Deformed supersymmetric gauge theories from String and M-Theory

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based on work with with D. Orlando, S. Hellerman, N. Lambert
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Introduction

In recent years, N=2 supersymmetric gauge theories and their deformations have played an important role in theoretical physics - very active research topic.

Examples:

2d gauge/Bethe correspondence (Nekrasov/Shatashvili): relates 2d gauge theories with twisted masses to integrable spin chains.

4d gauge/Bethe correspondence (Nekrasov/Shatashvili): relates Omega-deformed 4d gauge theories to quantum integrable systems.

AGT correspondence (Alday, Gaiotto, Tachikawa): relates Omega-deformed super-Yang-Mills theory to Liouville theory.

Introduction

All these examples have two things in common:

1. A deformed supersymmetric gauge theory is linked to an integrable system.

   Relation between two very constrained and well-behaved systems that can be studied separately with different methods.

   Transfer insights from one side to the other, cross-fertilization between subjects!

2. The deformed gauge theories in question can be realized in string theory via the fluxtrap background!

   The string theory construction provides a unifying framework and a different point of view on the gauge theory problems.

Introduction

Aim: Realize deformed supersymmetric gauge theories via string theory. Gauge theories encode fluctuations on the world-volume of D-branes. Many parameters can be tuned by varying brane geometry.

Here: Deform the string theory background ("fluxtrap") into which the branes are placed (Hellerman, Orlando, S.R.)

⇒ different brane set-ups give rise to different gauge theories with seemingly unrelated deformations!

Use the fluxtrap construction to unify and meaningfully relate and reinterpret a large variety of existing results.

Introduction

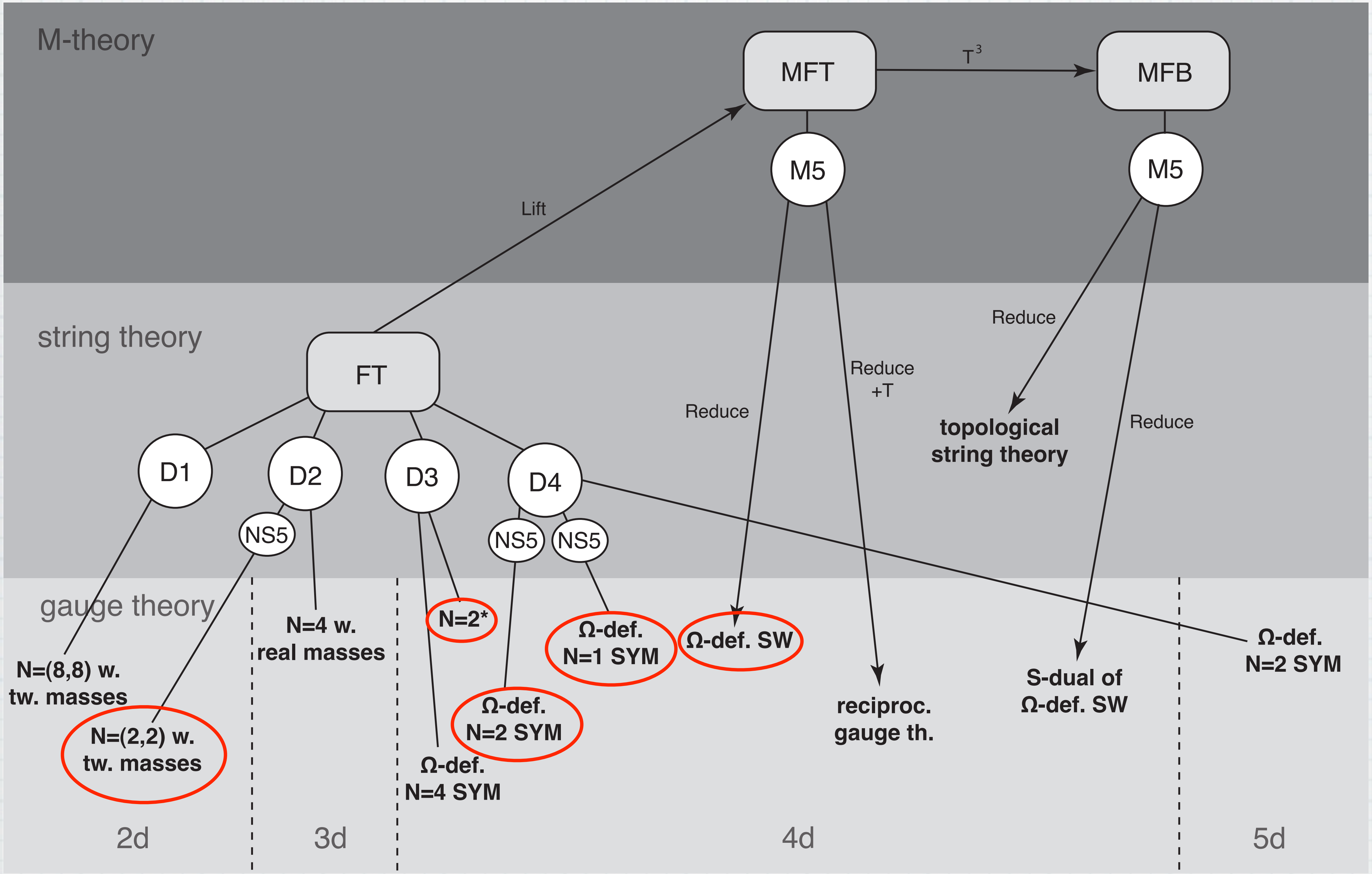
Our string theoretic approach enables us moreover to generate new deformed gauge theories in a simple and algorithmic way.

Today: panoramic overview over the many applications of the fluxtrap background:

- 2d effective gauge theories with deformations
- 4d effective gauge theories with deformations

Fluxtrap background as toolbox to generate deformed gauge theories and analyze them via string theoretic methods.

Introduction



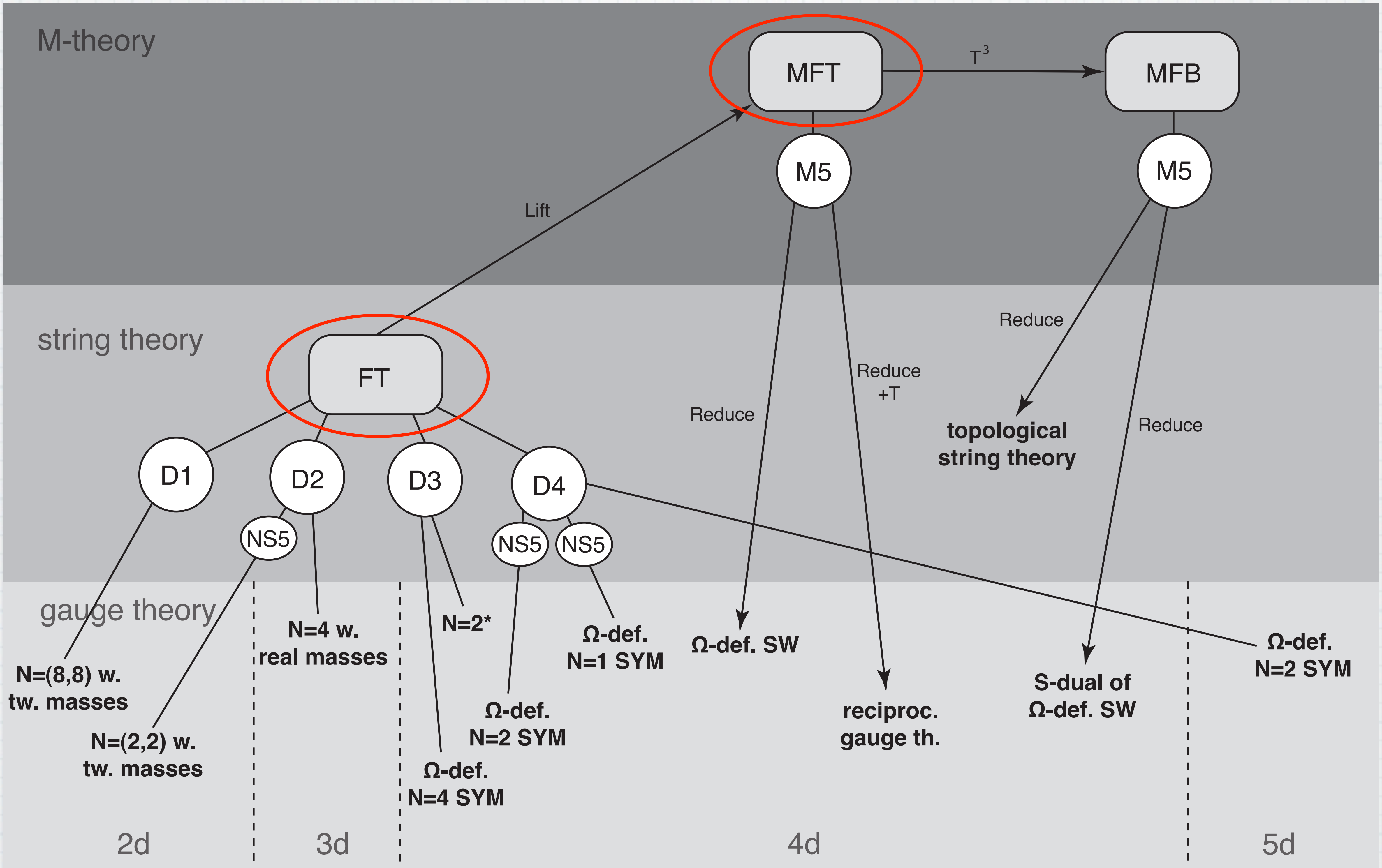
The same string theory background can give rise to many different deformations depending on how we place branes in it!

Outline

- Introduction, Motivation
- The Fluxtrap Background
- Deformed gauge theories
  - 2d Gauge Theories with twisted masses
  - N=2* theory
  - Polchinski/Strassler type gravity dual
  - Ω–deformed N=2 SYM
  - Ω–deformed N=1 SYM
  - Ω–deformed SW
- Summary

The Fluxtrap Background

The Fluxtrap Background



The Fluxtrap Background

Geometrical realization of Nekrasov's construction of the equivariant gauge theory.

Start with metric with 2 periodic directions and at least a U(1)xU(1) symmetry, no B-field, constant dilaton.

Fluxbrane background with 3 independent deformation parameters:

T^2

x	0	1	2	3	4	5	6	7	8	9
	(ρ_1, θ_1)		(ρ_2, θ_2)		(ρ_3, θ_3)		(ρ_4, θ_4)		v	
fluxbrane	ϵ_1		ϵ_2		ϵ_3		ϵ_4		$\circ$	$\circ$

$$\widetilde{x}^8 \simeq \widetilde{x}^8 + 2\pi \widetilde{R}_8$$
$$\widetilde{x}^9 \simeq \widetilde{x}^9 + 2\pi \widetilde{R}_9$$

Impose identifications: fluxbrane parameters

$$\begin{cases} \widetilde{x}^8 \simeq \widetilde{x}^8 + 2\pi \widetilde{R}_8 n_8 \\ \theta_k \simeq \theta_k + 2\pi \epsilon_k^R \widetilde{R}_8 n_8 \end{cases} \qquad \begin{cases} \widetilde{x}^9 \simeq \widetilde{x}^9 + 2\pi \widetilde{R}_9 n_9 \\ \theta_k \simeq \theta_k + 2\pi \epsilon_k^I \widetilde{R}_9 n_9 \end{cases}$$

This corresponds to the well-known Melvin or fluxbrane background.

The Fluxtrap Background

Introduce new angular variables with disentangled periodicities: $\phi_k = \theta_k - \epsilon_k^R \widetilde{x}^8 - \epsilon_k^I \widetilde{x}^9 = \theta_k - \mathrm{Re}(\epsilon_k \bar{\widetilde{v}})$

$$\epsilon_k = \epsilon_k^R + \mathrm{i}\, \epsilon_k^I \qquad \widetilde{v} = \widetilde{x}^8 + \mathrm{i}\, \widetilde{x}^9$$

Fluxbrane metric (T^2-fibration over Ω-deformed $\mathbb{R}^8$):

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}\vec{x}^2_{0\ldots7} &- \frac{V_i^R V_j^R \,\mathrm{d}x^i\, \mathrm{d}x^j}{1 + V^R \cdot V^R} - \frac{V_i^R V_j^R \,\mathrm{d}x^i\, \mathrm{d}x^j}{1 + V^R \cdot V^R} \\ &+ \left(1 + V^R \cdot V^R\right) \left[\mathrm{d}x^8 - \frac{V_i^R \,\mathrm{d}x^i}{1 + V^R \cdot V^R}\right]^2 \\ &+ \left(1 + V^I \cdot V^I\right) \left[\mathrm{d}x^9 - \frac{V_i^I \,\mathrm{d}x^i}{1 + V^I \cdot V^I}\right]^2 + 2 V^R \cdot V^I \,\mathrm{d}x^8\, \mathrm{d}x^9 \end{aligned}$$

Generator of rotations:

$$\begin{aligned} V = V^R + \mathrm{i}\, V^I = &\epsilon_1 \left(x^1\, \partial_0 - x^0\, \partial_1\right) + \epsilon_2 \left(x^3\, \partial_2 - x^2\, \partial_3\right) \\ &+ \epsilon_3 \left(x^5\, \partial_4 - x^4\, \partial_5\right) + \epsilon_4 \left(x^7\, \partial_6 - x^6\, \partial_7\right) \end{aligned}$$

The Fluxtrap Background

The general case breaks all supersymmetries.

Impose condition
$$\sum_{k=1}^{N} \epsilon_k = 0$$

Find preserved Killing spinor
$$K = \prod_k \exp\left[\phi_k \frac{\gamma_{\rho_k\theta_k}}{2}\right] \Pi_k^{\text{flux}} \eta$$

with projector
$$\Pi_k^{\text{flux}} = \tfrac{1}{2}\left(\mathbf{1} - \gamma_{\rho_k\theta_k\rho_N\theta_N}\right)$$

Each projector breaks half of the supersymmetries:

2^{6-N} susys are preserved

The Fluxtrap Background

T-dualize along torus directions and take decompactification limit to discard torus momenta:

Fluxtrap background

Before T-duality, locally, the metric was still flat, but some of the rotation symmetries were broken globally.

Bulk fields after T-duality (case $V^R \cdot V^I = 0,\ \epsilon_1 \in \mathbb{R},\ \epsilon_2 \in \mathrm{i}\,\mathbb{R},\ \epsilon_3 = \epsilon_4 = 0$):

not anymore flat

$$\mathrm{d}s^2 = \mathrm{d}\rho_1^2 + \frac{\rho_1^2\,\mathrm{d}\phi_1^2 + \mathrm{d}x_8^2}{1+\epsilon_1^2\rho_1^2} + \mathrm{d}\rho_2^2 + \frac{\rho_2^2\,\mathrm{d}\phi_2^2 + \mathrm{d}x_9^2}{1+\epsilon_2^2\rho_2^2} + \sum_{k=4}^{7}(\mathrm{d}x^k)^2,$$

B-field has appeared

$$B = \epsilon_1 \frac{\rho_1^2}{1+\epsilon_1^2\rho_1^2}\,\mathrm{d}\phi_1 \wedge \mathrm{d}x_8 + \epsilon_2 \frac{\rho_2^2}{1+\epsilon_2^2\rho_2^2}\,\mathrm{d}\phi_2 \wedge \mathrm{d}x_9\,,$$

$$\mathrm{e}^{-\Phi} = \frac{\sqrt{\alpha'}\,\mathrm{e}^{-\Phi_0}}{R}\sqrt{(1+\epsilon_1^2\rho_1^2)\,(1+\epsilon_2^2\rho_2^2)}$$

creates a potential that localizes instantons

The Fluxtrap Background

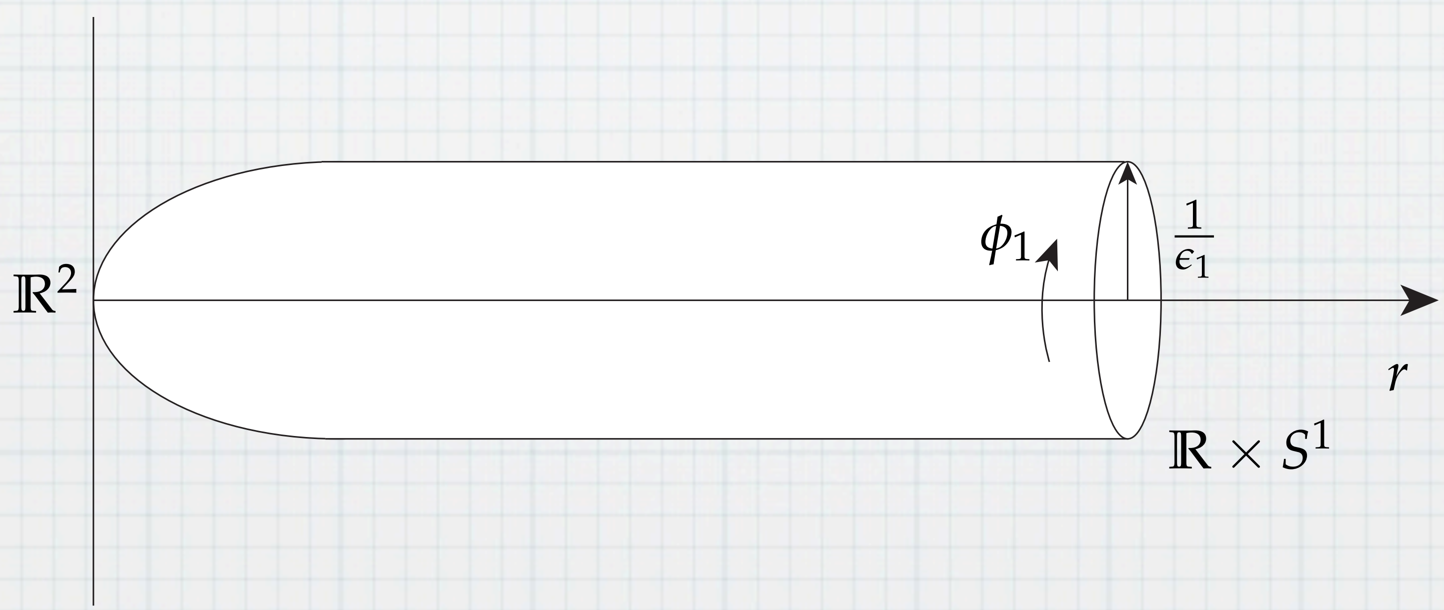
Study resulting geometry.

Space splits into

$$M_{10} = M_3(\epsilon_1) \times M_3(\epsilon_2) \times \mathbb{R}^4$$

$$\begin{array}{ccc} \mathbb{R}\langle x_8 \rangle & \longrightarrow & M_3(\epsilon_1) \\ & & \downarrow \\ & & \text{cigar } \langle \rho_1, \phi_1 \rangle \end{array}$$

R-foliation over the cigar



The generator of rotations is bounded (by asymptotic radius).

The Fluxtrap Background

Now we want to lift to M-theory:

$$\mathrm{d}s^2 = (\Delta_1\Delta_2)^{2/3}\left[\mathrm{d}\rho_1^2 + \frac{\epsilon_1^2\rho_1^2}{1+\epsilon_1^2\rho_1^2}\,\mathrm{d}\sigma_1^2 + \frac{\mathrm{d}x_8^2}{1+\epsilon_1^2\rho_1^2} + \mathrm{d}\rho_2^2 + \frac{\epsilon_2^2\rho_2^2}{1+\epsilon_2^2\rho_2^2}\,\mathrm{d}\sigma_2^2 + \frac{\mathrm{d}x_9^2}{1+\epsilon_2^2\rho_2^2}\right.$$
$$\left. + \,\mathrm{d}\rho_3^2 + \rho_3^2\,\mathrm{d}\psi^2 + \mathrm{d}x_6^2 + \mathrm{d}x_7^2\right] + (\Delta_1\Delta_2)^{-4/3}\,\mathrm{d}x_{10}^2\ ,$$

$$A_3 = \frac{\epsilon_1^2\rho_1^2}{1+\epsilon_1^2\rho_1^2}\,\mathrm{d}\sigma_1\wedge\mathrm{d}x_8\wedge\mathrm{d}x_{10} + \frac{\epsilon_2^2\rho_2^2}{1+\epsilon_2^2\rho_2^2}\,\mathrm{d}\sigma_2\wedge\mathrm{d}x_9\wedge\mathrm{d}x_{10}$$

$$\sigma_i = \frac{\phi_i}{\epsilon_i} \qquad \Delta_i^2 = 1+\epsilon_i^2\rho_i^2 \qquad x_{10} = x_{10} + 2\pi R_{10}$$

Consider only linear order in ϵ :

$$g_{MN} = \delta_{MN} + \mathcal{O}(\epsilon^2)\,,$$
$$G_4 = (\mathrm{d}z + \mathrm{d}\bar{z})\wedge(\mathrm{d}s + \mathrm{d}\bar{s})\wedge\omega$$

$$z = x^8 + \mathrm{i}\,x^9 \qquad\qquad s = x^6 + \mathrm{i}\,x^{10}$$

$$\omega = \epsilon_1\,\mathrm{d}x^0\wedge\mathrm{d}x^1 + \epsilon_2\,\mathrm{d}x^2\wedge\mathrm{d}x^3 + \epsilon_3\,\mathrm{d}x^4\wedge\mathrm{d}x^5$$

Deformed gauge theories

The type of deformation resulting from the fluxbrane background depends on how D-branes are placed into the fluxtrap with respect to the monodromies:

Deformation not on brane world-volume: mass deformation

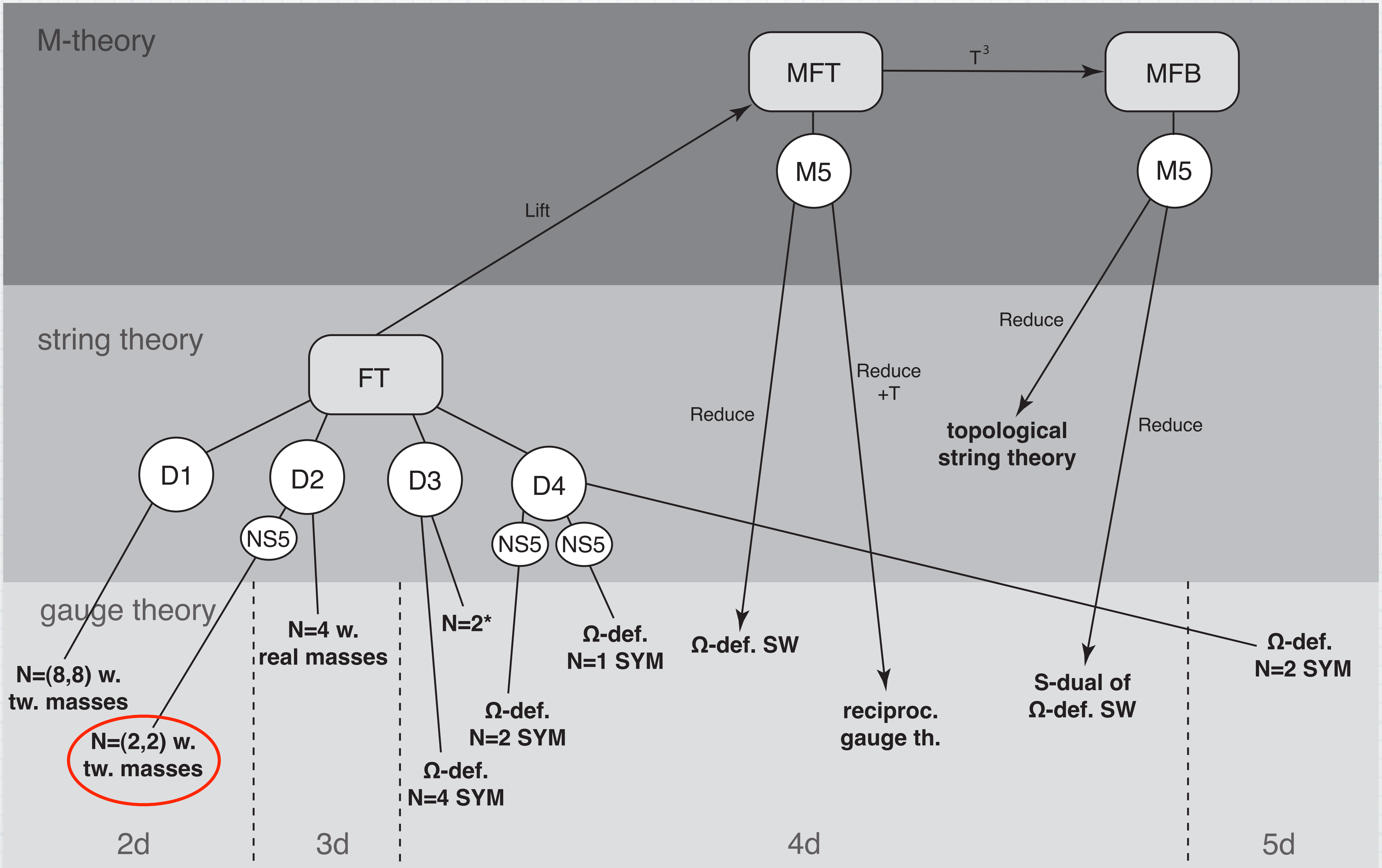
fluxtrap				ϵ_i	ϵ_j
D–brane	$\times$	$\times$	$\times$	ϕ_i	

Deformation on brane world-volume: Ω-type deformation, Lorentz invariance broken

fluxtrap	ϵ_i		ϵ_j	
D–brane	$\times$	$\times$	$\times$	$\times$

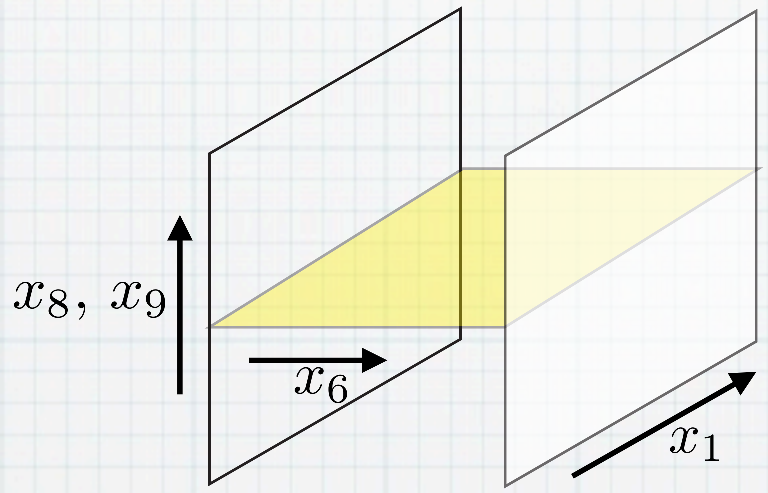
Examples: 2d gauge theory w. twisted mass

2d gauge theory w. twisted masses



2d Gauge Theories

We can construct N=2 gauge theories in 2d by studying the low energy theory on the world-volume of D2-branes suspended between NS5-branes.



x	0	1	2	3	4	5	6	7	8	9
fluxtrap	ϵ_1		ϵ_2		ϵ_3				$\circ$	$\circ$
D2–brane	$\times$	$\times$	ϕ				$\times$		σ	
NS5–brane	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$

Separation of NS5s in 6-direction: $1/g^2$

Separation of NS5s in 7-direction: FI-term

2d Gauge Theories

Why is the fluxtrap called a fluxtrap?

In the static embedding, $x^0 = \zeta^0,\ x^1 = \zeta^1,\ x^6 = \zeta^3$, the e.o.m. are solved for the D2-branes sitting in

$$x^2 = x^3 = x^4 = x^5 = x^7 = 0$$

The D2s are trapped at the origin.

Special case $\epsilon_2 = -\epsilon_3 = m$ preserves 16 supercharges. Adding only D2-branes to the fluxtrap preserves 8 supercharges (static embedding).

Adding also NS5-branes preserves 4 supercharges, N=(2,2)

Preserved Killing spinors:

$$\begin{cases} \epsilon_L = e^{-\Phi/8}\,(\mathbb{1}+\Gamma_{11})\,\Pi_-^{NS5}\Pi_-^{flux}\Gamma_{1608}\exp[\tfrac{1}{2}\,(\phi_1+\phi_2)\,\Gamma_{23}]\epsilon\,, \\ \epsilon_R = e^{-\Phi/8}\,(\mathbb{1}-\Gamma_{11})\,\Gamma_u\Pi_+^{NS5}\Pi_-^{flux}\exp[\tfrac{1}{2}\,(\phi_1+\phi_2)\,\Gamma_{23}]\epsilon\,. \end{cases}$$

$$\Pi_\pm^{NS5} = \tfrac{1}{2}\,(\mathbb{1} \pm \Gamma_{4567})$$

2d Gauge Theories

The fluxtrap deformation gives rise to the twisted masses!

Start with (kappa fixed) DBI action (democratic formulation):

$$S = -\mu_2 \int \mathrm{d}^3\zeta\, e^{-\Phi} \sqrt{-\det(g_{\alpha\beta} + B_{\alpha\beta})} \left[1 - \frac{1}{2} \bar{\psi} \left((g+B)^{\alpha\beta} \Gamma_\beta D_\alpha + \Delta^{(1)} \right) \psi \right]$$

$$D_\alpha = \partial_\alpha X^\mu \left(\nabla_\mu + \frac{1}{8} H_{\mu m n} \Gamma^{mn} \right) ,$$

$$\Delta^{(1)} = \frac{1}{2} \Gamma^m \partial_m \Phi - \frac{1}{24} H_{mnp} \Gamma^{mnp}$$

After expanding to quadratic order in the fields, we get

dilaton

B-field

$$S \propto \int \mathrm{d}^3\zeta \left[\partial^\mu \phi \partial_\mu \phi + m^2 |\phi|^2 + \bar{\psi} \partial\!\!\!/ \psi + \tfrac{m}{2} \bar{\psi} \Gamma_{45} \Gamma_8 \psi \right] + \dots$$

twisted mass terms!

2d Gauge Theories

An important ingredient of the Gauge/Bethe correspondence is the symmetry group of the integrable system, which also relates gauge theories with different gauge groups.

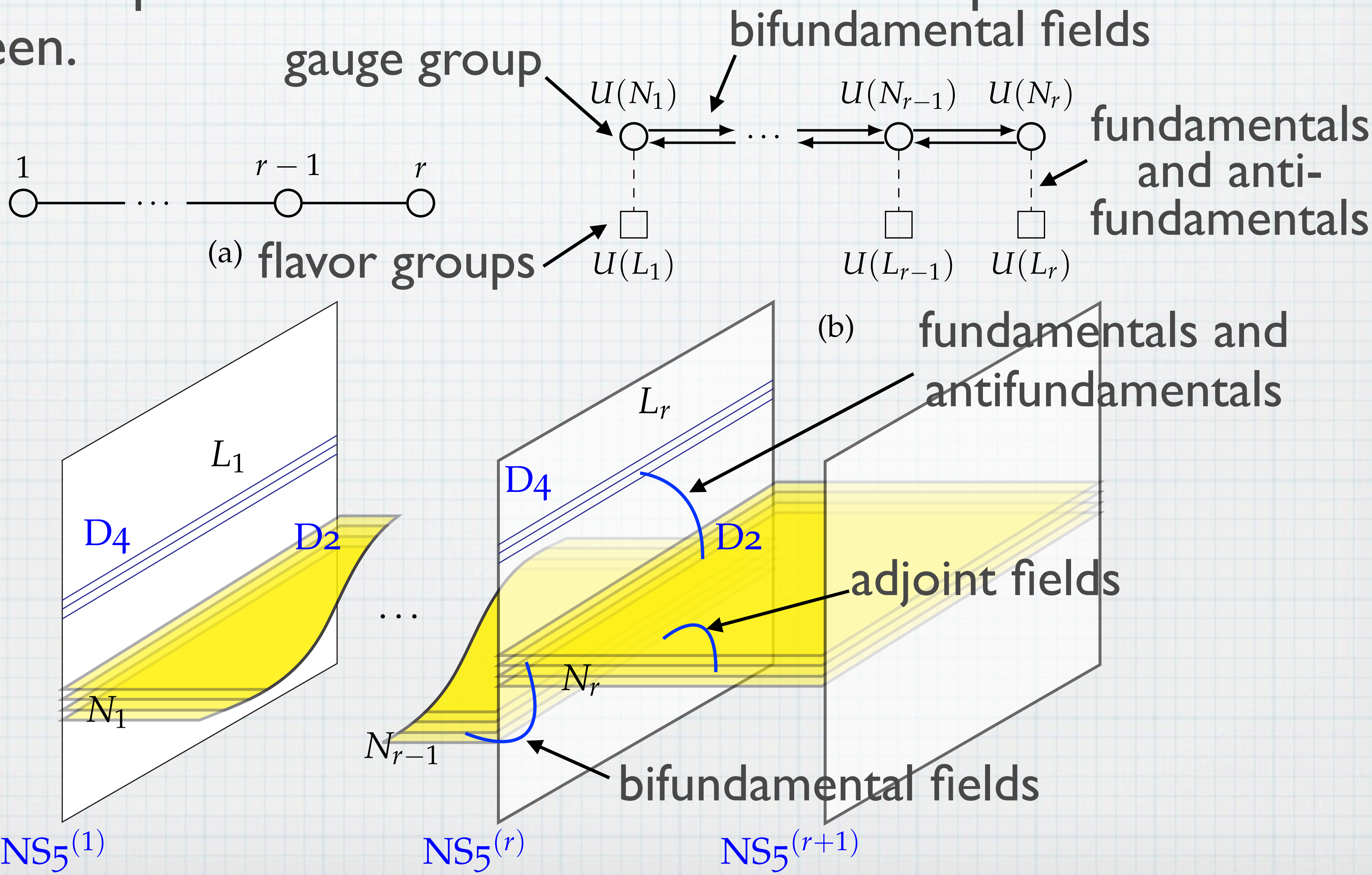
The example with two NS5-branes treated so far corresponds to the simplest case with symmetry group su(2).

Spin chains can have any Lie group as symmetry, even supergroups. Can we realize all those via a brane construction?

So far, we are able to reproduce the A and D-series.

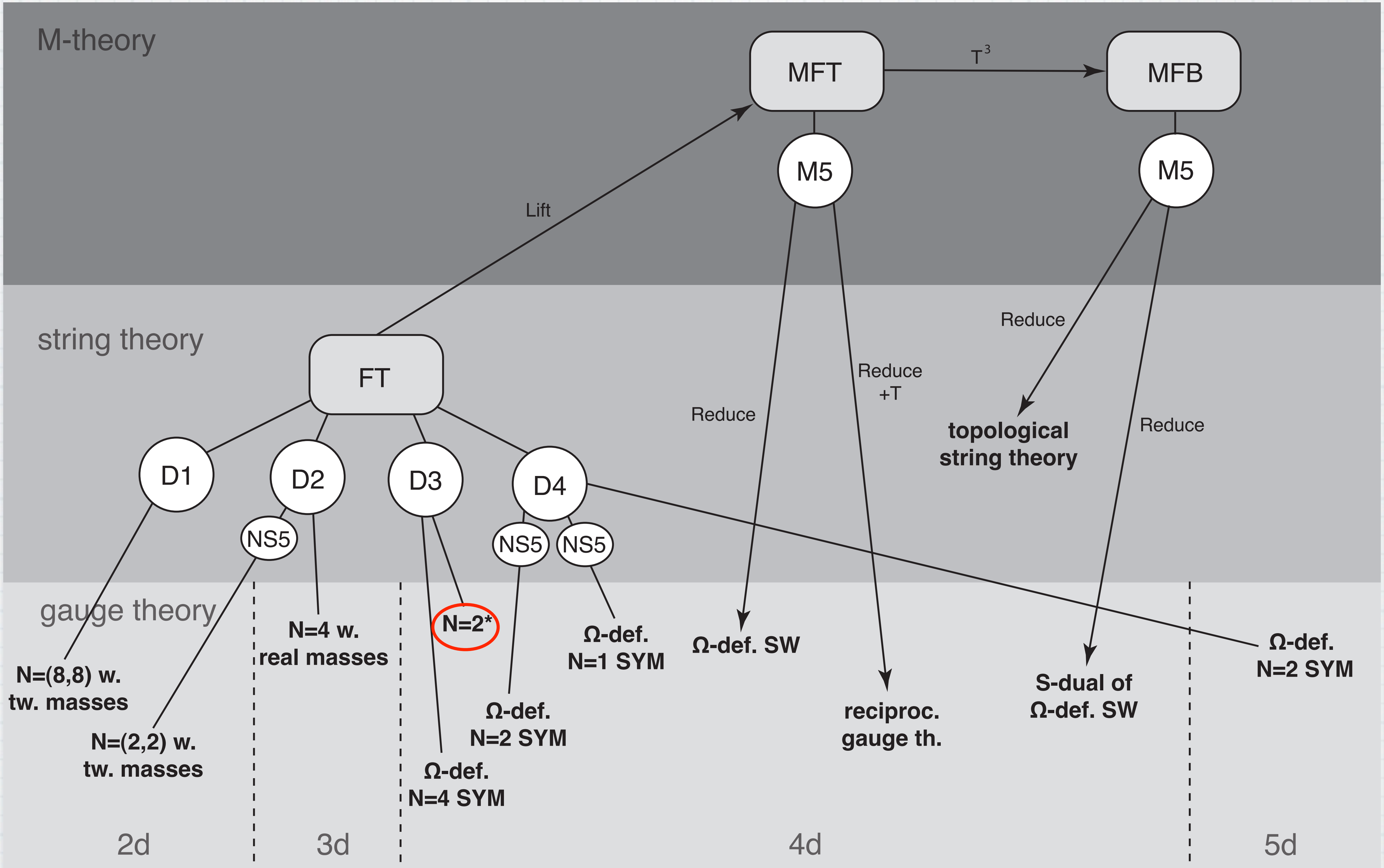
2d Gauge Theories

An SU(r) quiver gauge theory corresponds to a spin chain with SU(r) symmetry. Can be constructed by varying the brane set-up: r+1 NS5s with stacks of D2s suspended in between.



Examples: N=2* theory

N=2* theory


M-theory
MFT
T^3
MFB
M5
M5
Lift
string theory
FT
Reduce
Reduce +T
Reduce
topological string theory
Reduce
D1
D2
D3
D4
NS5
NS5
NS5
gauge theory
N=(8,8) w. tw. masses
N=(2,2) w. tw. masses
N=4 w. real masses
N=2*
Ω-def. N=4 SYM
Ω-def. N=2 SYM
Ω-def. N=1 SYM
Ω-def. SW
reciproc. gauge th.
S-dual of Ω-def. SW
Ω-def. N=2 SYM
2d
3d
4d
5d

N=2* theory

N=2* theory is obtained from N=4 SYM (4d) by giving equal masses to two of the scalar fields.

It is obtained from a D3-brane in the fluxtrap background with deformation parameters (8 conserved supercharges)

$\epsilon_1 = \epsilon_2 = 0$ $\qquad\qquad$ $\epsilon_3 = \epsilon_4 = \epsilon$

x	0	1	2	3	4	5	6	7	8	9
fluxtrap	ϵ_1		ϵ_2		ϵ_3		ϵ_4		$\circ$	$\circ$
D3–brane	$\times$	$\times$	$\times$	$\times$	ϕ_1		ϕ_2		ϕ_3	

$$\mathscr{L}_\Omega = \frac{1}{4g_{\mathrm{YM}}^2}\left[F_{ij}F^{ij} + \frac{1}{2}\sum_{k=1}^{3}\left(\partial^i\phi_k\right)\left(\partial_i\bar{\phi}_k\right) + \frac{1}{2}\left|\epsilon\right|^2\phi_1\bar{\phi}_1 + \frac{1}{2}\left|\epsilon\right|^2\phi_2\bar{\phi}_2\right]$$

Flows to N=2 in the IR (masses become infinite).

Different from Witten's construction (global BC).

Polchinski/Strassler-type solution

We have a string realization of a deformation of N = 4 SYM based on the dynamics of a D3–brane ⇒

What is the gravity dual of the Ω–deformed theory?

Gravity duals of massive deformations ⇒ Polchinski/Strassler

Gravity dual of the Ω–deformed N=4 SYM is given by the full backreaction of the D3–brane in the fluxtrap, which interpolates between the solution of Polchinski and Strassler in the near-horizon limit and the flat-space fluxtrap at infinity.

Example: Polchinski/Strassler-type solution for N=2* theory

Polchinski/Strassler-type solution

Start from standard D3-brane solution:

distance from center of the brane

$$ds^2 = H(r)^{-1/2}\, d\vec{x}^2_{0\ldots3} + H(r)^{1/2}\left(dr^2 + r^2\, d\Omega_5^2\right)$$

$$F_4 = dH(r)^{-1} \wedge dx^0 \wedge \ldots \wedge dx^3 + 4Q\, \omega_{S^5}$$

$$H(r) = a + Q/r^4$$

D-brane charge

a=0 at horizon

Lowest order deformation in ε:

1st order expansion of FT result

Polchinski/Strassler solution

$$B = aV \wedge dx^8 + \frac{Q}{r^4}\left(V \wedge dx^8 + x^8 \omega\right),$$

$$2\omega = dV$$

$$C_2 = -\frac{Q}{r^4}\left(V \wedge dx^9 + x^9 \omega\right)$$

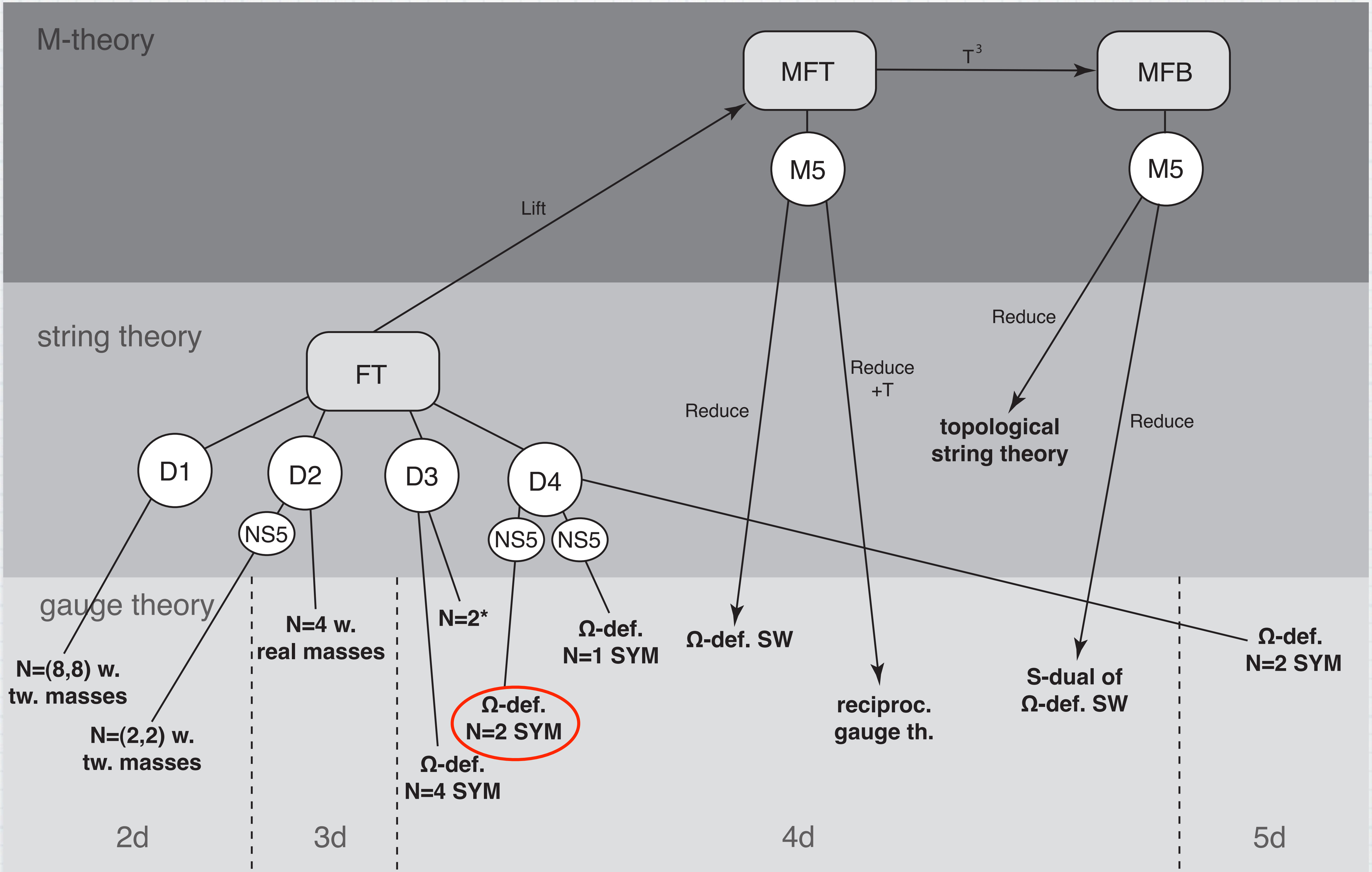
Conformal invariance is broken $\Rightarrow$ non-trivial dilaton and C_0 field in the near-horizon.

$$\begin{cases} \Phi = -\frac{aV\cdot V}{2} - \frac{Q\epsilon^2}{2}\frac{x_9^2 - x_8^2}{r^4} \\ C_0 = Q\epsilon^2 \frac{x^8 x^9}{r^4} \end{cases}$$

Metric undeformed at 1st order (expect Myers effect!)

Examples: Omega-deformed N=2 SYM

Omega-deformed N=2 SYM



Omega-deformed N=2 SYM

Original theory where the Ω-deformation was first introduced by Nekrasov.

x	0	1	2	3	4	5	6	7	8	9
fluxtrap	ϵ_1		ϵ_2		ϵ_3				$\circ$	$\circ$
D4–brane	$\times$	$\times$	$\times$	$\times$			$\times$		ϕ	
NS5–brane	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$

$$\mathscr{L}_\Omega = \frac{1}{4g_{\mathrm{YM}}^2}\left[F_{ij}F^{ij} + \frac{1}{2}\left(\partial^i\phi + \underbrace{V^k}_{\text{B-field}} F_k{}^i\right)\left(\partial_i\bar\phi + \underbrace{\bar V^k}_{\text{B-field}} F_{ki}\right) - \frac{1}{8}\left(\bar V^i\,\partial_i\phi - V^i\,\partial_i\bar\phi + \underbrace{V^k\bar V^l F_{kl}}_{\text{dilaton+metric}}\right)^2\right]$$

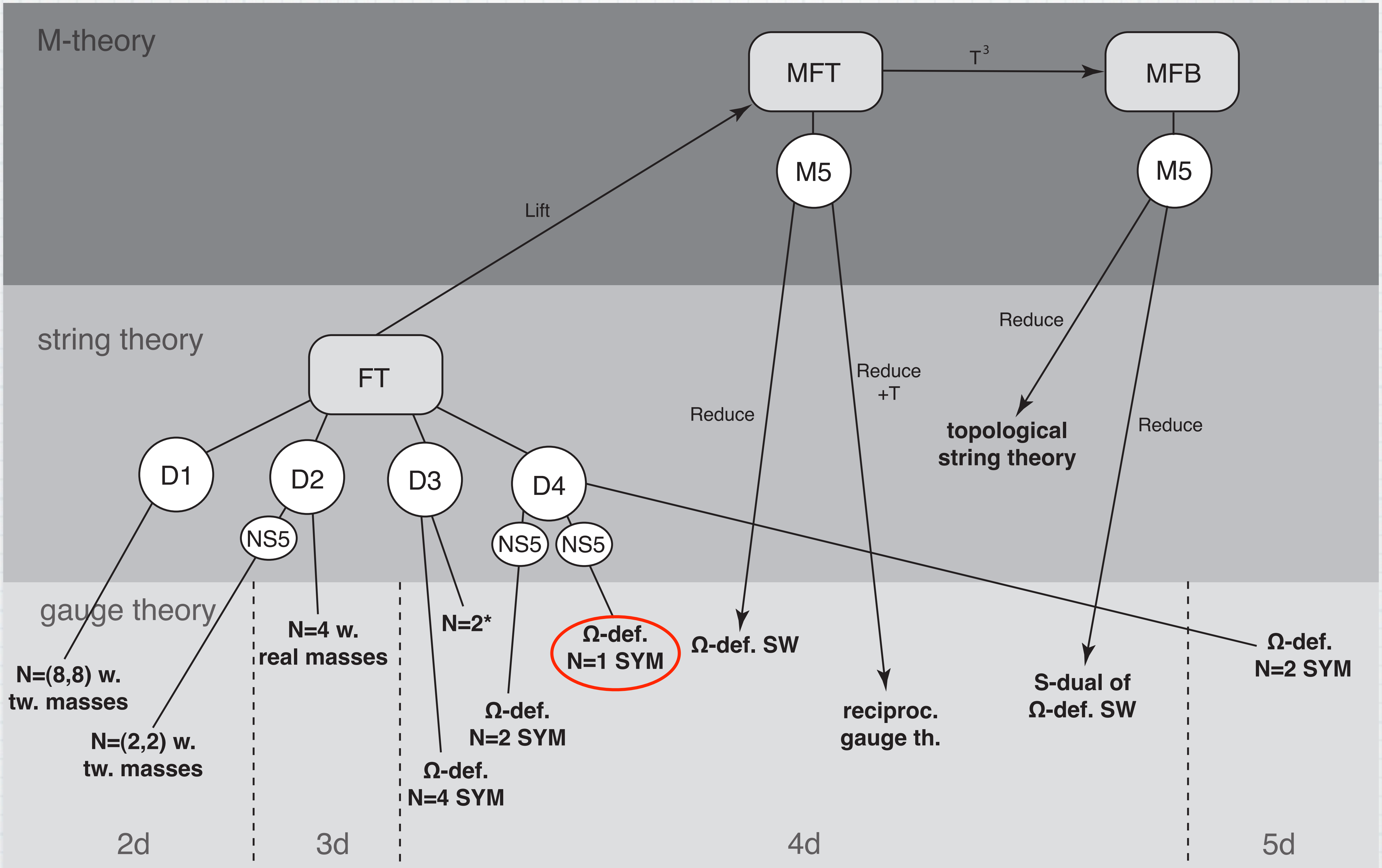
Interesting limits are

$\epsilon_1 = -\epsilon_2, \quad \epsilon_3 = 0$ reproduces top. string partition function, more supersymmetry

$\epsilon_1 = -\epsilon_3, \quad \epsilon_2 = 0$ Nekrasov/Shatashvili limit

Examples: Omega-deformed N=1 SYM

Omega-deformed N=1 SYM



Omega-deformed N=1 SYM

N=1 SYM in 4d requires a brane placement different from the previous examples.

x	0	1	2	3	4	5	6	7	8	9
fluxtrap	ϵ_1		ϵ_2		ϵ_3				$\circ$	$\circ$
D4–brane	$\times$	$\times$					$\times$		$\times$	$\times$
NS5–brane 1	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$
NS5–brane 2	$\times$	$\times$			$\times$	$\times$			$\times$	$\times$

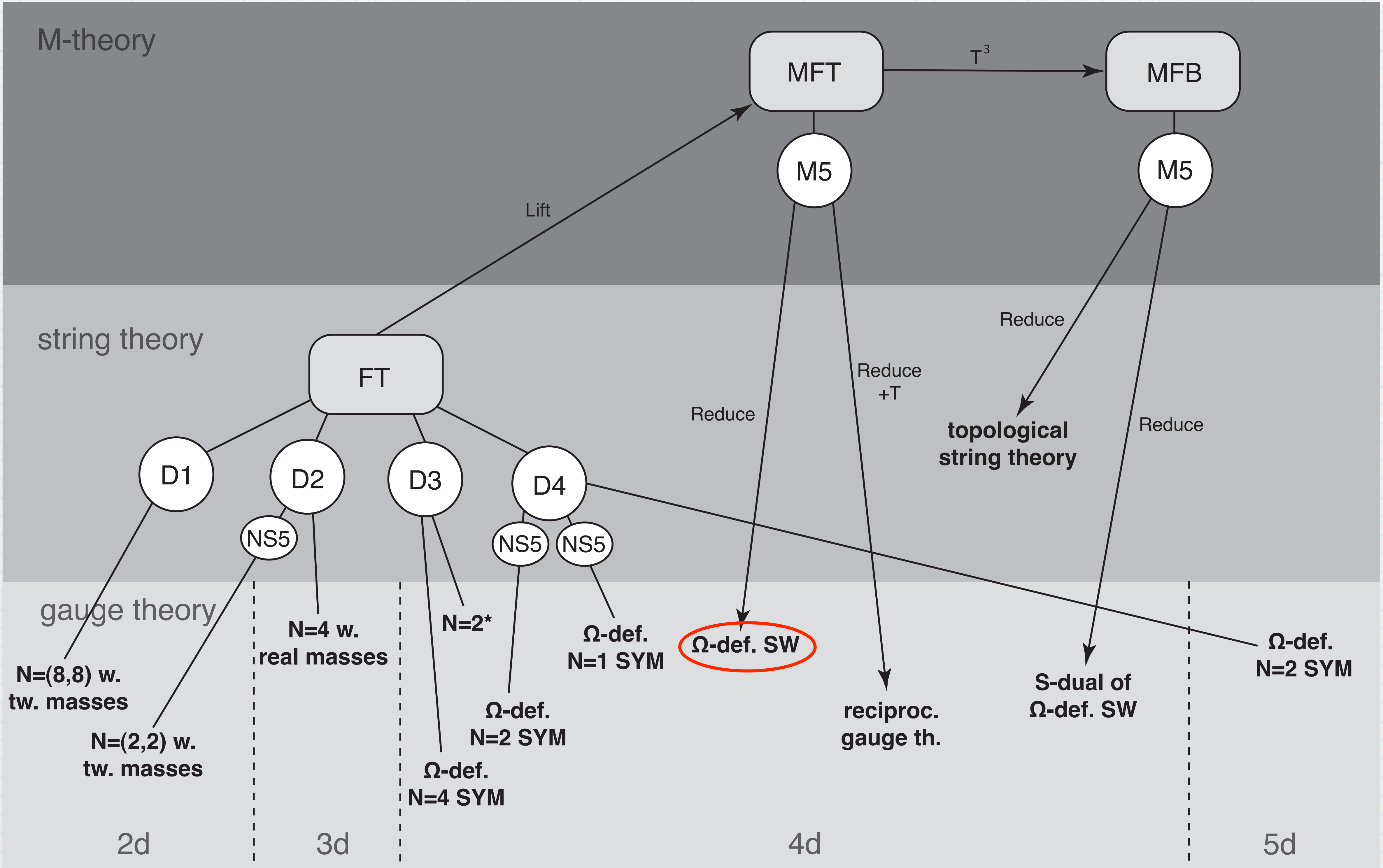
NS5-branes not parallel, only 3 deformation parameters possible, D4 extended in dual Melvin directions.
N=1 has no scalar fields, preserves 2 real supercharges.

$$\mathscr{L}_\Omega = \frac{1}{4g^2} F_{ij} F^{ij} + V_i^R F^{ij} \mathbf{e}_j^8 + V_i^I F^{ij} \mathbf{e}_j^9$$

unit vectors

Examples: Omega-deformed SW action

Omega-deformed SW action



Omega-deformed SW

Derive Omega-deformed Seiberg-Witten Lagrangian (eff. low energy action)

Use M-theory lift of fluxtrap BG.

Classical computation yields quantum result.

Embed M5-brane into fluxtrap BG.

- Self-dual three-form on the brane.
- Still wrapped on a Riemann surface at linear order.

Take vector and scalar equations of motion in 6d (not from an action!).

Integrate equations over Riemann surface.

4d equations of motion are Euler-Lagrange equations of an action.

This action reduces to the Seiberg-Witten action in the undeformed case.

Captures all orders of the 4D gauge theory.

Omega-deformed SW

Start with type IIA set-up of D4- and NS5-branes:

	0	1	2	3	4	5	6	7	8	9
fluxtrap	ϵ_1		ϵ_2		ϵ_3		$\times$	$\times$	$\circ$	$\times$
NS5	$\times$	$\times$	$\times$	$\times$					$\times$	$\times$
D4	$\times$	$\times$	$\times$	$\times$			$\times$			

Non-abelian generalization of bosonic world-volume action for D4-branes suspended between NS5-branes in fluxtrap BG:

$$\mathcal{L}_{D4} = \frac{1}{g_4^2} \operatorname{Tr} \Big[\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} (D_\mu \phi + \tfrac{1}{2} F_{\mu\lambda} V^\lambda)(D_\mu \bar{\phi} + \tfrac{1}{2} F_{\mu\rho} V^\rho) \\ - \tfrac{1}{2} \left([\phi, \bar{\phi}] - \tfrac{1}{2} V^\mu D_\mu (\phi - \bar{\phi}) \right)^2 \Big]$$

Lifts to single M5 extended in $x^0, \ldots, x^3$ and wrapping a 2-cycle in x^6, x^8, x^9, x^{10}.

Choose embedding preserving same susy as in type IIA.

Omega-deformed SW

Selfdual three-form on WV of the M5-brane:

selfdual

$$\mathrm{d}H_3 = -\tfrac{1}{4}\hat{G}_4 \qquad\qquad H_3 = h_3 + \mathcal{O}(h_3^3)$$

For $\epsilon = 0$, we have $h_3 = 0$ and the M5-brane wraps a Riemann surface $\bar{\partial} s = 0$.

At linear order, pullback only depends holomorphically on $s(z)$:

$$\hat{G}_4 = -\left(\partial s - \bar{\partial}\bar{s}\right)\mathrm{d}z \wedge \mathrm{d}\bar{z} \wedge \hat{\omega} + \mathcal{O}(\epsilon^2)\,.$$

From the susy condition, we find

$$\hat{\omega}^- = \frac{\epsilon_1 - \epsilon_2}{2}\left(\mathrm{d}x^0 \wedge \mathrm{d}x^1 - \mathrm{d}x^2 \wedge \mathrm{d}x^3\right) \qquad \hat{\omega}^+ = \frac{\epsilon_1 + \epsilon_2}{2}\left(\mathrm{d}x^0 \wedge \mathrm{d}x^1 + \mathrm{d}x^2 \wedge \mathrm{d}x^3\right)$$

$$h_3 = \tfrac{1}{4}\left(\bar{s} - \bar{z}\,\partial s\right)\mathrm{d}z \wedge \hat{\omega}^- + \tfrac{1}{4}\left(s - z\,\bar{\partial}\bar{s}\right)\mathrm{d}\bar{z} \wedge \hat{\omega}^+$$

M5 still embedded holomorphically, implicit form for SU(2): $t^2 - 2B(z|u)t + \Lambda^4 = 0\ ,$

$$t = \Lambda^2 e^{-s/R}$$

$$B(z|u) = \Lambda^4 z^2 - u$$

Witten

Riemann surface with modulus u:

$$\Sigma = \{(z,s)|s = s(z|u)\}$$

Omega-deformed SW

Want to describe the low energy dynamics of the fluctuations around the equilibrium.

Since we are interested in the 4d theory, we assume that:

- the geometry of the M5 is still a fibration of a Riemann surface over $\mathbb{R}^4$.
- for each point in $\mathbb{R}^4$ we have the same Riemann surface as above, but with a different value of the modulus u.

The modulus u of the Riemann surface is a function of the worldvolume coordinates and the embedding is still formally defined by the same equation:

$$s = s(z|u(x^\mu)) \qquad \partial_\mu s(z|u(x^\mu)) = \partial_\mu u \frac{\partial s}{\partial u}$$

$$z = x^8 + \mathrm{i}\, x^9 \qquad s = x^6 + \mathrm{i}\, x^{10}$$

Omega-deformed SW

6D Equations of Motion:

Dynamics can be obtained by evaluating the M5-brane equations of motion (bosonic fields).

scalar equation

$$\left(\hat{g}^{mn} - 16 h^{mpq} h^{n}{}_{pq}\right) \nabla_m \nabla_n X^I = -\frac{2}{3} \hat{G}^I{}_{mnp} h^{mnp} \,,$$

$$\mathrm{d}h_3 = -\tfrac{1}{4} \hat{G}_4 \,,$$

vector equation

Howe, Sezgin, West

These equations do not stem from an action in 6D.

General form of 3-form on brane:

pullback of bulk 3-form

$$h_3 = -\tfrac{1}{4} \left(\hat{C}_3 + \mathrm{i} *_6 \hat{C}_3 - \Phi \right)$$

selfdual 3-form, encodes fluctuations of 4d gauge field

Omega-deformed SW

Want to relate Φ to 4d gauge field: only components

$$(\mu,\nu,z),\ (\mu,\nu,\bar{z})$$

Ansatz:

$$\Phi = \frac{\kappa}{2}\mathcal{F}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}z + \frac{\bar{\kappa}}{2}\widetilde{\mathcal{F}}_{\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}\bar{z}$$
$$+\frac{1}{1+|\,\partial s|^2}\frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\left(\partial^\tau s\,\bar{\partial}\bar{s}\,\kappa\mathcal{F}_{\sigma\tau} - \partial^\tau\bar{s}\,\partial s\,\bar{\kappa}\widetilde{\mathcal{F}}_{\sigma\tau}\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\,.$$

to ensure self-duality

antiselfdual 2-form

$$*_4\mathcal{F} = -\mathcal{F}\,, \qquad *_4\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}$$

SW notation:

scalar field

$$a = \oint_A \lambda_{SW}\,, \qquad a_D = \oint_B \lambda_{SW}\,, \qquad \tau = \frac{\mathrm{d}a_D}{\mathrm{d}a}\,, \qquad \lambda = \frac{\partial\lambda_{SW}}{\partial u}$$

holomorphic 1-form on Riemann surface

holomorphic fn

$$\kappa = \frac{\mathrm{d}s}{\mathrm{d}a} = \left(\frac{\mathrm{d}a}{\mathrm{d}u}\right)^{-1}\lambda_z \qquad \lambda = \lambda_z\,\mathrm{d}z \qquad \frac{\mathrm{d}a}{\mathrm{d}u} = \oint_A \lambda$$

Omega-deformed SW

Integration over the Riemann surface of the 6d e.o.m. results in the 4d e.o.m. for the Omega-deformed SW theory:

The 3-form on the brane is the (generalized) pullback of the 3-form in the bulk.

Vector equation:

$$(\tau-\bar\tau)\left[\partial_\mu F_{\mu\nu}+\tfrac{1}{2}\,\partial_\mu(a+\bar a)\hat\omega_{\mu\nu}+\tfrac{1}{2}\,\partial_\mu(a-\bar a)^*\hat\omega_{\mu\nu}\right]$$
$$+\,\partial_\mu(\tau-\bar\tau)\left[F_{\mu\nu}+\tfrac{1}{2}\,(a-\bar a)\,{}^*\hat\omega_{\mu\nu}\right]-\partial_\mu(\tau+\bar\tau)\left[{}^*F_{\mu\nu}+\tfrac{1}{2}\,(a-\bar a)\,\hat\omega_{\mu\nu}\right]=0$$

Scalar equations: The M5 brane is a (generalized) minimal surface.

$$(\tau-\bar\tau)\,\partial_\mu\partial_\mu a+\partial_\mu a\,\partial_\mu\tau+2\frac{\mathrm{d}\bar\tau}{\mathrm{d}\bar a}\,(F_{\mu\nu}F_{\mu\nu}+F_{\mu\nu}{}^*F_{\mu\nu})$$
$$+\,4\frac{\mathrm{d}\bar\tau}{\mathrm{d}\bar a}\,(a-\bar a)\,\hat\omega^+_{\mu\nu}F_{\mu\nu}-4\,(\tau-\bar\tau)\,\hat\omega^-_{\mu\nu}F_{\mu\nu}=0\ ,$$

$$(\tau-\bar\tau)\,\partial_\mu\partial_\mu\bar a-\partial_\mu\bar a\,\partial_\mu\bar\tau-2\frac{\mathrm{d}\tau}{\mathrm{d}a}\,(F_{\mu\nu}F_{\mu\nu}-F_{\mu\nu}{}^*F_{\mu\nu})$$
$$+\,4\frac{\mathrm{d}\tau}{\mathrm{d}a}\,(a-\bar a)\,\hat\omega^-_{\mu\nu}F_{\mu\nu}-4\,(\tau-\bar\tau)\,\hat\omega^+_{\mu\nu}F_{\mu\nu}=0\ .$$

Consistent result justifies earlier assumptions about foliation structure, form of fluctuations and integration measure.

Omega-deformed SW

The vector and scalar e.o.m. are the Euler-Lagrange equations of the following Lagrangian:

generalized covariant derivative for the scalar a, non minimal coupling to the gauge field.

$$
\begin{aligned}
\mathrm{i}\,\mathscr{L} = &-(\tau_{ij}-\bar{\tau}_{ij})\Big[\tfrac{1}{2}\Big(\partial_\mu a^i + 2\left(\tfrac{\bar\tau}{\tau-\bar\tau}\right)_{ik}{}^*F^k_{\mu\nu}{}^*\hat{U}_\nu\Big)\Big(\partial_\mu \bar{a}^j - 2\left(\tfrac{\tau}{\tau-\bar\tau}\right)_{jl}{}^*F^l_{\mu\nu}{}^*\hat{U}_\nu\Big) \\
&+\left(F^i_{\mu\nu}+\tfrac{1}{2}\left(a^i-\bar{a}^i\right){}^*\hat\omega_{\mu\nu}\right)\left(F^j_{\mu\nu}+\tfrac{1}{2}\left(a^j-\bar{a}^j\right){}^*\hat\omega_{\mu\nu}\right)\Big] \\
&+(\tau_{ij}+\bar\tau_{ij})\left(F^i_{\mu\nu}+\tfrac{1}{2}\left(a^i-\bar{a}^i\right){}^*\hat\omega_{\mu\nu}\right)\left({}^*F^j_{\mu\nu}+\tfrac{1}{2}\left(a^j-\bar{a}^j\right)\hat\omega_{\mu\nu}\right)
\end{aligned}
$$

shift in the gauge field strength

$$\omega = \mathrm{d}U$$

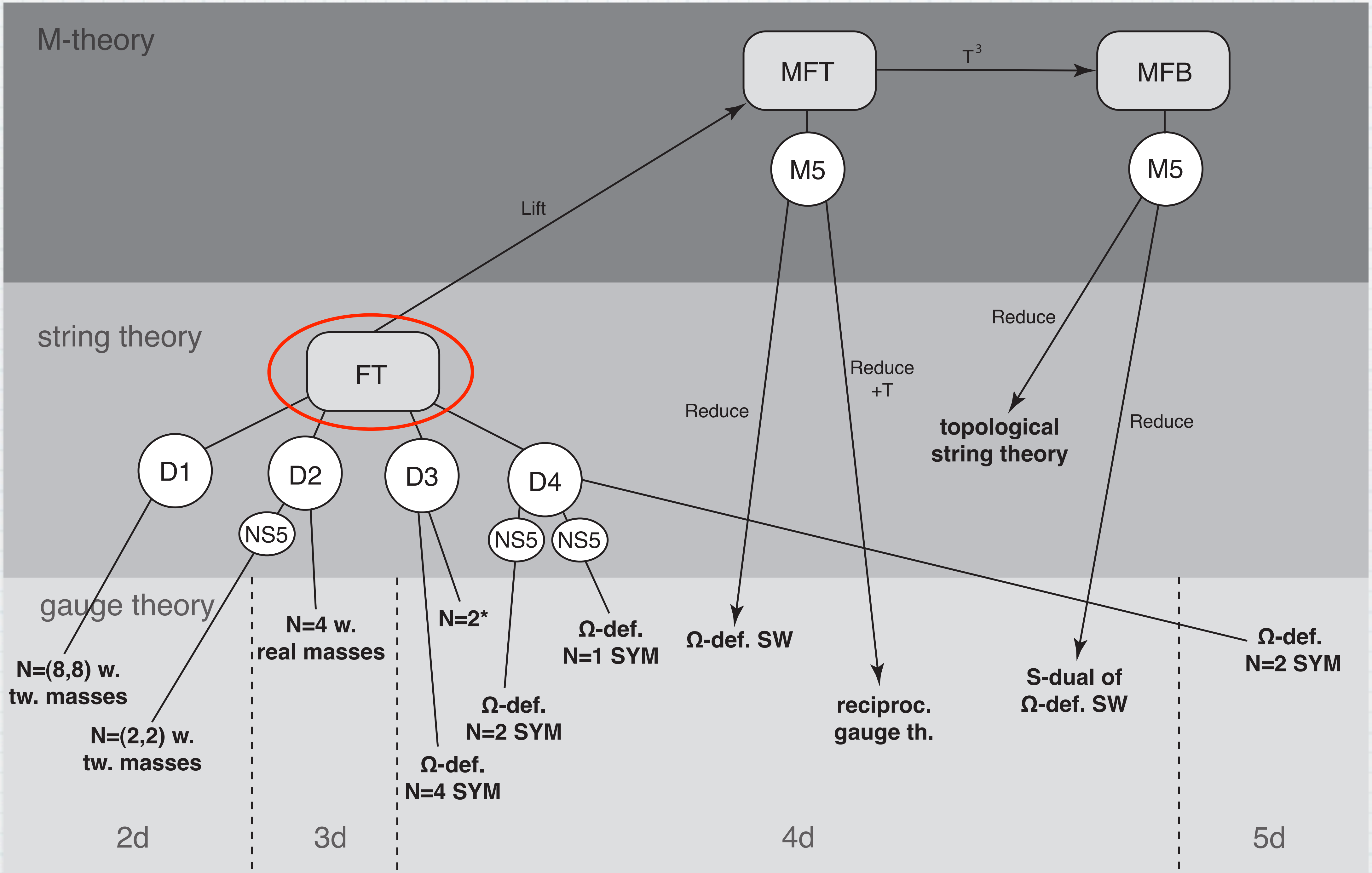
For $\epsilon = 0$, this reproduces the Seiberg-Witten Lagrangian.

Independent of compactification radius to IIA, which is related to gauge coupling in 4d → quantum result (all orders). True for any Riemann surface.

Summary

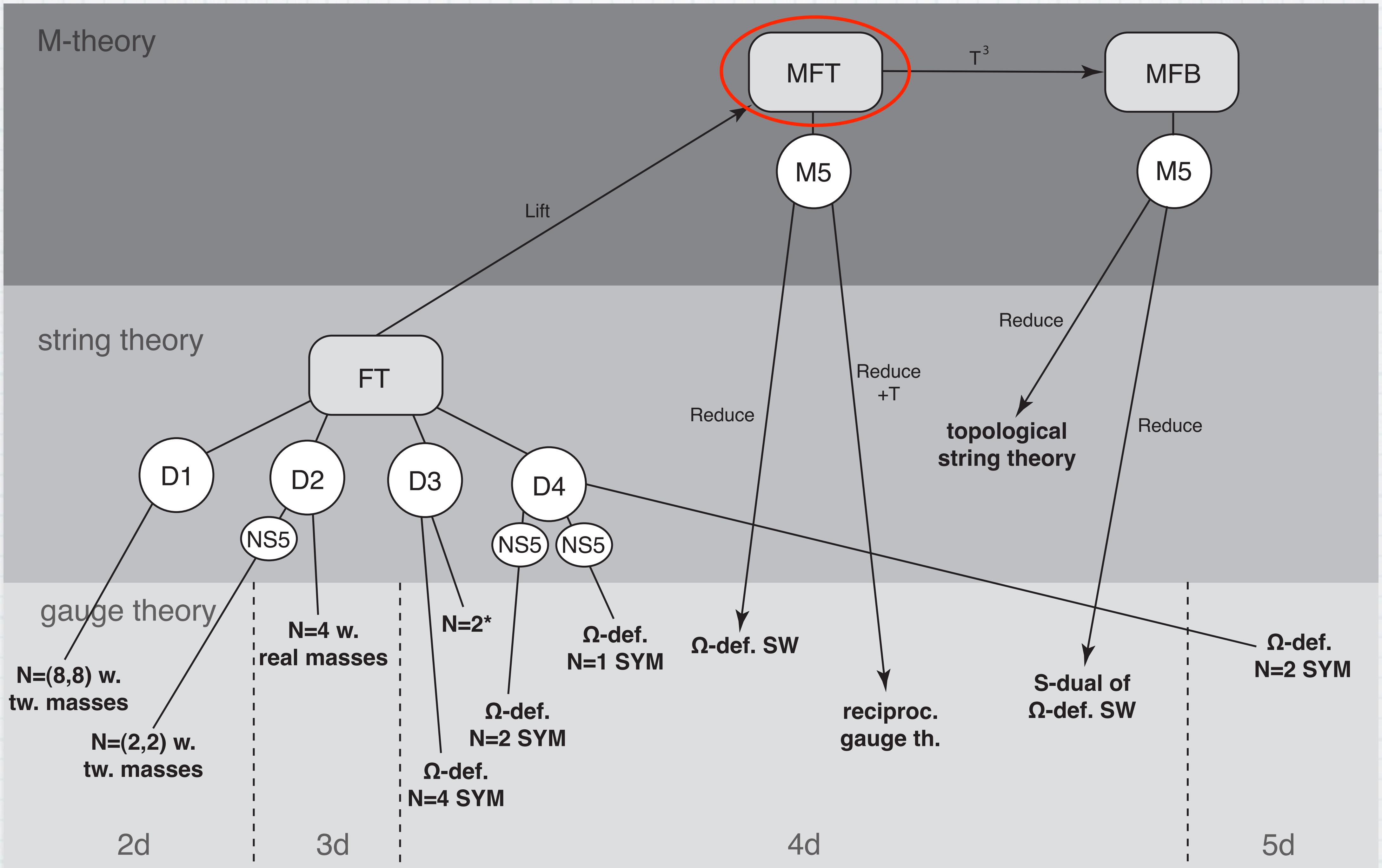
Summary

Constructed the fluxtrap background in string theory.



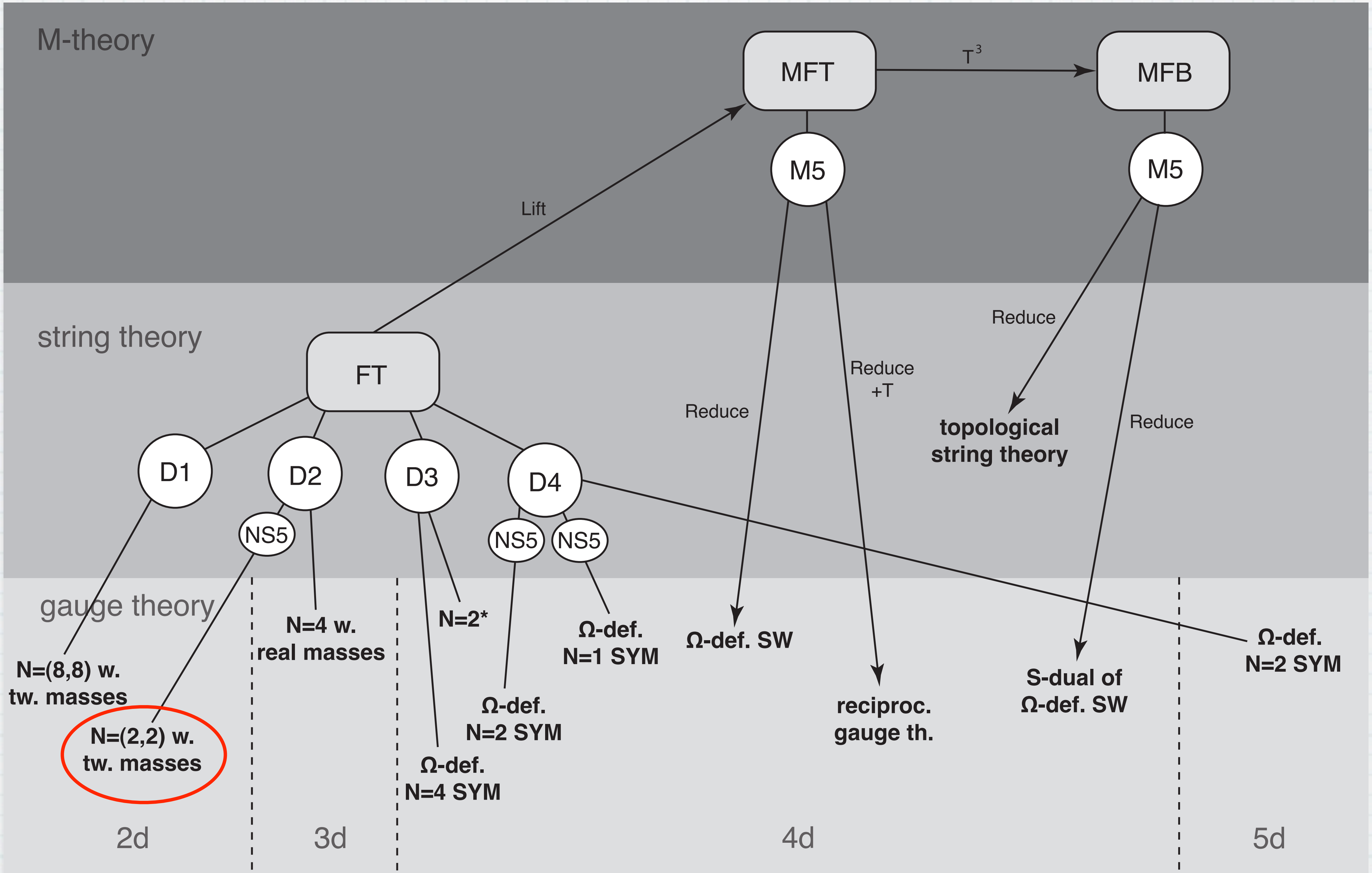
Summary

Can be lifted to M-theory: M-theory Fluxtrap



Summary

The fluxtrap construction has a variety of uses/applications.

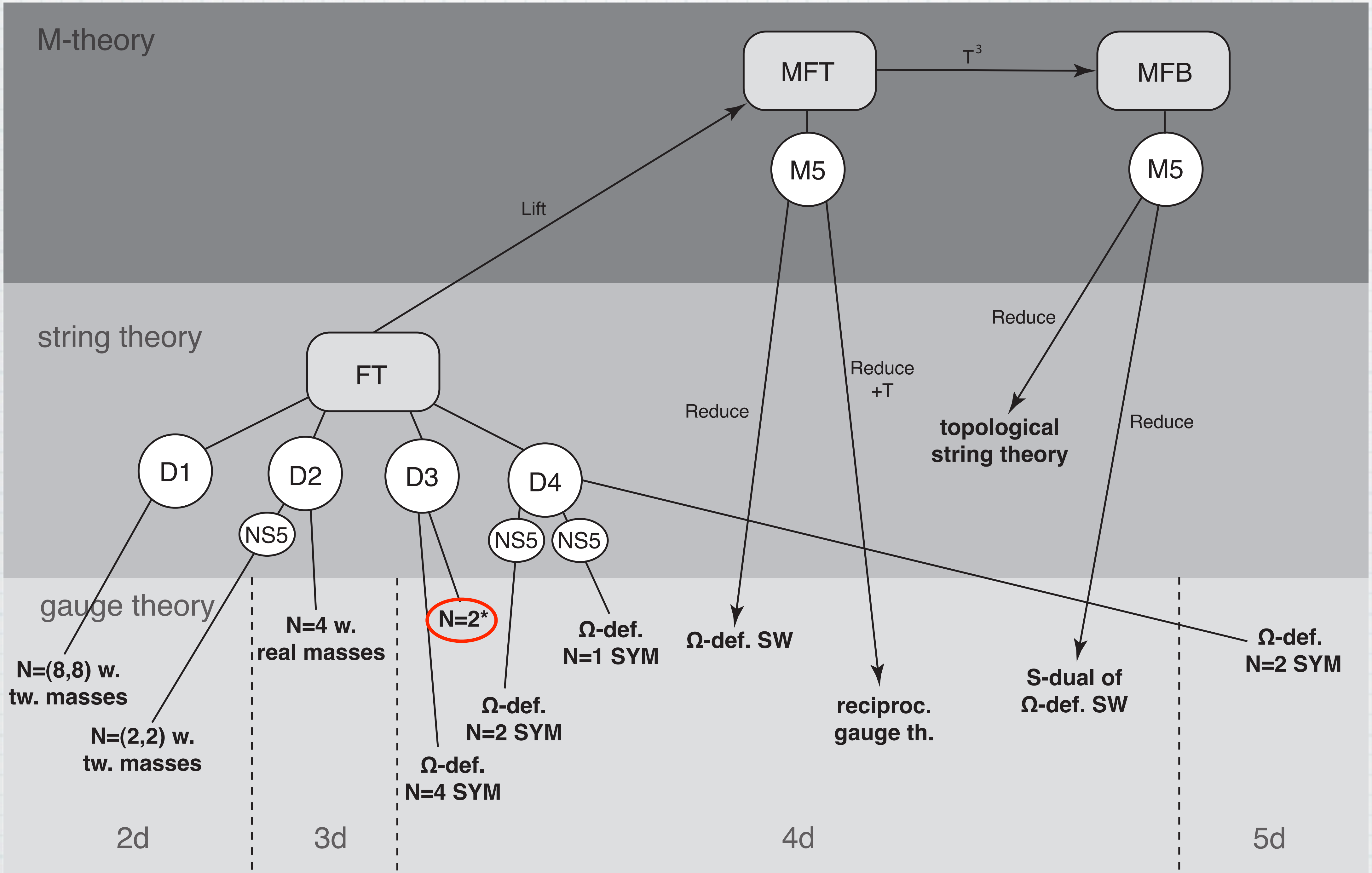


It captures the gauge theories with twisted masses of the 2d gauge/Bethe correspondence.

arXiv:1106.0279

Summary

We can construct the N=2* theory.

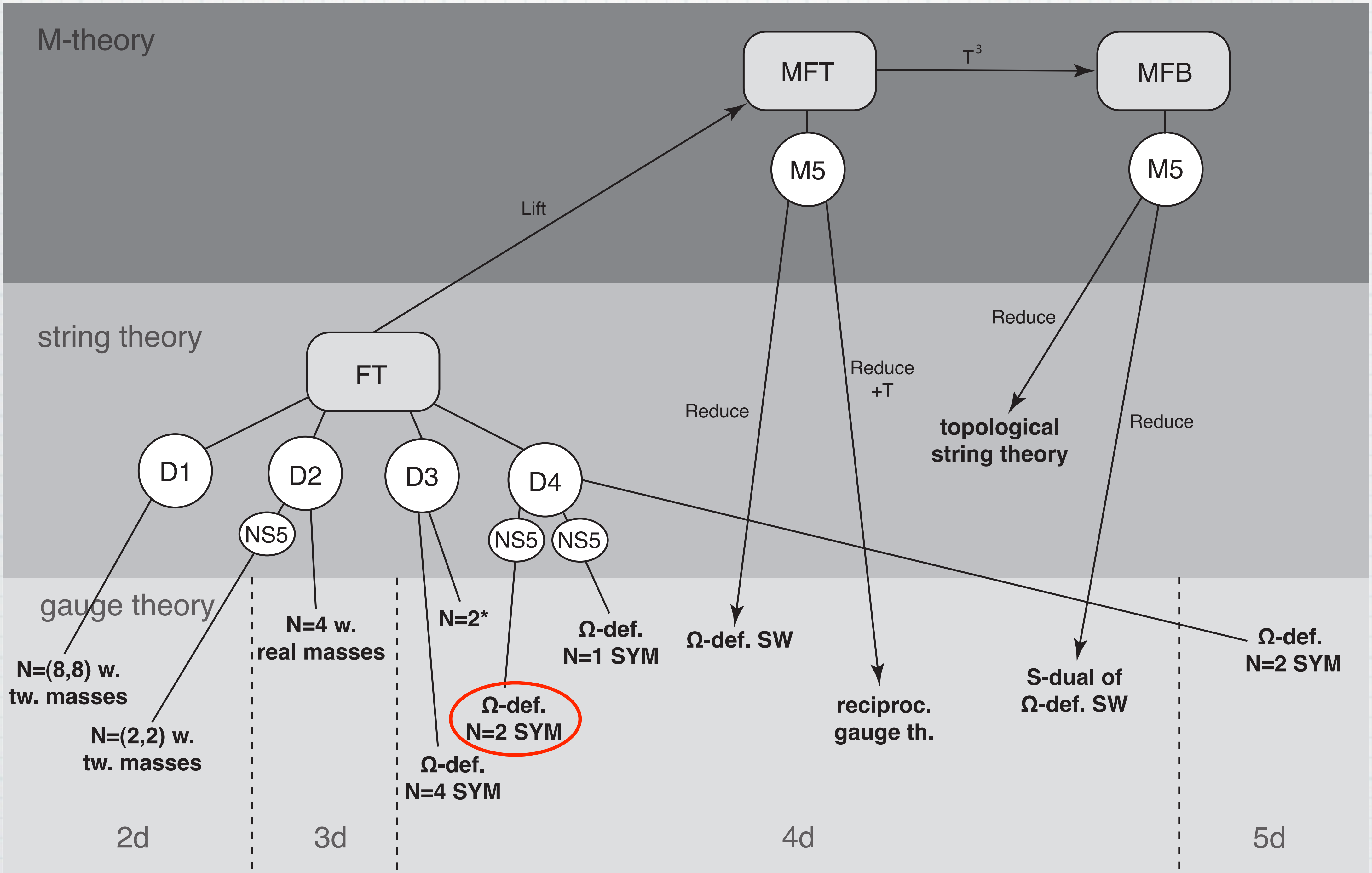


Construct gravity duals of deformed N=4 SYM

Summary

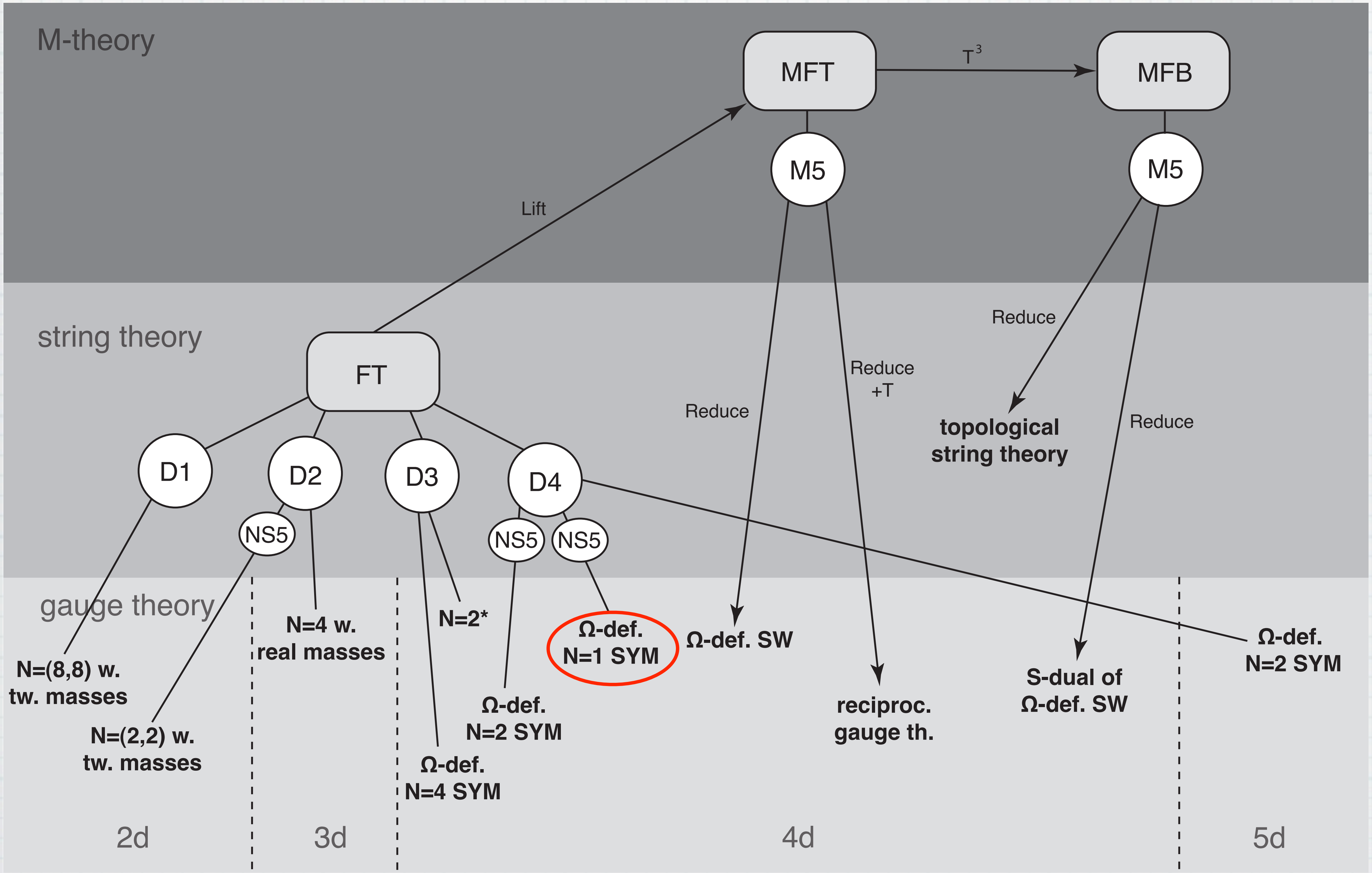
It captures the Omega-deformed gauge theories of the 4d gauge/Bethe correspondence.

arXiv:1204.4192



Summary

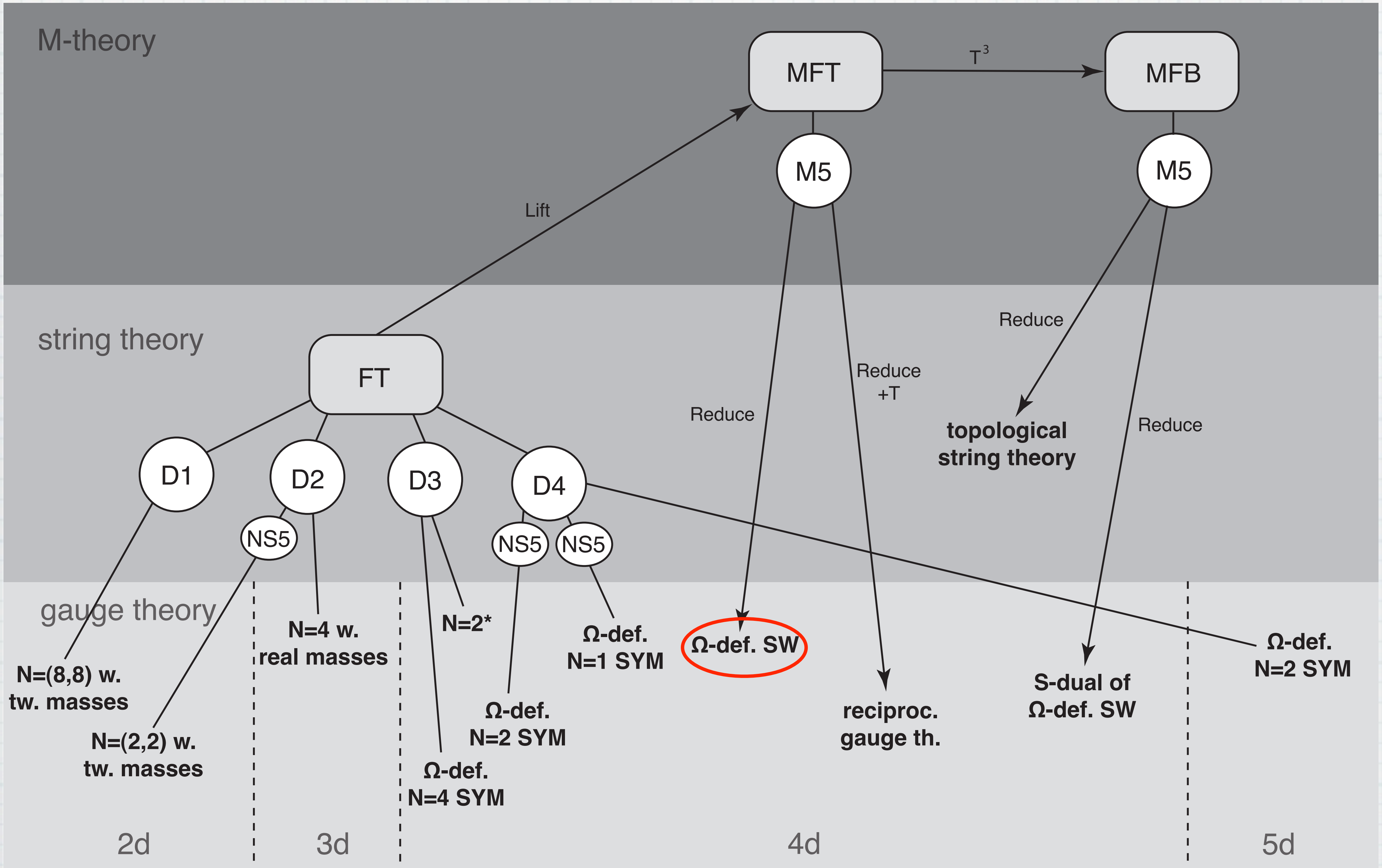
Can also construct Omega-deformed N=1 gauge theory.



Summary

Derive Omega-deformed Seiberg-Witten Lagrangian

arXiv:1304.3488



Summary

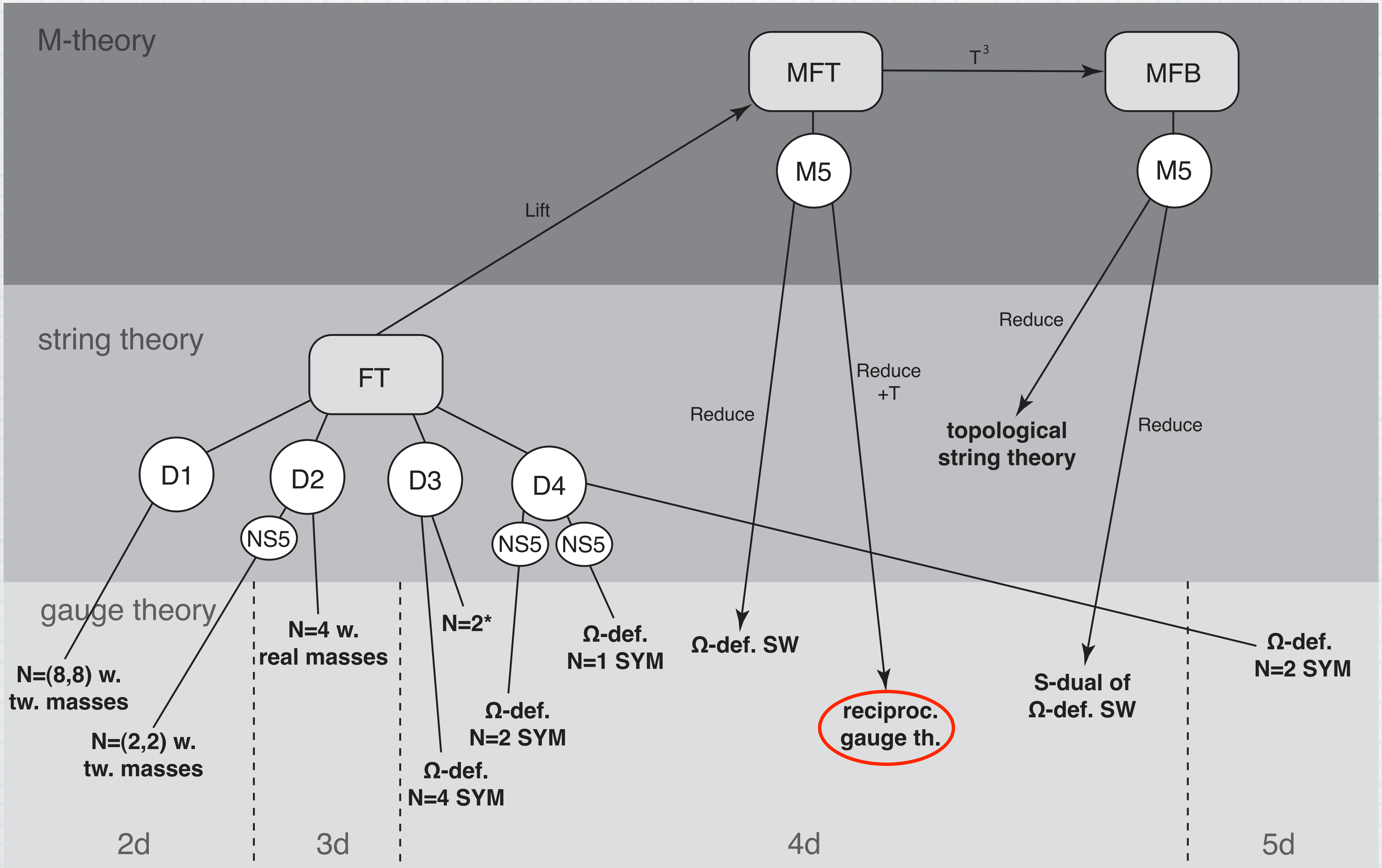
Use M-theory lift of fluxtrap BG, embed M5-brane, reduce 6d e.o.m. on Riemann surface.

The resulting 4d e.o.m. for the scalar and vector fields are Euler-Lagrange equations for a 4d action: Omega-deformed Seiberg-Witten Lagrangian!

Classical M-theory calculation yields quantum result, captures all orders of 4d gauge theory.

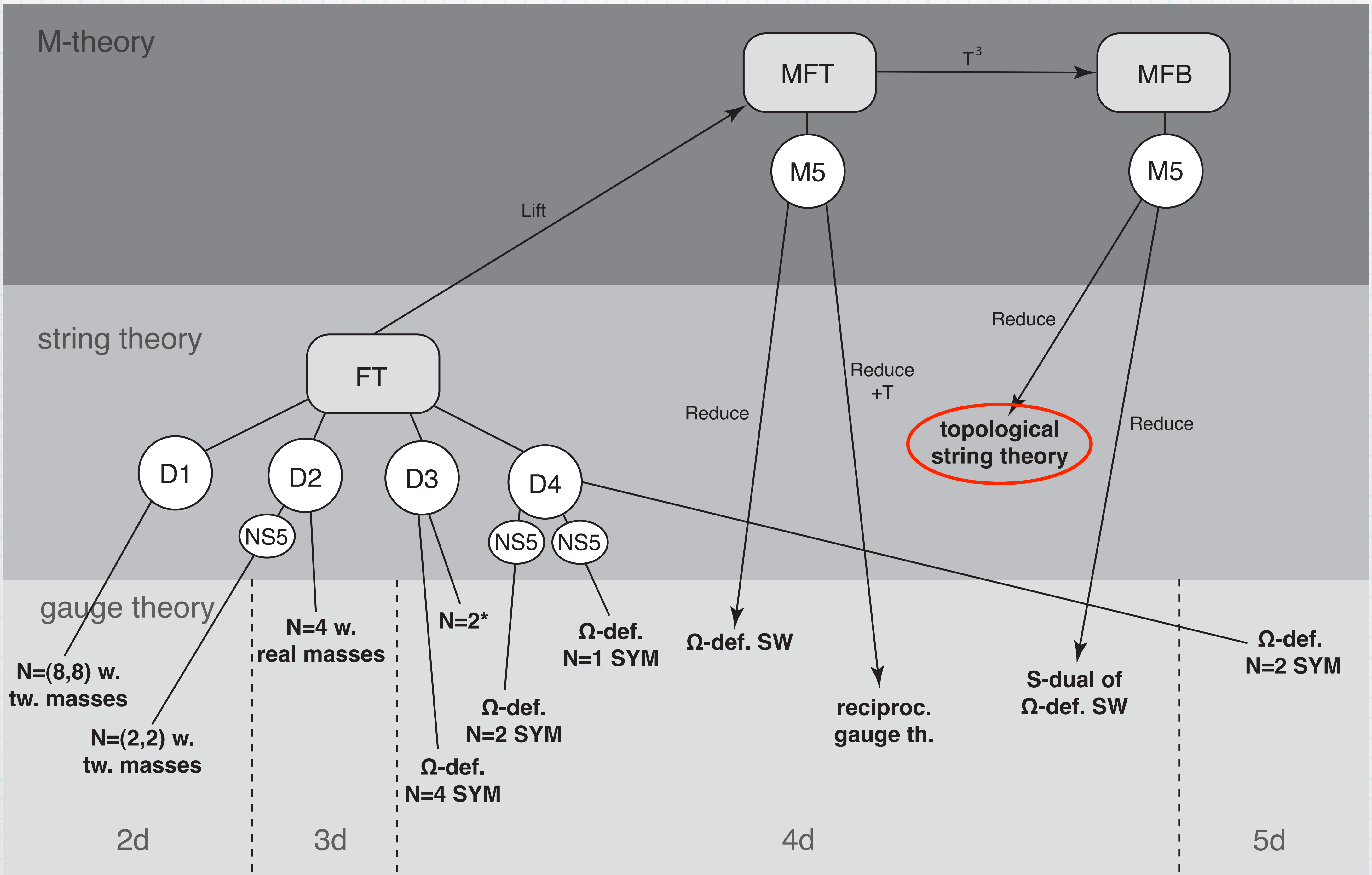
Summary

Starting point for understanding string theory formulation of AGT correspondence. arXiv:1210.7805



Summary

Connection to topological string theory.



Summary

The fluxtrap construction allows us to study different gauge theories of interest via string theoretic methods.

Omega deformation and (twisted) mass deformations have same origin in string theory.

The construction gives a geometrical interpretation for the Omega BG and its properties, such as localization etc.

Understanding of relation between deformation parameters and quantization of spectral curve.

Outlook

The area of N = 2 supersymmetric gauge theories and their connections to integrable models is a powerful laboratory to understand more realistic theories and holds great potential.

Use string theoretic fluxtrap construction of deformed supersymmetric gauge theory as a unifying paradigm.

Open questions:

- string-theoretical realization of the AGT correspondence
- identify BPS states in the AGT correspondence and in the Nekrasov/Shatashvili limit
- Topological string theory from the fluxtrap BG
- Geometric representation theory and gauge theories
- construct gravity duals to deformed gauge theories

Thank you for your attention!