\
(\nabla_\mu + iA_\mu)\tilde{\zeta} &= -\tfrac{1}{2}H\gamma_\mu\tilde{\zeta} + iV_\mu\tilde{\zeta} + \tfrac{i}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde{\zeta}
\end{aligned} \tag{1}$$

They were analyzed in [Klare, Tomasiello, Zaffaroni; CKDF].

Recently a complete component formulation of $\mathcal{N} = 2$ New Minimal Sugra in 3D and its coupling to matter was made available
[Kuzenko, Lindstrom, Rocek, Sachs, Tartaglino-Mazzucchelli]

Example S^3 [Kapustin,Willet,Yaakov; Jafferis,...]

$\mathcal{N} = 2$ theory with $\mathrm{U}(1)_R$ on round S^3 is SUSY for

$$H = -\frac{i}{r}, \qquad V = A = 0$$

There are 4 supercharges with superalgebra $SU(2|1)_\ell \times SU(2)_r$

Not reflection positive unless H decouples (theory is conformal).

When additional $U(1)_f$ flavor symmetries are present

- We can add real masses m_f
- $\mathrm{U}(1)_R$ can be shifted by $\mathrm{U}(1)_f$ (Improvements of $\mathcal{R}$ multiplet)

$$j^R_\mu \to j^R_\mu + t\, j^f_\mu\ , \ldots$$

- They appear as $t + i\ m_f$

Classification of SUSY geometries

Classifying Supersymmetric Manifolds

Consider the 4d New Minimal Sugra Killing spinor equation.

On which Riemannian Manifolds $(\mathcal{M}, g)$ are there solutions of

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{i}{2}V^\nu \sigma_\mu \bar{\sigma}_\nu \zeta \ , \qquad (\nabla_\mu + iA_\mu)\bar{\zeta} = \frac{i}{2}V^\nu \bar{\sigma}_\mu \sigma_\nu \bar{\zeta}$$

for some choice of background fields A_μ and V_μ?

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- One Killing spinor ζ exists if and only if $\mathcal{M}$ is complex.
- Two Killing spinors of opposite R-charge ζ and $\bar{\zeta}$ are present only on torus fibrations over a Riemann surface Σ.
- Two Killing spinors of the same R-charge require $SU(2)$ holonomy (compact case).
- Four supercharges are only present on R^4 or $S^3 \times R$ or $H^3 \times R$ (and their compactifications)

One supercharge in 4d with $U(1)_R$

$$(\nabla_\mu - iA_\mu)\zeta = -\frac{i}{2} V^\nu \sigma_\mu \bar{\sigma}_\nu \zeta$$

A solution ζ is everywhere nonzero hence we can form the tensor

$$J^\mu{}_\nu = \frac{2i}{|\zeta|^2} \zeta^\dagger \sigma^\mu{}_\nu \zeta$$

- $J^\mu{}_\nu$ is an almost complex structure $J^\mu{}_\nu J^\nu{}_\rho = -\delta^\mu_\rho$
- $J^\mu{}_\nu$ is metric compatible. $g_{\rho\lambda} J^\rho{}_\mu J^\lambda{}_\nu = g_{\mu\nu}$
- $J^\mu{}_\nu$ is integrable because ζ is Killing.

The triple $(\mathcal{M}, \; g_{\mu\nu}, \; J^\mu{}_\nu)$ defines an Hermitian manifold.

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The triple $(\mathcal{M},\ g_{\mu\nu},\ J^\mu{}_\nu)$ defines an Hermitian manifold.

Conversely on any Hermitian manifold $\exists$ a solution ζ such that

$$J^\mu{}_\nu = \frac{2i}{|\zeta|^2}\zeta^\dagger \sigma^\mu{}_\nu \zeta$$

[Klare, Tomasiello, Zaffaroni; Dumitrescu, Seiberg, GF]

One supercharge in 4d with $U(1)_R$

In the Kähler case we could set $V_\mu = 0$ and get $(\nabla_\mu - iA_\mu)\zeta = 0$

The holonomy of the Levi-Civita connection is in $U(1)_l \times SU(2)_r$ we can twist away its $U(1)_l$ part using A_μ.

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In the general case:

- The auxiliary vector field V_μ encodes the failure of $\mathcal{M}$ to be Kähler.

$$V_\mu = -\frac{1}{2}\nabla_\nu J^\nu{}_\mu + (*W)_\mu, \quad W_{ij\bar{j}}, \quad dW = 0$$

- The Chern connection has holonomy in $U(1)_l \times SU(2)_r$. A_μ twists away its $U(1)_l$ part.

The Superalgebra generated by ζ is $\{Q_\zeta, Q_\zeta\} = 0$.

Q_ζ is a scalar under holomorphic complex coordinate changes followed by appropriate R-transformations.

Background gauge fields

If the flat space theory has some global symmetry (say $U(1)$) it possesses a conserved current j_μ part of a linear multiplet

$$(J,\ j^\mu,\ j_\alpha,\ \bar{j}_{\dot\alpha})$$

We can couple it to a background gauge multiplet

$$(D,\ a_\mu,\ \lambda_\alpha = 0,\ \bar\lambda_{\dot\alpha} = 0)$$

This background preserves a supercharge ζ if

$$f^{(0,2)}_{\mu\nu} = 0, \qquad D = -\frac{1}{2} J^{\mu\nu} f_{\mu\nu}$$

Hence a_μ is connection on a holomorphic line bundle

Two Supercharges in 4d

If a second solution $\bar{\zeta}$ is present there are further restrictions on the metric. Consider the complex vector field

$$K^\mu = \bar{\zeta}\bar{\sigma}^\mu\zeta, \qquad Re(K^\mu) = X^\mu, \qquad Im(K^\mu) = Y^\mu$$

- K^μ is Killing.
- $X^\mu X_\mu = Y^\mu Y_\mu$ and $X^\mu Y_\mu = 0$
- $J^\mu{}_\nu$ is determined by K^μ and the metric.
- If $[X, Y] \neq 0$ the manifold is locally isometric to $S^3 \times R$
- If $[X, Y] = 0$ the two Killing vector fields X and Y generate translations on a T^2 fibered over a Riemann surface Σ

Two Supercharges 4d/3d

The superalgebra generated by ζ and $\bar{\zeta}$ is

$$\{Q_\zeta, Q_\zeta\} = 0, \quad \{Q_{\bar{\zeta}}, Q_{\bar{\zeta}}\} = 0, \quad \{Q_{\bar{\zeta}}, Q_\zeta\} = \delta_K$$

$$[\delta_K, Q_\zeta] = [\delta_K, Q_{\bar{\zeta}}] = 0$$

By reducing along one direction on the T^2 we obtain the following:

Any $\mathcal{N} = 2$ field theory with a $U(1)_R$ symmetry in 3d can be placed on a circle bundle over Σ preserving two supercharges.

All squashed 3-spheres in the literature are in this class.

One supercharge in 3d

The analysis of the Killing Sinor equation is similar to 4D

$$(*) \quad (\nabla_\mu - iA_\mu)\zeta = -\frac{1}{2}H\gamma_\mu\zeta - iV_\mu\zeta - \frac{i}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta$$

Again a nontrivial solution has no zeros and we define:

$$\eta^\mu = \frac{1}{|\zeta|^2}\zeta^\dagger\gamma^\mu\zeta, \qquad \Phi^\mu{}_\nu = \epsilon^\mu{}_{\nu\rho}\eta^\rho$$

- $\eta_\mu\eta^\mu = 1$
- $g_{\mu\nu} = \eta_\mu\eta_\nu - \Phi_{\mu\rho}\Phi^\rho{}_\nu$

This gives rise to an almost contact metric structure on $\mathcal{M}$
Because of the Killing equation (*) it satisfies the condition

$$\Phi^\mu{}_\rho(\mathcal{L}_\eta\Phi)^\rho{}_\nu = 0$$

[Klare, Tomasiello, Zaffaroni; Closset, Dumitrescu, Komargodski, GF]

One supercharge in 3d

The foliation generated by η^μ is Transversely Holomorphic.

This condition is similar to the integrability condition for $J^\mu{}_\nu$. We can choose coordinates $\tau \in R$ and $z \in C$ such that

- $\eta^\mu \partial_\mu = \partial_\tau$
- $\Phi^z{}_{\bar{z}} = \Phi^{\bar{z}}{}_z = 0$ while $\Phi^z{}_z = i$ and $\Phi^{\bar{z}}{}_{\bar{z}} = -i$
- $ds^2 = (d\tau + h(\tau, z, \bar{z})dz + \bar{h}(\tau, z, \bar{z})d\bar{z})^2 + c(\tau, z, \bar{z})\, dzd\bar{z}$
- Two such coordinate charts are related by

$$\tau' = \tau + t(z, \bar{z}), \qquad z' = f(z)$$

Given a metric $g_{\mu\nu}$ on an orientable three manifold $\mathcal{M}$ it is always possible to find η^μ which satisfies the constraint in a patch.
However there are global obstructions.

One supercharge in 3d

As in 4D there are connections (with torsion) such that

- $\nabla^c_\mu g_{\nu\rho} = \nabla^c_\mu \eta_\nu = 0$
- The holonomy of ∇^c_μ is contained in $U(1) \subset SU(2)$

Given a $(\Phi^\mu{}_\nu,\ \eta_\mu)$ satisfying the constraint we can rewrite the Killing spinor equation as

$$(\nabla^c_\mu - iA^c_\mu)\zeta = 0, \qquad A^c_\mu = A_\mu + \dots$$

Again we use A^c_μ to twist away the $U(1)$ holonomy and find a Killing spinor ζ.

H and V_μ are determined by $\nabla_\mu \eta_\nu$.

More Results I

The analysis in 4d and 3d has been extended in several directions:

- Classification of Susy geometries for theories with an FZ multiplet [Dumitrescu, GF]. Susy on S^4
- The New Minimal case in 4D has been considered also in Lorentzian signature. Killing spinors are related by c.c. They are present whenever there is a nonzero null Killing vector. [Cassani , Klare, Martelli, Tomasiello, Zaffaroni]
- The situation is similar in 3D for $\mathcal{N} = 2$ theories. Preserving Susy is related to the presence of a nonzero Killing vector which can be either timelike or null. [Hristov, Tomasiello, Zaffaroni]

More Results II

[Klare, Zaffaroni] considered $\mathcal{N} = 2$ conformal Sugra both in Euclidean and Lorentzian Signature.

- In the Lorentzian case rigid Susy requires the presence of a timelike or null conformal Killing vector.
- Similarly in Euclidean signature we can preserve Susy when there is a conformal Killing vector K. Then $Q^2 = K$.
- In Euclidean signature there are more ways to preserve susy. E.g. the topological twist using the $SU(2)$ R-symmetry.

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In some cases the theory need not be conformal. For instance it is possible to preserve one supercharge on a squashed S^4 with $U(1) \times U(1)$ isometry [Hama, Hosomichi]. This squashing is not $\mathcal{N} = 1$ supersymmetric.

Different Dimensions

In 2D [Benini, Cremonesi; Doroud, Gomis, LeFloch Lee] considered $\mathcal{N} = (2,2)$ theories on round and squashed S^2.

For $\mathcal{N} = 1$ In 5D we lack a classification of supercurrent multiplets nevertheless many interesting examples of susy manifolds have been considered.

- $S^4 \times S^1$ [Kim, Kim, Lee; Terashima]
- S^5 is a Hopf fibration of S^1 over CP^2. There are Susy theories on the round S^5 with 8 supercharges. Q^2 corresponds to a translation along a Hopf fiber. A background Sugra interpretation is possible. [Hosomichi, Seong Terashima]

Different Dimensions

[Imamura] considered squashing S^5 by changing the S^1 radius. The resulting manifold has $SU(3) \times U(1)$ isometry. He showed how to place an $\mathcal{N} = 1$ theory on this manifold preserving either 6 or 2 supercharges. In the first case the Q's do not square to translations along the Hopf fiber.

A classification program for $\mathcal{N} = 1$ supersymmetric geometries relying on background Sugra has been initiated by [Pan]. Besides S^5 and $S^4 \times S^1$ it allows to consider $S^3 \times S^2$, $T^3 \times S^2$, $T^2 \times S^3$....

Different Dimensions

It is possible to define a twisted version of an $\mathcal{N} = 1$ theory on a contact manifold (with compatible metric) [Källén, Zabzine]. For the round S^5 the relation of the twisted theory to the one in [HST] was worked out explicitly [Källén, Qiu Zabzine].

Finally supersymmetric manifolds for $\mathcal{N} = (1, 0)$ theories in 6D have been studied using background Sugra resulting in a set of necessary or sufficient conditions. [Samtleben, Sezgin, Tsimpis]

Dependence of Observables on Geometry

Dependence of Observables on Geometry

- Start with analyzing the partition function of $\mathcal{N} = 1$ theories on compact complex manifolds $\mathcal{M}_4$
- Here we will consider the limit of large $\mathcal{M}_4$. A linearized analysis around flat space is applicable
- Only classical considerations.

Linearized Analysis

Choose a single supercharge Q_ζ in flat space $\Rightarrow$ choice of $J^\mu{}_\nu$.

Small variations of the geometry $\delta g_{\mu\nu}$, $\delta J^\mu{}_\nu$ and of $W_{\mu\nu\rho}$ couple to the R-multiplet while changes in the background gauge fields δa_μ couple to the corresponding linear multiplet.

$$\delta\mathcal{L} = -\frac{1}{2}\delta g^{\mu\nu} T_{\mu\nu} + \delta A^\mu j_\mu^{(R)} + \delta V^\mu \mathcal{A}_\mu - \delta a_\mu j^\mu + \delta D J \ .$$

Some terms in $\delta\mathcal{L}$ are Q-exact and do not contribute to Z.

Q_ζ is a scalar under complex coordinate changes $\Rightarrow$ the results holds also at the nonlinear level.

Deformations of Complex structures

Choose a c.s $J^{\mu}{}_{\nu}$ on $\mathcal{M}_4$ and deform it by adding $\delta J^{\mu}{}_{\nu}$.

In complex coordinates adapted to $J^{\mu}{}_{\nu}$ the requirement that $J + \delta J$ is an almost complex structure implies that at linear level

$$\delta J^{i}{}_{j} = \delta J^{\bar{i}}{}_{\bar{j}} = 0$$

The remaining components are constrained by the integrability requirement

$$\partial_{\bar{j}} \delta J^{i}{}_{\bar{i}} - \partial_{\bar{i}} \delta J^{i}{}_{\bar{j}} = 0$$

δJ generated by diffeomorphisms are trivial $\delta J^{i}{}_{\bar{i}} \sim \delta J^{i}{}_{\bar{i}} + 2i\partial_{\bar{i}}\epsilon^{i}$
Hence, at first order, deformations of $J^{\mu}{}_{\nu}$ are determined by

$$\Theta^{i} = \delta J^{i}{}_{\bar{i}} d\bar{z}^{\bar{i}}, \qquad [\Theta^{i}] \in H^{0,1}(\mathcal{M}_4, T^{1,0}(\mathcal{M}_4))$$

Deformations of metric and background gauge fields

Variations of the metric are constrained by the change in complex structure

- $\delta g_{i\bar{j}}$ are unconstrained
- $\delta g_{ij} = \frac{i}{2}\left(g_{i\bar{k}}\Delta J^{\bar{k}}{}_{j} + g_{j\bar{k}}\Delta J^{\bar{k}}{}_{i}\right)$

In the same way for Abelian background gauge fields we must have

$$\partial_{\bar{i}}\delta a_{\bar{j}} - \partial_{\bar{j}}\delta a_{\bar{i}} = 0$$

modulo gauge transformations $\delta a_\mu = \partial_\mu \epsilon$. Hence the holomorphic line bundle moduli are in $H^{0,1}(\mathcal{M}_4)$.

The deformed Lagrangian

We can express the deformation of the Lagrangian $\delta\mathcal{L}$ in terms of the variations of $\delta J^{\mu}{}_{\nu}$ $\delta g_{\mu\nu}$ and δa_{μ}. (We set $W = d\tilde{B}$)

$$\delta\mathcal{L} = Q_{\zeta}(\mathcal{I}) + \delta J^{i}{}_{\bar{i}}\mathcal{O}^{\bar{i}}_{i} + \delta a_{\bar{i}}\mathcal{J}^{\bar{i}}$$

- $\delta g_{i\bar{j}}$ appears in Q_{ζ} exact terms.
  $Z(\mathcal{M}_4)$ does not depend on the Hermitian metric.
- Varying $W = d\tilde{B}$ does not change $Z(\mathcal{M}_4)$.
  Dependence on $W_{\mu\nu\rho}$ is at most cohomological.

Invariance under diffeomorphisms and gauge transformations implies that for $\delta J^{i}{}_{\bar{i}} = 2i\partial_{\bar{i}}\epsilon^{i}$ and $\delta a_{\bar{i}} = \partial_{\bar{i}}\epsilon$

$$\delta\mathcal{L}_{\text{trivial}} = Q_{\zeta}(\mathcal{I}') + \text{total der}$$

- The partition function depends holomorphically on the moduli of the complex structure and of the holomorphic line bundle.
  Z can however be singular.

Example $S^3 \times S^1$

Display $S^3 \times S^1$ as a complex manifold by a quotient of C^2–$\{(0,0)\}$.

$$(z_1, z_2) \sim (pz_1, qz_2), \qquad 0 < |p| \leq |q| < 1$$

p, q are complex structure moduli.

We will denote this branch of the moduli space of complex structures on $S^3 \times S^1$ by $\mathcal{M}_4^{p,q}$.

- There exists an Hermitian metric that allows to preserve 2 supercharges for any (p, q).
- For $p = q^*$ we can preserve four supercharges.

Example $S^3 \times S^1$

The partition function on $S^3 \times S^1$ is the supersymmetric index

$$\mathcal{I}(p, q, u) = Tr_{S^3}\left((-1)^F p^{J_3+J_3'-R/2} q^{J_3-J_3'-R/2} u^{Q_f}\right)$$

- The fugacities p, q can be identified with the moduli of $\mathcal{M}_4^{p,q}$
- u the fugacity for Q_f is an holomorphic line bundle modulus.
- The index is meromorphic in p, q and u.
- It does not depend on the choice of Hermitian metric.

Conclusions

Placing supersymmetric theories on different manifolds preserving Susy provides a new set of tools to study the dynamics of strongly coupled theories.

Turning on background values for the fields in the supergravity multiplet and taking the rigid limit allows a general description of rigid SUSY in curved space.

Conclusions

Placing supersymmetric theories on different manifolds preserving Susy provides a new set of tools to study the dynamics of strongly coupled theories.

Turning on background values for the fields in the supergravity multiplet and taking the rigid limit allows a general description of rigid SUSY in curved space.

The $(\mathcal{M}, g)$ allowing for SUSY can be identified independently from the matter content.

This allows a classification of supersymmetric geometries.

We can study the dependence on the geometry of supersymmetric observables. We find they are "almost" topological.

Thank You!