$3d$ $\mathcal{N} = 4$ Gauge Theories, Hilbert series and Hall-Littlewood Polynomials

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April 1, 2014

Based on the following works:

- [arXiv:1403.0585, 1403.2384] with S. Cremonesi, A. Hanany, A. Zaffaroni
- [arXiv:1402.0016] with A. Dey, A. Hanany, P. Koroteev
- [arXiv:1205.4741] with A. Hanany and S. Razamat
- [arXiv:1111.5624] with C. Keller, J. Song and Y. Tachikawa
- [arXiv:1110.6203] with A. Hanany
- [arXiv:1005.3026] with S. Benvenuti and A. Hanany

Features of $3d$ $\mathcal{N} = 4$ gauge theories

- Two branches of the moduli space:
  - Higgs branch: VEVs of scalars components of the hypermultiplets.
    Classically exact.
  - Coulomb branch: VEVs of scalars components of the vector multiplets.
    Receives quantum corrections.
  - Both are hyperKähler spaces.
- R-symmetry: $SU(2)_H \times SU(2)_V$.
- A quantum description of the Coulomb branch involves monopole operators.
  [Kapustin et al. from '02]
- Mirror symmetry exchanges the Higgs branch of one theory with the quantum Coulomb branch of another theory, and vice-versa. [Intriligator-Seiberg '97]
  - As a working assumption, it's very useful for studying moduli spaces of various theories.
  - E.g. $3d$ Gaiotto's type theories, whose mirrors have known Langrangians.
    [Benini-Tachikawa-Xie '10]

Overview of the talk

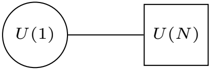
1. The Higgs branch of $3d$ $\mathcal{N} = 4$ gauge theories.
2. The Coulomb branch and its quantum description.
3. Hilbert series as a generating function of the gauge invariant quantities on the moduli space.
4. $3d$ Sicilian theory as a theory of class $\mathcal{S}$ ($4d$ Gaiotto's theory) compactified on S^1.
    - Their mirror theories and the constructions.
    - The Coulomb branch of these mirror theories.
5. Connections with the moduli spaces of instantons.
    - New technology in computing instanton partition functions.

Part I: Higgs branch

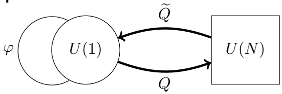
Higgs branch of a $3d$ $\mathcal{N} = 4$ gauge theory

- Translate the theory in $3d$ $\mathcal{N} = 2$ language: The F and D terms give rise to the moment map equations of the hyperKähler quotient.
- A suitable description is in terms of gauge invariant quantities subject to the F term relations.
- A Hilbert series is a generating function that counts these gauge invariant quantities wrt. a $U(1)$ global symmetry that is a generator of $SU(2)_H$ and wrt. the flavour symmetry.
- For the Higgs branch Hilbert series, the global $U(1)$ symmetry can be taken as a generator of the $SU(2)_H$ R-symmetry.

Example 1: $U(1)$ gauge theory with N flavours



- In $3d$ $\mathcal{N} = 2$ notation, the above quiver becomes



with the superpotential $W = \widetilde{Q}_i \varphi Q^i$.

- The relevant F-term for the Higgs branch is $\partial_\varphi W = \widetilde{Q}_i Q^i = 0$.

- The Higgs branch is parametrised by the gauge invariant quantities

$$M^i{}_j = Q^i \widetilde{Q}_j \qquad \text{with} \quad \mathrm{tr}\, M = M^i{}_i = 0 \ .$$

They transform in the adjoint rep., $\mathbf{Adj} = [1, 0, \ldots, 0, 1]$, of $SU(N)$.

- Thanks to the F-term, the square of matrix $M^2 = 0$, *i.e.* M is nilpotent:

$$M^i{}_j M^j{}_k = Q^i \widetilde{Q}_j Q^j \widetilde{Q}^k = 0 \ .$$

Example 1: $U(1)$ gauge theory with N flavours

- The Higgs branch is

$$\{M : M \text{ an } N \times N \text{ matrix, } \operatorname{tr} M = 0 \text{ and } M^2 = 0\}.$$

  This space is also
  - the 'reduced' moduli space of 1 $SU(N)$ instanton on $\mathbb{C}^2$;
  - minimal nilpotent orbit of $SU(N)$.
- Any gauge invariant is a product of a matrix M, which carries charge 2 under the J_3 generator of $SU(2)_H$.
  - The operators with charge p transform in $\mathrm{Sym}^p \mathbf{Adj} = \mathrm{Sym}^p [1, 0, \ldots, 0, 1]$ of $SU(N)$.
  - The (minimal) nilpotency kills all representations but $[p, 0, \ldots, 0, p]$ in $\mathrm{Sym}^p \mathbf{Adj}$.

    [e.g. Kronheimer '90; Vinberg-Popov '72; Garfinkle '73; Gaiotto, Neitzke, Tachikawa '08, Benvenuti-Hanany-NM '10]
- The Higgs branch Hilbert series is

$$H_{\text{red. } 1 \; SU(N) \text{ inst. } \mathbb{C}^2}(t; \boldsymbol{y}) = \sum_{p=0}^{\infty} \chi^{SU(N)}_{[p,0,\ldots,0,p]}(\boldsymbol{y}) \, t^{2p} \; .$$

Example 2: One G instanton on $\mathbb{C}^2$

- The Hilbert series has a uniform expression for one instanton in any simple group G:

$$H_{\text{red. } 1\ G \text{ inst. } \mathbb{C}^2}(t; \boldsymbol{y}) = \sum_{p=0}^{\infty} \chi^G_{p\cdot\mathbf{Adj}}(\boldsymbol{y})\ t^{2p}\ .$$

[Benvenuti-Hanany-NM '10]

- For an instanton moduli space, the Hilbert series has an interpretation of the instanton contribution to the partition function of $5d$ $\mathcal{N}=1$ pure SYM with gauge group G on $S^1 \times \mathbb{R}^4$. [Nekrasov-Okounkov ' 03; Keller-NM-Song-Tachikawa '11]

- $4d$ Nekrasov: With $t = e^{-\frac{1}{2}\beta(\epsilon_1+\epsilon_2)}$, $x = e^{-\frac{1}{2}\beta(\epsilon_1-\epsilon_2)}$, $y_i = e^{-\beta a_i}$,

$$H_{1\ G \text{ inst. } \mathbb{C}^2}(t; x; \boldsymbol{y}) = \frac{1}{(1-tx)(1-tx^{-1})} \sum_{p=0}^{\infty} \chi^G_{p\cdot\mathbf{Adj}}(\boldsymbol{y})\ t^{2p}\ ,$$

reduces to the Nekrasov partition function [Keller-NM-Song-Tachikawa '11]

$$Z_{1 \text{ inst.}}(\epsilon_1, \epsilon_2, \boldsymbol{a}) = -\frac{1}{\epsilon_1\epsilon_2} \sum_{\gamma \in \Delta_l} \frac{1}{(\epsilon_1+\epsilon_2+\boldsymbol{\gamma}\cdot\boldsymbol{a})(\boldsymbol{\gamma}\cdot\boldsymbol{a}) \prod_{\gamma^\vee\cdot\alpha=1,\ \alpha\in\Delta}(\boldsymbol{\alpha}\cdot\boldsymbol{a})}\ ,$$

where Δ and Δ_l are the sets of the roots and the long roots, and $\gamma^\vee = \frac{2\gamma}{\gamma\cdot\gamma}$.

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Part II: Coulomb branch

Coulomb branch of a $3d$ $\mathcal{N} = 4$ gauge theory

- A v-plet contains a gauge field A_μ and 3 real scalars; all in the adjoint rep. of the gauge symmetry G. Combine the latter into a real scalar σ and a complex scalar φ.
- Generic VEVs of σ_j $(j = 1, \ldots, r_G)$ breaks the gauge group G to $U(1)^{r_G}$.
- For each $U(1)$ factor, A_μ can be dualised into a periodic scalar a.
- We have chiral fields: $\exp\left(\frac{\sigma_j}{g^2(\sigma)} + i a_j\right)$.

Quantum description of the Coulomb branch.

- Replace

$$\exp\left(\frac{\sigma_j}{g^2(\sigma)} + i a_j\right) \quad \longrightarrow \quad \text{monopole operators } X_j$$

- A monopole operator for the gauge group G is specified by the magnetic fluxes $\boldsymbol{m} = (m_1, \ldots, m_{r_G})$ at the insertion point.

Coulomb branch of a $3d$ $\mathcal{N} = 4$ gauge theory

- The magnetic flux $\boldsymbol{m}$ in $U(1)^{r_G} \subset G$ is labelled by a weight of the GNO dual group $G^\vee$, modulo the Weyl transformations $W_{G^\vee}$.
- Turning on the monopole flux $\boldsymbol{m}$ breaks G to a residual gauge symmetry $H_{\boldsymbol{m}}$.
  - **Example:** $G = U(2)$, $\boldsymbol{m} = (m_1, m_2)$.
    Up to a Weyl transformation, we can take $m_1 \geq m_2 > -\infty$.
    $H_{\boldsymbol{m}} = U(2)$ if $m_1 = m_2$, and $H_{\boldsymbol{m}} = U(1)^2$ if $m_1 \neq m_2$.
- The monopole operator can be dressed by all possible products of φ that are invariant under the action of $H_{\boldsymbol{m}}$. This is a quantum description of the chiral ring.

Coulomb branch of a $3d$ $\mathcal{N} = 4$ gauge theory

A topological symmetry

- A monopole operator may carry charge under a topological symmetry, which is the centre of the gauge group G.
- For $G = U(N_c)$, there is a topological $U(1)_J$ symmetry, associated with the current $J = *F$ associated with the $U(1)$ factor in $U(N_c)$.
  - A monopole op. with fluxes $\boldsymbol{m}$ carries charge $m_1 + m_2 + \ldots + m_{N_c}$ under $U(1)_J$.
- The topological symmetry (or a collection of them) can enhance to a larger non-abelian symmetry.

Coulomb branch Hilbert series

- Operators on the Coulomb branch: Monopole operators dressed with powers of scalars in the residual gauge group.
- Count these operators with respect to a $U(1)$ global symmetry, which is a generator of $SU(2)_V$ and wrt. the topological $U(1)_J$ charges.
- The **monopole formula** for the Coulomb branch HS: [Cremonesi, Hanany, Zaffaroni '13]

$$H(t;\boldsymbol{z}) = \sum_{\boldsymbol{m}\in\Gamma_{G^\vee}/W_{G^\vee}} t^{\Delta(\boldsymbol{m})}\boldsymbol{z}^{J(\boldsymbol{m})}P_G(t;\boldsymbol{m}) ,$$

  - Fugacity t keeps track of the $U(1)$ global symmetry.
  - Fugacities $\boldsymbol{z} = (z_1, z_2, \ldots)$ keep track of a collection of the $U(1)_J$ charges, $\boldsymbol{J}(\boldsymbol{m})$.
  - $P_G(t;\boldsymbol{m}) = \prod_i 1/(1-t^{2d_i})$, with d_i the degrees of independent Casimirs of $H_{\boldsymbol{m}}$.
  - Dimension of monopole operators:

$$\Delta(\boldsymbol{m}) = \left(\sum_{\text{all hypers}} \sum_{\substack{\boldsymbol{w}:\text{ weights} \\ \text{of rep of each hyper}}} |\boldsymbol{w}(\boldsymbol{m})| \right) - 2\sum_{\boldsymbol{\alpha}\in\Delta_G^+} |\boldsymbol{\alpha}(\boldsymbol{m})| .$$

[Gaiotto-Witten '08; Kim '09, Benna-Klebanov-Klose '09, Bashkirov-Kapustin '10]

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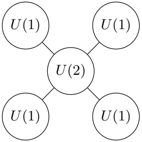
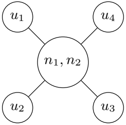
Example: Coulomb branch of the affine D_4 quiver



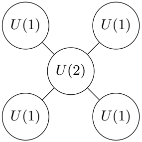


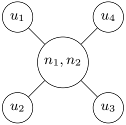
- There's an overall $U(1)$ that decouples. Can remove this from any node, say $U(2)$.
- The Coulomb branch Hilbert series:

$$
\begin{aligned}
& H_{\widehat{D}_4}(t; z_0, z_1, \ldots, z_4) \\
&= \sum_{u_1,\ldots,u_4 \in \mathbb{Z}} \sum_{n_1 \geq n_2 = 0} t^{-2|n_1-n_2| + \sum_{i=1}^4 \sum_{j=1}^2 |u_i - n_j|} \left(z_0^{n_1+n_2} z_1^{u_1} \ldots z_4^{u_4} \right) \times \\
&\qquad [P_{U(1)}(t)]^{-1} P_{U(2)}(t; n_1, n_2) [P_{U(1)}(t)]^4
\end{aligned}
$$

  - $P_{U(1)}(t) = (1-t^2)^{-1}$ and $P_{U(2)}(t; n_1, n_2) = \begin{cases} (1-t^2)^{-2}, & n_1 \neq n_2 \\ (1-t^2)^{-1}(1-t^4)^{-1} & n_1 = n_2 \end{cases}$.
  - z_0 keeps track of the topological charge for $U(2)$ gauge group, and z_i keep track of topological charges for each $U(1)$.

Example: Coulomb branch of the affine D_4 quiver





- An overall $U(1)$ $\Rightarrow$ shift symmetry $n_{1,2} \to n_{1,2} + 2,\ u_{1,\ldots,4} \to u_{1,\ldots,4} + 1.$

  This requires $z_0^2 z_1 z_2 z_3 z_4 = 1.$

- The power series in t admits an $SO(8)$ **character expansion**:

$$H_{\widehat{D}_4}(t; \boldsymbol{z}) = \sum_{p=0}^{\infty} \chi^{SO(8)}_{[0,p,0,0]}(\boldsymbol{z}) t^{2p} \ .$$

  The four $U(1)$ topological symmetries enhance to $SO(8)$.

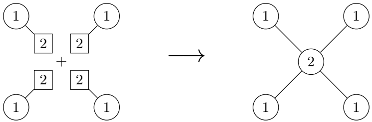
- This is the Hilbert series of
  - the Higgs branch of $SU(2)$ gauge theory with 4 flavours;
  - the reduced moduli space of 1 $SO(8)$ instantons on $\mathbb{C}^2$.

  This agrees with the prediction of mirror symmetry.

Gluing the Coulomb branch Hilbert series

A number of quivers can be constructed from 'gluing' together basic building blocks.

Example:



The Coulomb branch Hilbert series can be computed as follows:

1. Compute the HS of each basic building block with 'background fluxes' $\boldsymbol{n}_F$ turned on for a global flavour symmetry G_F:

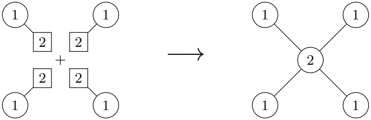
$$H_{G,G_F}(t;\boldsymbol{z};\boldsymbol{n}_F) = \sum_{\boldsymbol{m}\in\Gamma_{G^\vee}/W_{G^\vee}} t^{\Delta(\boldsymbol{m};\boldsymbol{n}_F)} \boldsymbol{z}^{J(\boldsymbol{m})} P_G(t;\boldsymbol{m}) \ ,$$

2. Glue each basic building block together via the common global symmetry G_F:

$$H(t;\boldsymbol{z}^{(1)},\boldsymbol{z}^{(2)},\ldots) = \sum_{\boldsymbol{n}_F\in\Gamma_{G_F^\vee}/W_{G_F^\vee}} t^{-\delta_{G_F}(\boldsymbol{n}_F)} P_{G_F}(t;\boldsymbol{n}_F) \prod_i H^{(i)}_{G,G_F}(t;\boldsymbol{z}^{(i)};\boldsymbol{n}_F) \ ,$$

with $\delta_{G_F}(\boldsymbol{n}_F) = 2\sum_{\boldsymbol{\alpha}\in\Delta_G^+} |\boldsymbol{\alpha}(\boldsymbol{m})|$.

Gluing the Coulomb branch Hilbert series



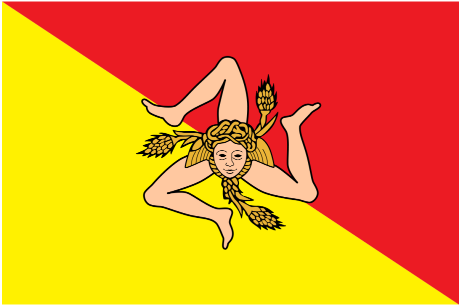
1. The HS for each building block $(1)-[2]$ with background fluxes $\boldsymbol{n}=(n_1,n_2)$ is

$$H_{(1)-[2]}(t;z;\boldsymbol{n}) = \sum_{m=-\infty}^{\infty} t^{|m-n_1|+|m-n_2|} z^m P_{U(1)}(t)$$
$$= \frac{t^{n_1-n_2}\left(t^2\left(z^{-n_1+n_2+1}-z^{n_1-n_2-1}\right)+z^{n_1-n_2+1}-z^{-n_1+n_2-1}\right)}{\left(z-\frac{1}{z}\right)\left(1-\frac{t^2}{z^2}\right)(1-t^2z^2)} .$$

2. Glue four copies of $(1)-[2]$ together:

$$H_{\widehat{D}_4}(t;z_1,\ldots,z_4) = \sum_{n_1\geq n_2=0} t^{-2(n_1-n_2)} [P_{U(1)}(t)]^{-1} P_{U(2)}(t;\boldsymbol{n}) \prod_{i=1}^{4} H_{(1)-[2]}(t;z_i;\boldsymbol{n})$$
$$= \sum_{p=0}^{\infty} \chi^{SO(8)}_{[0,p,0,0]}(\boldsymbol{z})\, t^{2p} .$$

Part III: $3d$ Sicilian theories and beyond!



Picture taken from http://en.wikipedia.org/wiki/Flag_of_Sicily

Credit for the name: [Benini, Tachikawa, Wecht '09; Benini, Tachikawa, Xie '09]

$3d$ Sicilian theories

- These are $3d$ theories obtained from the $6d$ $(2,0)$ theory (of type A, D or E) compactified on S^1 times a Riemann surface with punctures.
  - For type A_{N-1}, each puncture is accompanied by a partition of N.

    The partition $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots)$ induces an embedding of $su(2)$ into $su(N)$ such

$$\underbrace{N}_{N\text{-dim rep of } su(N)} = \underbrace{\rho_1 + \rho_2 + \ldots}_{\text{dim of irreps of } su(2)}$$

  - For type D_N, each puncture is specified by a partition $\boldsymbol{\rho}$ such that

$$\underbrace{2N}_{2N\text{-dim rep of } so(2N)} = \underbrace{\rho_1 + \rho_2 + \ldots}_{\text{dim of irreps of } su(2)}$$

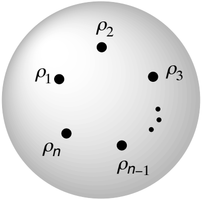
    subject to the condition that *each even ρ_k appears even times*: **D-partition**.
  - For type E_6, see a recent paper [arXiv:1403.4604] by Chacaltana, Distler, Trimm.
- Upon the compactification, one may introduce a twist. This gives rise to other types of embedding and hence other types of partitions, e.g. B and C partitions.

  [Chacaltana, Distler, Tachikawa '12, Chacaltana, Distler, Trimm '13]

$3d$ Sicilian theories

- Given a puncture $\boldsymbol{\rho}$ associated with group H, there is a global symmetry $G_{\boldsymbol{\rho}}$ associated with it. Let r_k be the number of times that k appears in the partition $\boldsymbol{\rho}$.

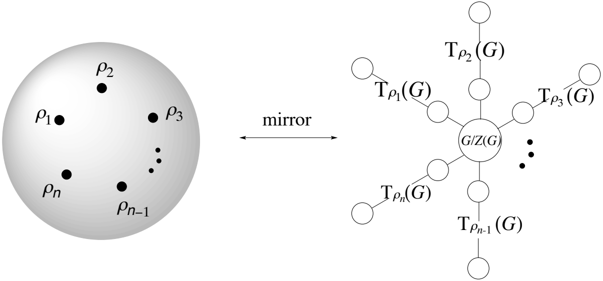
$$G_{\boldsymbol{\rho}} = \begin{cases} S\left(\prod_k U(r_k)\right) & H = U(N)\ , \\ \prod_{k \text{ odd}} SO(r_k) \times \prod_{k \text{ even}} USp(r_k) & H = SO(2N+1) \text{ or } SO(2N)\ , \\ \prod_{k \text{ odd}} USp(r_k) \times \prod_{k \text{ even}} SO(r_k) & H = USp(2N)\ . \end{cases}$$



- For a Riemann surface with a collection of punctures $\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots\}$, the global symmetry is $\prod_i G_{\boldsymbol{\rho}_i}$. This may enhance to a larger group.

A mirror of a $3d$ Sicilian theory

[Benini, Tachikawa, Xie '10]



- For a Sicilian theory with partitions $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_n$ associated with a *classical group*, its mirror theory admits a Lagrangian description.
- A mirror of the theory associated with a *sphere* with punctures $\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots\}$ $=$ a quiver theory formed by gluing basic building blocks, $T_{\boldsymbol{\rho}_1}(G), T_{\boldsymbol{\rho}_2}(G), \ldots$, via their common symmetry $G/Z(G)$, with $Z(G)$ the centre of G.
- For a genus g Riemann surface, the mirror theory is the same but with g additional adjoint hypers under gauge group G.

The $T_\rho(G)$ theory [Gaiotto-Witten ' 08]

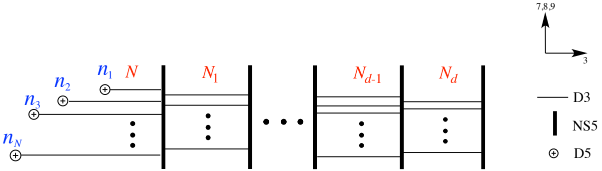
- $T_\rho(G)$ is constructed as a boundary theory of $4d$ $\mathcal{N}=4$ SYM on a half-space, with the half-BPS boundary condition specified by $\rho : su(2) \to \mathrm{Lie}(G^\vee)$.
  - ρ can be classified, up to conjugation, by the nilpotent orbits of $\mathrm{Lie}(G^\vee)$.
- For a classical group G, $T_\rho(G)$ is a quiver theory.
- The quiver for $T_\rho(SU(N))$ is

$$[U(N)] - (U(N_1)) - (U(N_2)) - \cdots (U(N_d))$$

with $\rho = (N - N_1,\ N_1 - N_2,\ \ldots,\ N_{d-1} - N_d,\ N_d)$ and ρ is non-increasing:

$N - N_1 \geq N_1 - N_2 \geq \ldots \geq N_{d-1} - N_d \geq N_d > 0.$

- A brane configuration [Hanany-Witten '97]:



ρ is the set of linking numbers of each NS5-brane.

The $T_{\rho}(G)$ theory

- For $G = SO(N)$, $USp(2N)$, one can consider such a system with D3-branes on top of an appropriate O3-planes [Feng-Hanany '00; Gaiotto-Witten ' 08].

  The partition ρ is associated with the GNO dual group $G^{\vee}$ of G.

- **Example:** For $G = SO(5)$, $G^{\vee} = USp(4)$.

$$
\begin{aligned}
T_{(4)}(SO(5)): &\qquad [SO(5)] \\
T_{(2,2)}(SO(5)): &\qquad [SO(5)] - (USp(2)) - (O(1)) \\
T_{(2,1,1)}(SO(5)): &\qquad [SO(5)] - (USp(2)) - (O(3)) \\
T_{(1,1,1,1)}(SO(5)): &\qquad [SO(5)] - (USp(4)) - (O(3)) - (USp(2)) - (O(1))
\end{aligned}
$$

- **Special case:** $\boldsymbol{\rho} = (1, 1, \ldots, 1)$. The Higgs branch is a nilpotent orbit of the Lie algebra of $G^{\vee}$, and the Coulomb branch is a nilpotent orbit of the Lie algebra of G.

The Coulomb branch of $T_{\rho}(G)$

- The monopole formula for the Coulomb branch Hilbert series works well in most cases for $T_{\rho}(G)$, except that
  1. For an exceptional group G, the Lagrangian description (and quiver) is not available! No starting point for the monopole formula!
  2. For a theory such as
     $T_{(1,1,1,1)}(SO(5)) : \ [SO(5)] - (USp(4)) - (O(3)) - (USp(2)) - (O(1))$
     the monopole formula blows up to infinity, because the dimension $\Delta(\boldsymbol{m})$ for the monopole operator in $USp(2)$ vanishes when $\boldsymbol{m} \neq 0$:
     'Bad theory' in the sense of [Gaiotto-Witten '08].
- In general, a theory $T_{\rho}(G)$ is 'bad' occurs if it contains
  - $U(N_c)$ gauge group with $N_f < 2N_c - 1$;
  - $SO(N_c)$ gauge group with $N_f < N_c - 1$;
  - $USp(2N_c)$ gauge group with $N_f < 2N_c + 1$.

  A 'bad' theory does not flow to a standard IR critical point: the R-symmetry is not the one as can be seen in the UV. [Gaiotto-Witten '08]

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The Hall-Littlewood formula for Coulomb branch of $T_{\rho}(G)$

[Cremonesi, Hanany, NM, Zaffaroni '14]

The HL formula for the Coulomb branch HS of $T_{\boldsymbol{\rho}}(G)$:

$$H[T_{\boldsymbol{\rho}}(G^\vee)](t;\boldsymbol{z};\boldsymbol{n}) = t^{\delta_{G^\vee}(\boldsymbol{n})}(1-t^2)^{r_G} K^G_{\boldsymbol{\rho}}(\boldsymbol{x};t)\Psi^{\boldsymbol{n}}_G(\boldsymbol{a}(t,\boldsymbol{x});t)$$

- $\delta_{G^\vee}(\boldsymbol{n}) := \sum_{\boldsymbol{\alpha}\in\Delta_+(G^\vee)} |\boldsymbol{\alpha}(\boldsymbol{n})|$.
- The Hall-Littlewood (HL) polynomial associated with a group G:

$$\Psi^{\boldsymbol{n}}_G(x_1,\ldots,x_{r_G};t) := \sum_{w\in W_G} \boldsymbol{x}^{w(\boldsymbol{n})} \prod_{\boldsymbol{\alpha}\in\Delta_+(G)} \frac{1-t\boldsymbol{x}^{-w(\boldsymbol{\alpha})}}{1-\boldsymbol{x}^{-w(\boldsymbol{\alpha})}}$$

- The argument $\boldsymbol{a}(t,\boldsymbol{x})$ of the HL poly comes from the decomposition

$$\chi^G_{\mathbf{fund}}(\boldsymbol{a}) = \sum_k \chi^{G_{\rho_k}}_{\mathbf{fund}}(\boldsymbol{x}_k)\chi^{SU(2)}_{\rho_k\text{-dim irrep}}(t)$$

where $G_{\rho_k} \subset G_{\boldsymbol{\rho}}$ associated with the number k in $\boldsymbol{\rho}$ that appears r_k times.

- $K^G_{\boldsymbol{\rho}}(\boldsymbol{x};t)$ is fixed by the decomposition of adjoint rep of G into those of $SU(2)\times G_{\boldsymbol{\rho}}$:

$$\chi^G_{\mathbf{Adj}}(\boldsymbol{a}) = \sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}} \chi^{SU(2)}_{[2j]}(t)\chi^{G_{\boldsymbol{\rho}}}_{\mathbf{R}_j}(\boldsymbol{x}_j)\ , \quad K^G_{\boldsymbol{\rho}}(\boldsymbol{x};t) = \mathrm{PE}\left[\sum_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}} t^{2j+2}\chi^{G_{\boldsymbol{\rho}}}_{\mathbf{R}_j}(\boldsymbol{x}_j)\right] .$$

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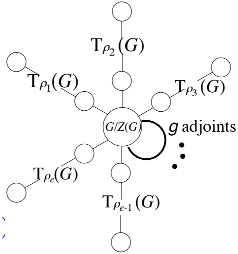
Coulomb branch of a $3d$ Sicilian theory of the A type

Gluing: The Coulomb branch HS of a mirror of the Sicilian theory of the A_{N-1} type on a genus g surface with e punctures $\{\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_e\}$.

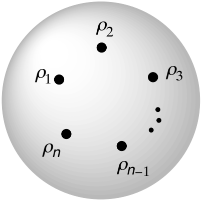


$$
\begin{aligned}
&H[\text{mirror } g, \{\boldsymbol{\rho}_i\}_{i=1}^{e}](t; \boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(e)}) \\
&= \sum_{n_1 \geq \cdots \geq n_{N-1} \geq 0} t^{(e+2g-2)\sum_{j=1}^{N-1}(N+1-2j)n_j} (1-t^2)^{eN+1} \times \\
&\quad P_{U(N)}(t; n_1, \ldots, n_{N-1}, 0) \prod_{j=1}^{e} K_{\boldsymbol{\rho}_j}(\boldsymbol{x}^{(j)}; t) \Psi_{U(N)}^{(n_1, \ldots, n_{N-1}, 0)}(\boldsymbol{x}^{(j)} t^{\boldsymbol{w}_{\boldsymbol{\rho}_j}}; t)
\end{aligned}
$$

where

- $\Psi^{\boldsymbol{n}}_{U(N)}(\boldsymbol{x} t^{\boldsymbol{w}_{\boldsymbol{\rho}}}; t) := \Psi^{(n_1, \ldots, n_N)}_{U(N)}(x_1 t^{\boldsymbol{w}_{\rho_1}}, x_2 t^{\boldsymbol{w}_{\rho_2}}, \ldots, x_{d+1} t^{\boldsymbol{w}_{\rho_{d+1}}}; t)$.
- $t^{\boldsymbol{w}_r} = (t^{r-1}, t^{r-3}, \ldots, t^{-(r-3)}, t^{-(r-1)})$.
- $P_{U(N)}(t; \boldsymbol{n})$ is the generating function for the indep Casimirs in the residue group left unbroken by $\boldsymbol{n}$.

Agreement with the Hall-Littlewood limit of the superconformal index



- For genus $g = 0$, this formula agrees with the Hall-Littlewood index for the $4d$ Sicilian theory with punctures $\{\boldsymbol{\rho}_i\}_{i=1}^{e}$. [Gadde, Rastelli, Razamat, Yan '11]
  - This is equal to the **Higgs branch HS** for the corresponding $3d$ Sicilian theory.
  - Agrees with mirror symmetry.
- This Coulomb branch HS can also be computed for mirrors of D-, twisted A- and twisted D-type Sicilian theories. Agree with the Higgs branch computations from [Lemos, Peelaers, Rastelli '12; Chacaltana, Distler, Tachikawa '12; Chacaltana, Distler, Trimm '13]

Application:

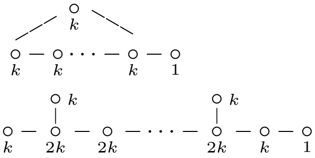
The moduli spaces of k G instantons on $\mathbb{C}^2$

The moduli spaces of k G instantons on $\mathbb{C}^2$

- For a simple group G, such a moduli space can be realised from the Coulomb branch of the affine Dynkin diagram of G with a $U(1)$ node attached at the imaginary root.
- If G is simply-laced (ADE), such Dynkin diagrams correspond to quiver diagrams **with Lagrangian descriptions**. Each node ℓ denotes a $U(\ell)$ group and each line denotes a hypermultiplet. [Intriligator-Seiberg '97]

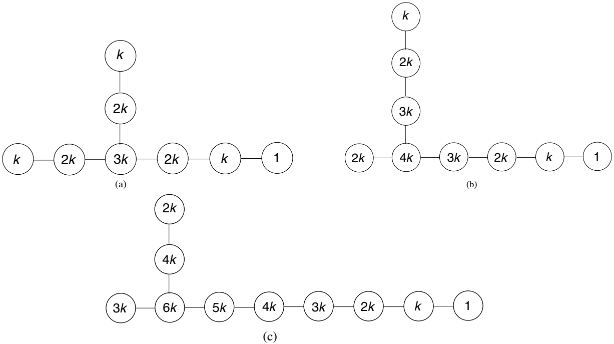
$SU(N)$:

$SO(2N)$:



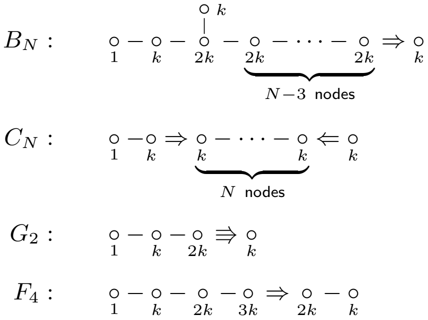
The moduli spaces of k $E_{6,7,8}$ instantons on $\mathbb{C}^2$

[Gaiotto-Razamat '12]



	$\boldsymbol{\rho}_1$	$\boldsymbol{\rho}_2$	$\boldsymbol{\rho}_3$
E_6	(k, k, k)	(k, k, k)	$(k, k, k-1, 1)$
E_7	(k, k, k, k)	$(2k, 2k)$	$(k, k, k, k-1, 1)$
E_8	$(3k, 3k)$	$(2k, 2k, 2k)$	$(k, k, k, k, k, k-1, 1)$

Non-simply laced groups



- These diagrams do not correspond to theories with known Lagrangian descriptions.
- For G_2 and F_4, the theories themselves and their mirrors have no known Lagrangian. Mirror symmetry does not help in these cases!
- Nevertheless, these quivers have brane configurations in terms of D3, NS5 and ON-planes. Magnetically charged BPS objects can be identified. [Hanany, Troost '01]

Non-simply laced groups

- Each node ℓ still represents the $U(\ell)$ root system. But the weights for the bi-fundamental hypers are modified as follows:

Quiver	Weights of the bi-fund rep of $U(k_1) \times U(k_2)$
$\underset{k_1}{\circ} - \underset{k_2}{\circ}$	$\{m_i - n_j \mid i = 1, \ldots, k_1,\ j = 1, \ldots, k_2\}$
$\underset{k_1}{\circ} \Rightarrow \underset{k_2}{\circ}$	$\{2m_i - n_j \mid i = 1, \ldots, k_1,\ j = 1, \ldots, k_2\}$
$\underset{k_1}{\circ} \Rrightarrow \underset{k_2}{\circ}$	$\{3m_i - n_j \mid i = 1, \ldots, k_1,\ j = 1, \ldots, k_2\}$

- Gluing technique still applies.
  - **Example:** The quiver for k G_2 instantons on $\mathbb{C}^2$

$$\underbrace{\underset{1}{\circ} - \underset{k}{\circ} - \underset{2k}{\Box}}_{\boldsymbol{\rho}_1=(k,k-1,1)} + \underbrace{\underset{2k}{\Box} \Rrightarrow \underset{k}{\circ}}_{\boldsymbol{\rho}_2=(k,k)} \longrightarrow \underset{1}{\circ} - \underset{k}{\circ} - \underset{2k}{\circ} \Rrightarrow \underset{k}{\circ}$$

- Using the monopole/HL formula, one can compute the Coulomb branch HS for all quivers listed above.

Conclusions

- Study moduli spaces of $3d$ $\mathcal{N}=4$ gauge theories using Hilbert series.
- Understand quantum corrections to the Coulomb branch chiral rings using the **monopole formula**.
- Compute Coulomb branch Hilbert series of $T_{\boldsymbol{\rho}}(G)$ and $3d$ Sicilian theories using the **Hall-Littlewood formula**.
- Provide more tests for $3d$ mirror symmetry.
- Connections between Coulomb branches and instanton moduli spaces
- Connections between Hilbert series and $5d$ instanton partition functions.
- Hilbert series for k G-instantons on $\mathbb{C}^2$ can be computed for any simple group G.