# Constraints on the spectra of 2D CFTs

Joshua Qualls

Al Shapere

arXiv:1312.0038

KITP 1/16/2014

**Conformal bootstrap**

The program of constructing conformal field theories using nothing but conformal invariance and unitarity.

Has had important successes in the past few years in 3D and 4D
- where crossing symmetry of 4-point functions has been used to bound the dimensions of low-lying operators.

Was used with great success to study 2D CFTs in the 80's
- Much work focused on $c < 1$
- Famously, the constraints imposed by conformal invariance are strong enough to allow a complete classification in this case.

We'll be concerned with 2D CFTs with $c > 1$.
- Relatively poorly understood
  - mainly due to the complexity of the conformal blocks.
- Worth understanding better
  - they include string compactifications, holographic CFTs, etc.

**Abstract**: We will obtain upper bounds on dimensions of the lowest few primary operators, valid for all $c, \tilde{c} > 1$, and a lower bound on the number of primaries of dimension less than $\frac{c+\tilde{c}}{12}$ valid in the limit of large left- and right-moving central charges $c, \tilde{c}$.

Primary tool: modular invariance of the partition function.

Method due to Cardy, used by Hellerman [0902.2790] to bound lowest primary. Complementary to the approach of Ratazzi et. al. who used crossing symmetry of 4-point functions – another form of modular invariance which generalizes more easily to dimensions larger than 2.

"Complementary" because there are two necessary and sufficient conditions for a CFT to be defined on all Riemann surfaces:

- crossing symmetry of 4-point functions on sphere
- modular invaiance of partition function and 1-pt fcns on the torus.

If these are satisfied, can glue elements together to make any surface.

(It's possible that our results could be strengthened by combining with data coming from 4-point functions.)

## Hellerman's bound on $\Delta_1$

Start by reviewing Hellerman's proof that the lowest nontrivial primary must have dimension $\Delta_1 \leq \frac{c+\tilde{c}}{12} + .473695..$

Partition function of a 2D CFT on torus:

$$Z(\tau, \bar{\tau}) = \mathrm{tr}\left(e^{2\pi i\tau(L_0 - c/24)} e^{-2\pi i\bar{\tau}(\bar{L}_0 - \tilde{c}/24)}\right)$$

Hellerman makes the following assumptions about Z:

- no chiral primary fields, other than the identity
- so no extended chiral algebra, just Virasoro
- vacuum is unique; cluster decomposition
- $c, \tilde{c} > 1$

In particular, $c, \tilde{c} > 1$ implies that the conformal blocks take a specific form, so that we can write the partition function as

$$Z(\tau, \bar{\tau}) = Z_{id}(\tau, \bar{\tau}) + \sum_{A} Z_A(\tau, \bar{\tau})$$

where

$$Z_{id}(\tau, \bar{\tau}) = q^{-\frac{c}{24}} \bar{q}^{-\frac{\tilde{c}}{24}} \prod_{m=2}^{\infty} (1 - q^m)^{-1} \prod_{n=2}^{\infty} (1 - \bar{q}^n)^{-1}$$

$$Z_A(\tau, \bar{\tau}) = q^{h_A - \frac{c}{24}} \bar{q}^{h_A - \frac{\tilde{c}}{24}} \prod_{m=1}^{\infty} (1 - q^m)^{-1} \prod_{n=1}^{\infty} (1 - \bar{q}^n)^{-1}$$

are the conformal blocks of the identity and other operators indexed by $A$.

The modular invariance condition we wish to impose is

$$Z(\tau, \bar{\tau}) = Z(-1/\tau, -1/\bar{\tau})$$

The conformal blocks $Z_A$ are not modular invariant by themselves; the condition that their sum be invariant is nontrivial and imposes constraints on the conformal dimensions $h_A$.

Strategy: expand this condition around the point $\tau = i$ and rewrite it as a set of differential constraints on $Z$. For this purpose it is convenient to change variables to $s$ defined by $\tau = i\,\exp(s)$ . Then modular invariance becomes invariance under $s \to -s$:

$$Z\left(i\exp(s), i\exp(\bar{s})\right) = Z(i\exp(-s), i\exp(-\bar{s}))$$

*i.e.* $Z$ is an even function of $s$.

Now restrict attention to purely imaginary $\tau \equiv i\frac{\beta}{2\pi}$ , or purely real $s$. The statement that $Z$ is an even function of $s$ is equivalent to an infinite set of differential constraints

$$\left(\frac{\partial}{\partial s}\right)^p Z(\beta)\bigg|_{s=0} = 0, \quad p \text{ odd}$$

which are in turn equivalent to

$$\left(\beta\frac{\partial}{\partial \beta}\right)^p Z(\beta)\bigg|_{\beta=2\pi} = 0, \quad p \text{ odd}$$

Plugging in the expression for $Z$ decomposed into conformal blocks, the $p$th equation takes the form

$$\sum_{A=1}^{\infty} f_p(\Delta_A + \hat{E}_0)\exp(-2\pi\Delta_A) = -b_p(\hat{E}_0)$$

The $p$th equation takes the form

$$\sum_{A=1}^{\infty} f_p(\Delta_A + \hat{E}_0)\exp(-2\pi\Delta_A) = -b_p(\hat{E}_0)$$

where $\Delta \equiv h + \tilde{h}$ $\qquad \hat{E}_0 \equiv E_0 + \frac{1}{12} = \frac{1}{12} - \frac{c+\tilde{c}}{24}$

and $f_p$ and $b_p$ are polynomials of degree $p$ :

$$f_0(z) = 1$$
$$f_1(z) = (2\pi z) - \frac{1}{2}$$
$$f_2(z) = (2\pi z)^2 - 2(2\pi z) + \left(\frac{7}{8} + 2r_{20}\right)$$
$$f_3(z) = (2\pi z)^3 - \frac{9}{2}(2\pi z)^2 + \left(\frac{41}{8} + 6r_{20}\right)(2\pi z) - \left(\frac{17}{16} + 3r_{20}\right)$$

etc…

Hellerman considers the ratio of the first two of these conditions, for $p = 1, 3$:

$$\frac{\sum_{A=1}^{\infty} f_3(\Delta_A + \hat{E}_0)\exp(-2\pi\Delta_A)}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0)\exp(-2\pi\Delta_B)} = \frac{b_3(\hat{E}_0)}{b_1(\hat{E}_0)}$$

Defining the RHS to be $F_0(\hat{E}_0)$ and rearranging gives

$$\frac{\sum_{A=1}^{\infty} \left[f_3(\Delta_A + \hat{E}_0) - F_0(\hat{E}_0) f_1(\Delta_A + \hat{E}_0)\right] \exp(-2\pi\Delta_A)}{\sum_{B=1}^{\infty} f_1(\Delta_B + \hat{E}_0)\exp(-2\pi\Delta_B)} = 0$$

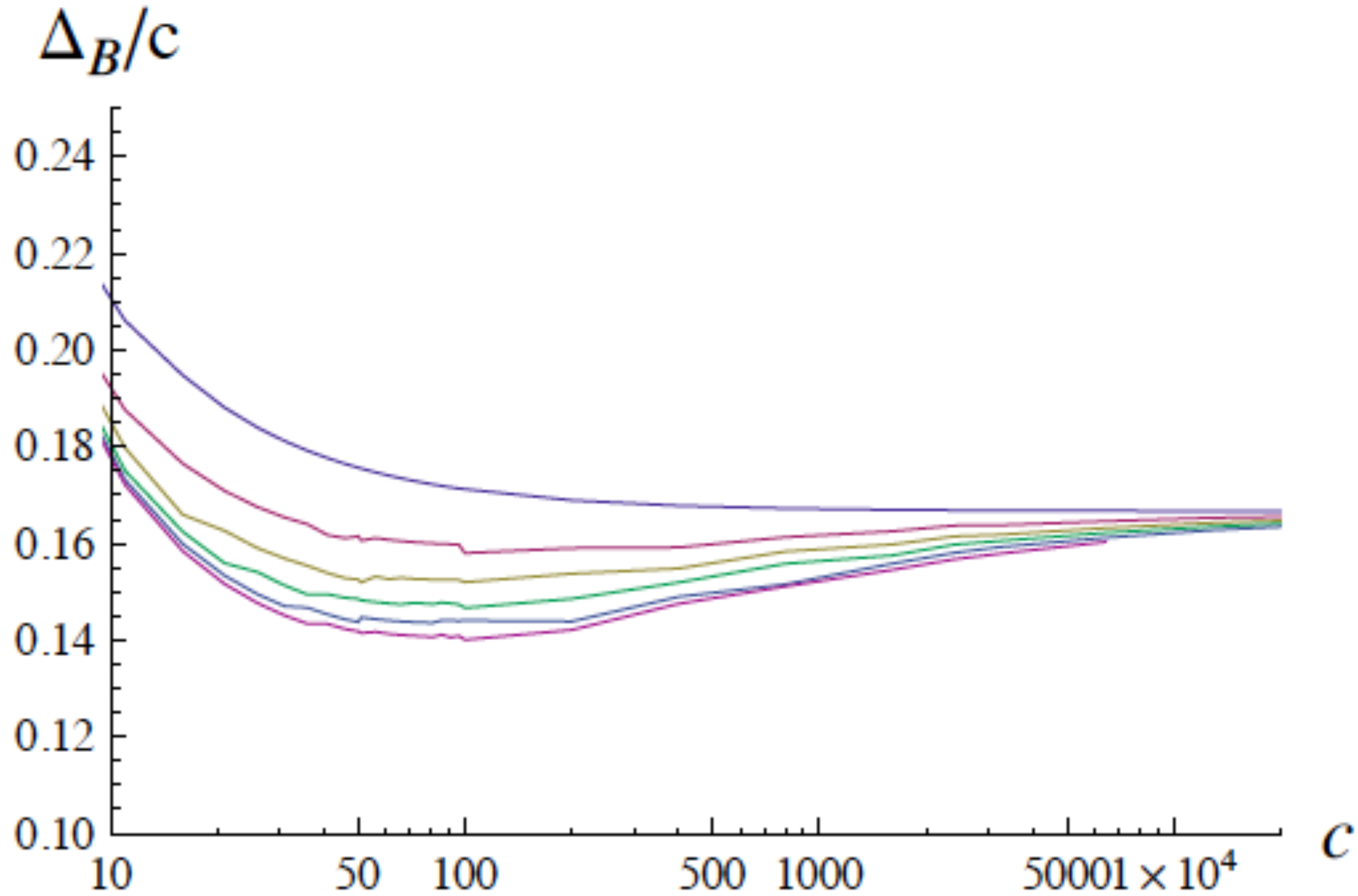
The expression in brackets is a cubic polynomial in $\Delta_A$, with a positive leading coefficient. Let $\Delta_1^+$ be its largest root, and suppose that $\Delta_1 > \Delta_1^+$. Then since $\Delta_A \geq \Delta_1 > \Delta_1^+$, every term in the numerator is positive, so the LHS can not possibly vanish. Thus it must be that $\Delta_1 \leq \Delta_1^+$.

Maximizing $\Delta_1^+$ over all possibilities gives the result

$$\boxed{\Delta_1 \leq \frac{c_{tot}}{12} + .4737}$$

Comments:

- Valid for all $c, \tilde{c} > 1$
- We have only used a fraction of the information available from constraints on $Z$: only pure-imaginary $\tau$, not $\tau \to \tau + 1$, only $p = 1, 3$.
- Can improve bound somewhat for finite $c$ by including higher-order constraints, $p = 1, 3, 5, 7, 9$ .... [Friedan & Keller 1307.6562]
- However, in all cases the bound asymptotes to c/12 .



Δ_B/c
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
10
50
100
500
1000
5000
1 × 10^4
c

**New bounds**

Similar methods may be used to derive bounds on $\Delta_2$ , $\Delta_3$ .
The main difference in the derivation for $\Delta_2$ is that we put all terms involving $\Delta_1$ on the RHS:

$$\frac{\sum_{A=2}^{\infty} f_3(\Delta_A + \hat{E}_0)e^{-2\pi\Delta_A}}{\sum_{B=2}^{\infty} f_1(\Delta_B + \hat{E}_0)e^{-2\pi\Delta_B}} = \frac{f_3(\Delta_1 + \hat{E}_0)e^{-2\pi\Delta_1} + b_3(\hat{E}_0)}{f_1(\Delta_1 + \hat{E}_0)e^{-2\pi\Delta_1} + b_1(\hat{E}_0)} \equiv F_1(\Delta_1, c_{tot})$$

This leads to

$$\frac{\sum_{A=2}^{\infty} \left[f_3(\Delta_A + \hat{E}_0) - f_1(\Delta_A + \hat{E}_0)F_1\right] \exp(-2\pi\Delta_A)}{\sum_{B=2}^{\infty} f_1(\Delta_B + \hat{E}_0)\exp(-2\pi\Delta_B)} = 0$$

and finally (after a bit of algebra)

$$\Delta_2 \leq \frac{c_{tot}}{12} + 0.533845...$$

Likewise,

$$\Delta_3 \leq \frac{c_{tot}}{12} + 0.879532...$$

Are we beginning to see a pattern?

A new wrinkle occurs for $\Delta_4$ .

We can no longer prove such a bound under the assumption $c_{tot} >$ 2, because the denominator vanishes when $c_{tot}$ = 2.34..
However, if we assume $c_{tot} > 2.5$ then

$$\Delta_4 \leq \frac{c_{tot}}{12} + 1.0795...$$

## Large $n$

It turns out that to obtain bounds on $\Delta_n$ for larger $n$, we must keep raising the lower limit for $c_{tot}$ in order for our method to work. The lower limit grows logarithmically with $n$:

$$c_{tot} \gtrsim \frac{12}{\pi} \log n$$

With this assumption, we can show that, asymptotically,

$$\Delta_n \leq \frac{c_{tot}}{12} + O(1)$$

where the $O(1)$ term depends weakly on $n$.

The fact that all these bounds have the same form indicates that there are a large number of operators with dimensions approximately less than $\frac{c_{tot}}{12}$.
In fact, the number *N* of such operators must be quite large:

$$\log N \geq \frac{\pi c_{tot}}{12} + O(1)$$

or

$$N \gtrsim \exp \frac{\pi c_{tot}}{12}$$

up to a prefactor of order *c*. It's surprising to find so many low-lying states!

Or maybe not…

**Holographic Interpretation**

If the CFT has a gravitational dual, it becomes a statement about the number of gravitational bound states, or more precisely, about the number of BTZ black holes without boundary excitations ("primary"). And we know there are a lot of these.

According to AdS/CFT, we identify

$$c + \tilde{c} = \frac{3L}{G_N}$$

We also match the rest energy of the *n*th gravitational solution

$$M_n = \Delta_n / L$$

(with no boundary excitations) with the dimension of the *n*th primary in the CFT.

Then using the bound on $\Delta_n$ gives

$$M_n \leq \frac{1}{4G_N} + \frac{D_n}{L}$$

where $D_n$ is an $O(1)$ constant.

In the limit $L \to \infty$ we get

$$M_n \leq \frac{1}{4G_N}$$

Our CFT result implies that the number of states satisfying this inequality is of order

$$\log N \geq \frac{\pi c_{tot}}{12} + O(1) = \frac{\pi L}{4G_N} + O(1)$$

This is comparable to the entropy of a black hole of mass $1/4G_N$ :

$$S(M = \tfrac{1}{4G_N}, J = 0) = \frac{\pi L}{\sqrt{2} G_N}$$

## Comments

It's suggestive that $1/4G_N$ is also the maximum possible ADM mass of any localized configuration of 2+1 gravity coupled to matter, for $\Lambda = 0$. Thus higher excitations of the CFT can not correspond to local excitations of an effective gravitational theory, although they might correspond to extended (stringy or braney) excitations.

Hellerman has advertised his bound on $\Delta_1$ as a proof of a form of the Weak Gravity Conjecture, according to which the lightest state in a quantum theory of gravity can't be heavier than a certain value proportional to the Planck mass. Along these lines, we have proved a significant extension generalization of the WGC: that there must be an enormous number of states lighter than a certain value proportional to the Planck mass.

To conclude, our lower bound on the number of primaries of dimension $c/12$ is of a similar order to what we would expect on holographic grounds. The difference is, our bound is universally valid, whether or not the CFT has a gravitational dual.

**Questions**

1. What about the assumption of no nontrivial chiral primaries?

   We believe it can be relaxed, and one can obtain similar bounds on the dimensions of primaries, still of order $c/12$. A new feature is that when extended chiral algebras are present, primary operators are defined with respect to the full chiral algebra, including both the Virasoro and current algebras. Our methods would give bounds on *Virasoro* primaries. Since primaries of the full algebra are also Virasoro primaries, they will also obey bounds of order $c/12$.
   If this turns out to be the case, our results will apply to any unitary, modular invariant CFT.

2. Are our bounds optimal?

   There's a lot of additional information we haven't used: full modular invariance of torus, crossing symmetry of 4-point functions, etc. Maybe one can do better by taking these into account.

3. Are there CFTs that saturate our bounds?

It's hard to find CFTs with large values of $\Delta_1$ .
Examples with the largest known gaps are the Höhn-Witten "monster" CFTs, proposed as duals of pure 3D gravity. Assuming they exist, these CFTs have gap $\Delta_1 = \frac{c_{tot}}{48} + 1$ and an exponential large density of states. They are holomorphically factorized, and thus highly nongeneric. There is no reason why non-holomorphically factorized CFTs should not permit a larger gap, but on the other hand I don't know of any examples.