Bit Threads for Multiple Regions

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Geometry from the Quantum
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Based on

1604.00354 with Michael Freedman

1808.05234 with Shawn Cui, Patrick Hayden, Temple He, Bogdan Stoica, Michel Walter

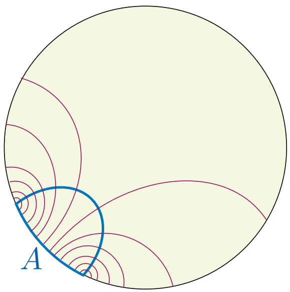
1906.05970 with Jonathan Harper

& forthcoming work with Jesse Held, Joel Herman

Contents

1 Bit threads

In this talk: holographic theory dual to classical Einstein gravity; static bulk spacetime; pure state



Bit threads are unoriented 1d objects in bulk

1. Don't split or join, end only on boundary
2. Density $\leq 1/4G_{\rm N}$ (threads have Planckian thickness)

Ryu-Takayanagi: $$S(A) = \frac{1}{4G_{\rm N}} \min_{m\sim A} \text{area}(m) = \max N_{A:\bar{A}}$$

$N_{A:\bar{A}} = \#$ threads connecting A to $\bar{A}$

Max thread configuration seems to represent distilled A-$\bar{A}$ entanglement; 1 thread = 1 Bell pair

Casini-Huerta-Magán-Pontello '19: Threads are **intertwiners** between algebras of observables in A and $\bar{A}$

Riemannian max flow-min cut theorem [Federer '74, Strang '83, Headrick-Hubeny '17]

Intuition: Add threads until tightly packed on minimal surface $m(A)$

Proof: Maximizing $N_{A:\bar{A}}$ can be written as convex problem; strongly dual to:

Minimize $\int \sqrt{g}\,\lambda$ for function λ subject to $\lambda \geq 0$, $\int_{\mathcal{C}} ds\,\lambda \geq 1$ for any curve $\mathcal{C}$ from A to $\bar{A}$

Solution: $\lambda = \delta$-function on $m(A)$ $\quad\Rightarrow\quad$ max $N_{A:\bar{A}} = \int \sqrt{g}\,\lambda = \text{area}(m(A))$

2 Three boundary regions

What if we divide boundary into 3 regions and try to connect them with as many threads as possible?

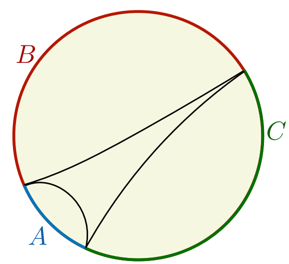
$$N_{A:B} + N_{A:C} = N_{A:\bar{A}} \leq S(A)$$
$$N_{A:B} + N_{B:C} = N_{B:\bar{B}} \leq S(B)$$
$$N_{A:C} + N_{A:C} = N_{C:\bar{C}} \leq S(C)$$

Theorem: These bounds are collectively tight; there exists a thread configuration such that

$$N_{A:\bar{A}} = S(A)\,, \qquad N_{B:\bar{B}} = S(B)\,, \qquad N_{C:\bar{C}} = S(C)$$

Proof for networks: Kupershtokh '71, Lovász '76, Cherkassky '77
Proof for Riemannian manifolds: Cui-Hayden-He-MH-Stoica-Walter '18

$$N_{\text{tot}} = N_{A:B} + N_{A:C} + N_{B:C} \leq \frac{1}{2}\left(S(A) + S(B) + S(C)\right)$$



Maximizing N_{tot} can be written as convex program; strongly dual to:

Minimize $\int \sqrt{g}\,\lambda$ subject to $\lambda \geq 0$,
$\int_{\mathcal{C}} ds\,\lambda \geq 1$ for any path $\mathcal{C}$ connecting different regions

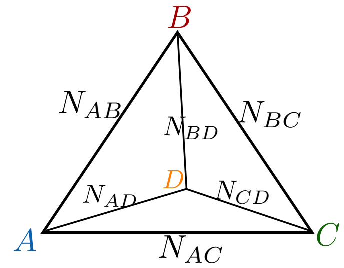
Solution is $\frac{1}{2}\delta$-function on each RT surface:

$$\lambda = \frac{1}{2}(\delta_{m(A)} + \delta_{m(B)} + \delta_{m(C)}) \quad \Rightarrow \quad \max N_{\text{tot}} = \int \sqrt{g}\,\lambda = \frac{1}{2}\left(S(A) + S(B) + S(C)\right)$$

Theorem generalizes to any number of regions...

3 Four boundary regions & MMI

There exists a thread configuration that computes S for each region:



$$
\begin{aligned}
S(A) &= N_{A:\bar{A}} = N_{A:B} + N_{A:C} + N_{A:D} \\
S(B) &= N_{B:\bar{B}} = N_{A:B} + N_{B:C} + N_{C:D} \\
S(C) &= N_{C:\bar{C}} = N_{A:C} + N_{B:C} + N_{C:D} \\
S(D) &= N_{D:\bar{D}} = N_{A:D} + N_{B:D} + N_{C:D}
\end{aligned}
$$

However, composite regions are not necessarily saturated:

$$
\begin{aligned}
S(AB) &\geq N_{AB:CD} = N_{A:C} + N_{B:C} + N_{A:D} + N_{B:D} \\
S(AC) &\geq N_{AC:BD} = N_{A:B} + N_{B:C} + N_{A:D} + N_{C:D} \\
S(AD) &\geq N_{AD:BC} = N_{A:B} + N_{A:C} + N_{B:D} + N_{C:D}
\end{aligned}
$$

Summing:

$$S(AB) + S(AC) + S(AD) \geq S(A) + S(B) + S(C) + S(D)$$

monogamy of mutual information (MMI) [Hayden-MH-Maloney '11]

(See [Hubeny '18] for alternative proof, [Agon-de Boer-Pedraza '18] for explicit constructions)

4 Higher inequalities

Generalizations [MH-Held-Herman, forthcoming]:

The good news:

Consider a set of *composite* regions R_α that do not *cross* (partially overlap); e.g. A, B, C, D, AB, ABC

Theorem 1: There exists a thread configuration saturating all the R_α:

$$N_{R_\alpha:\bar{R}_\alpha} = S(R_\alpha)$$

A *bundle* consists of all threads connecting two elementary regions

Theorem 2: Bundles can be chosen not to overlap

Proves conjecture by Hubeny '18

The bad news:

When regions *cross* (e.g. AB, BC), a saturating configuration does *not* necessarily exist

(Statements true on graphs do *not* hold on manifolds)

Higher entropy-cone inequalities [Bao et al '15] have crossing regions on RHS; e.g. 5-party dihedral inequality

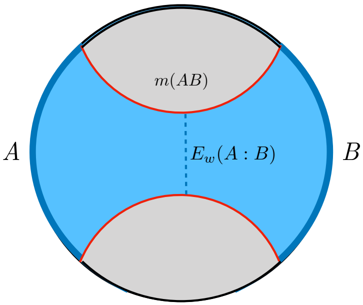
$$\begin{aligned} &S(ABC) + S(BCD) + S(CDE) + S(DEA) + S(EAB) \\ &\qquad \geq S(AB) + S(BC) + S(CD) + S(DE) + S(EA) + S(ABCDE) \end{aligned}$$

Therefore RHS is not calculated by any single thread configuration

Understanding such inequalities in terms of bit threads requires something more complicated

5 Entanglement wedge cross sections

[Harper-MH '19; see also Du-Chen-Shu '19; Bao-Chatwin-Davies-Pollack-Remmen '19]



$E_w(A:B) :=$

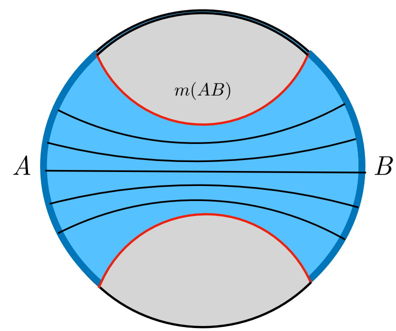
minimal cross section of AB homology region (entanglement wedge)

Conjectured to equal

- entanglement of purification [Takayanagi-Umemoto '17, Nguyen-Devakul-Halbasch-Zaletel-Swingle '17]
- entanglement negativity [Kudler-Flam-Ryu '18]
- odd entropy [Tamaoka '18]
- reflected entropy [Dutta-Faulkner '19]

Can be computed by threads, not allowing threads to end on $m(AB)$

All geometric properties of E_w follow simply



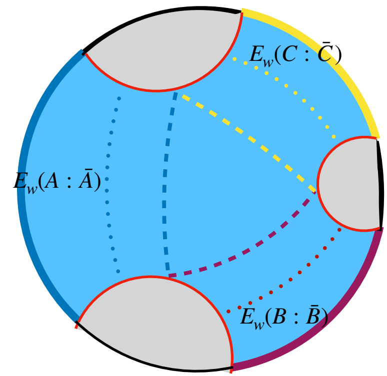
6 Multi-region cross sections

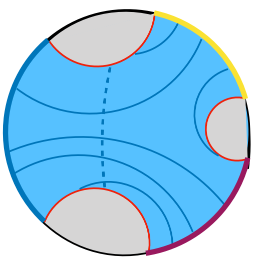
Conjectured to equal multipartite entanglement of purification [Umemoto-Zhou '18; see also Bao-Cheng '19 for alternative conjecture]:

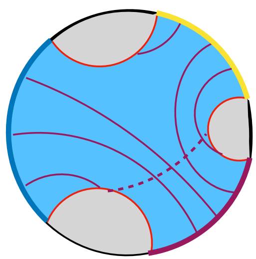
$$E_p(\{A_i\}) := \min_{\substack{\text{purification} \\ |\psi\rangle}} \sum_i S(\psi_{A_i A_i'}) \qquad \text{Note:} \quad E_p(\{A_i\}) \geq \sum_i E_p(A_i : \bar{A}_i)$$

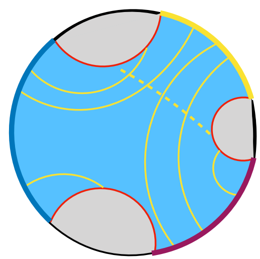
To calculate with threads:

- Different "species" of thread for each boundary region
- Threads of different species do not interact in bulk
- Threads can attach to $m(\{A_i\})$ — but only all species together
- Number of A_i threads attached to A_i equals $E_w(A_i : \bar{A}_i)$
- Rest of threads are attached to $m(\{A_i\})$; may represent "truly multipartite" part of $E_p(\{A_i\})$

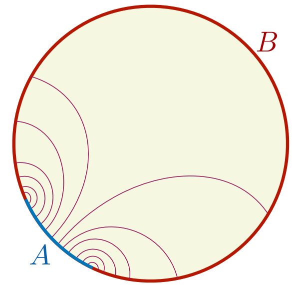



v_A


v_B

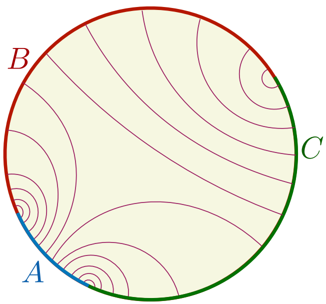

v_C

7 Bipartite dominance



For 2 regions, threads can be thought of Bell pairs after entanglement distillation:

$$N_{A:B} = S(A) = S(B) = \frac{1}{2} I(A:B)$$



For 3 regions,

$$N_{A:B} = \frac{1}{2} I(A:B)\,, \qquad N_{A:C} = \frac{1}{2} I(A:C)\,, \qquad N_{B:C} = \frac{1}{2} I(B:C)$$

Three-party distillation into Bell pairs?
Possible only if state contains only bipartite (no tripartite) entanglement (up to order-1 corrections)

— no matter how boundary is decomposed into A, B, C

"**Bipartite dominance**"

Do there exist quantum states with this property? Yes:

Simple example: 4-party perfect tensor (4PT)

More interesting: Random stabilizer tensor networks [Nezami-Walter '16]

Bipartite dominance implies that, for any 4-region decomposition, there is only bipartite + 4PT entanglement

Simplest non-trivial entanglement structure consistent with MMI

8 Bipartite dominance vs. cross sections

Bipartite dominance implies $I(A:B)$ is entirely due to entanglement (no classical correlations)

This fixes both entanglement of purification & reflected entropy:

$$\mathsf{EOP} = \frac{1}{2}\mathsf{RE} = \frac{1}{2}I(A:B) \tag{1}$$

But typically

$$E_w(A:B) > \frac{1}{2}I(A:B) \qquad \text{(at order in } G_{\mathrm{N}}^{-1}\text{)}$$

Thus conjectures relating E_w to EOP/RE apparently contradict bipartite dominance

Three ways out:

- Bipartite dominance is wrong
- $E_w \neq$ EOP, RE
- Order-1 corrections in bipartite dominance spoil (1)
  Akers-Rath '19 argued against this on grounds of continuity