

Entanglement entropy in JT gravity

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Geometry from the Quantum

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- Introduction
- Factorization map
- Defect operator
- Implications

Gravitational entropy

- The area/entropy relation can be derived from the gravity path integral. It is dominated by weakly curved saddles well-described by the low energy theory.
- The Lorentzian meaning is not obvious because the euclidean time circle shrinks in the interior, so there is no obvious interpretation as a trace.

Defining entanglement entropy

- In QFT and the gravity effective theory, the Hilbert space doesn't factorize into subregions.
- Lattice calculations give, in the continuum limit, a map to a split Hilbert space defined with boundary conditions.
- Potentially easier in gravity because the exact answer is finite and can't depend on b.c. choice.

All orders quantum HRT

- The quantum HRT formula instructs us to extremize the generalized entropy. However, it is not clear how to define the gravity contribution when the bulk region is not bounded by a geometrically extremal surface. Gauge choices have to be made to specify the meaning of the location of the region, and some boundary conditions for gravitons have to be picked.
- This affects the EE result at second order in the G_N expansion.

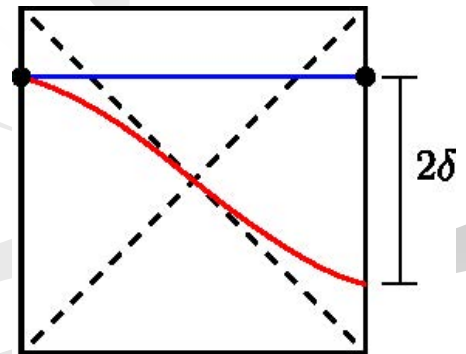
Lorentzian JT gravity

- For simplicity, take the eternal black hole in JT gravity.

$$S_{\text{JT}} = \phi_0 \left(\int_M d^2x \sqrt{-g} R + 2 \int_{\partial M} dt \sqrt{-\gamma} K \right) + \int_M d^2x \sqrt{-g} \phi (R+2) + 2 \int_{\partial M} dt \sqrt{-g} \phi (K-1)$$

- The phase space is two dimensional, labelled by the black hole mass $H/2 = \phi_h^2/\phi_b$, and the relative time shift between the two boundaries.

- The time shift is not a good operator in quantum mechanics (and is teleological in spacetime), so one can use other variables.



Liouville quantum mechanics

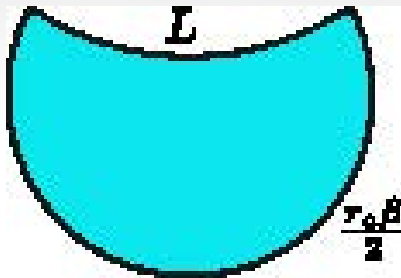
- Take maximal volume slices to fix temporal gauge – in other words, geodesics. As time evolves up along both boundaries, the geodesic grows. Its regularized length is a good variable.

$$H = \frac{P^2}{2\phi_b} + \frac{2}{\phi_b} e^{-L}$$

- The system is just Liouville quantum mechanics (thanks to Henry Lin for finding a typo in our paper).
- Scattering problem with continuous spectrum due to the infinite range of L .

Black hole state

- This is computed in L basis by this path integral



- Performing that exactly, and going to $s = \phi_h$ basis,

$$\Psi_\beta(s) = \sqrt{2s \sinh(2\pi s)} e^{-\beta \frac{s^2}{4\phi_b}}$$

[Z. Yang]

- No factorization, but the normalization equals the Schwarzian path integral computed by [Stanford Witten](#)

Entanglement entropy in QFT

- The definition of entanglement entropy in continuum quantum field theory requires introducing boundary conditions to factorize the Hilbert space. This appears automatically in lattice regulators, when ending the lattice. The map depends on the finite split, ε , taken between the regions, which is sent to 0 at the end.

$$i : \mathcal{H} \rightarrow \mathcal{H}_L \otimes \mathcal{H}_R$$

- What's different in gravity is thus not the lack of factorization of the Hilbert space *per se*, but rather, as we shall see, a topological obstruction to finding such a boundary condition that obeys the appropriate rules.

Cutting rules

- An obvious property that the splitting must obey is that it is an isometry in the limit the regulator is removed.

$$\lim_{\epsilon \rightarrow 0} i_{\epsilon}^{\dagger} i_{\epsilon} = 1$$

- In addition, the local algebra strictly on the left must act only on the split left factor, and likewise for the right.

Ambiguity in JT

- Those two rules do not lead to unique answers. In the JT case, the left and right algebras are equal, and are just generated by the Hamiltonian. Therefore the splitting $|E\rangle \rightarrow \sum_{i=1}^{d(E)} |E, i\rangle |E, i\rangle$ is allowed for any $d(E)$ and the resulting EE is arbitrary. Likewise for QFT.
- Require a stronger notion of locality – the cutting map should be defined using the local boundary condition.

A simple example

- One natural way to split the Hilbert space is to think of the original system as two boundary points (at a given time) given abstractly as a pair of spacelike separated points in a fixed global AdS_2 , modulo $SL(2, \mathbb{R})$ isometries. That gives an 8d phase space minus 6 from the gauging, resulting in the original 2.
- The split Hilbert space is obtained by removing the constraint.

Schwarzian quantum mechanics

- This results in the Schwarzian QM with action

$$S = \phi_b \int dt \left[-\dot{T}^2 + \left(\frac{\ddot{T}}{\dot{T}} \right)^2 \right]$$

- Introducing $e^\chi = \dot{T}$ exactly in the path integral, one derives the Hamiltonian

$$H = \frac{1}{2\phi_b} \left(\frac{p_\chi^2}{2} + p_T e^\chi + \frac{e^{2\chi}}{2} \right)$$

- There is a unique energy preserving isometric cutting map whose image are states that vanish at timelike separation.

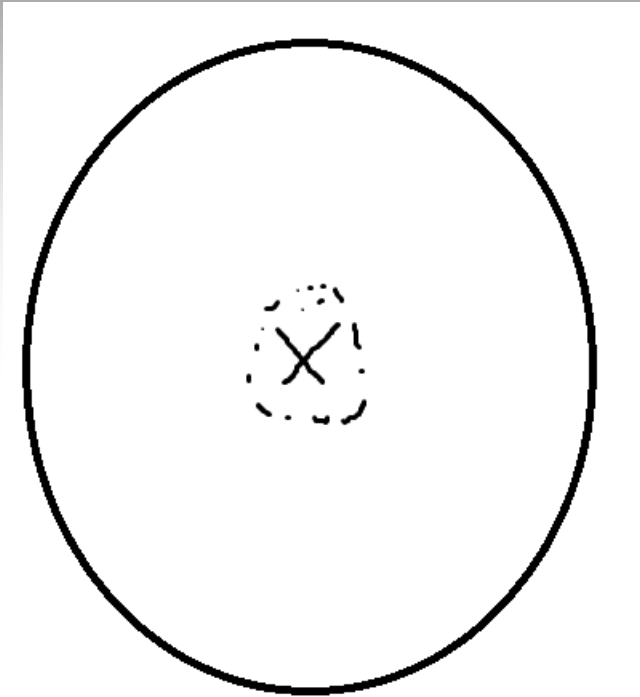
$$\Psi_s(T_L, \chi_L, T_R, \chi_R) = \begin{cases} \frac{\sqrt{2s \sinh(2\pi s)}}{2\pi^2} \frac{e^{i(e^{\chi_R} - e^{\chi_L}) \tan(\frac{T_L - T_R}{2})}}{\cos(\frac{T_R - T_L}{2})} K_{2is} \left(\frac{2e^{\frac{\chi_L + \chi_R}{2}}}{\cos(\frac{T_R - T_L}{2})} \right), & |T_L - T_R| < \pi, \\ 0, & |T_L - T_R| > \pi. \end{cases}$$

Bulk interpretation

- The same Hilbert space is obtained from a bulk boundary condition that fixes the metric and extrinsic curvature to 0.
- In the first order formalism of $SL(2, \mathbb{R})$ BF theory, this is just the $A_t = 0$ boundary condition.
- Setting the metric to 0 is good – this says that boundary in the euclidean path integral shrinks away.

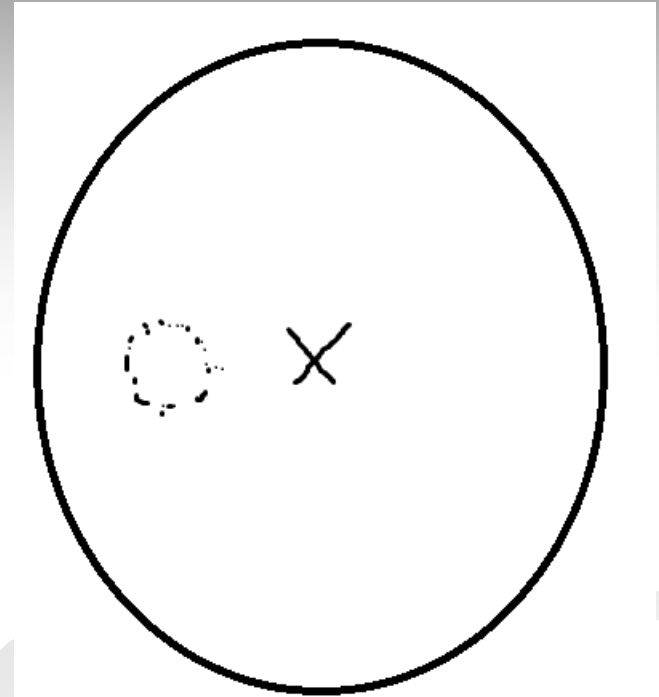
Topological defect

- The problem is that the integral of the extrinsic curvature around a smooth point should be 2π , not 0. Euclidean JT gravity is $SL(2, \mathbb{R})$ BF theory with only certain topology of bundles included, equivalently, the $SO(2)$ spin connection should be decompactified, and exactly one defect operator should be inserted in the disk. The defect is the operator around which the holonomy is 2π , ie. the generating central element of the universal cover of $SL(2, \mathbb{R})$.



Excise the defect itself. No such local boundary condition.

$$Z = \text{Tr}(e^{-\beta H})$$



Split the Hilbert space at a simple boundary.

$$Z = \text{Tr}(D e^{-\beta H})$$

Cutting the black hole state

- The splitting defined by the simple boundary condition is not an isometry. Combining it with half the defect operator gives the isometric map discussed before. This leads to Renyi entropies

$$Z_n = \text{Tr}(D e^{-n\beta H}) \quad \frac{Z_n}{Z_1^n} = \text{Tr}(D^{1-n} \rho_\beta^n)$$

- Thus the defect contributes to the von Neumann entropy for a general state $\int ds f(s) |s\rangle$

$$-\partial_n \left(\frac{\tilde{Z}_f[n]}{\tilde{Z}_f[1]^n} \right) \Big|_{n=1} = \log V + \int_0^\infty ds \left[-|f(s)|^2 \log |f(s)|^2 + |f(s)|^2 \log(2s \sinh 2\pi s) \right].$$

$$S_{HH}(\beta) = \frac{\pi \phi_b}{2G_N \beta} - \frac{3}{2} \log \frac{16\pi G_N \beta}{\phi_b} + \frac{3}{2} + \log(4\pi^{3/2}).$$

Quantum HRT formula

- The classical term in the entropy formula is the logarithm of the defect operator. That is exactly the generator of time translation along the cut boundary. Here, this is the same as the local boost of the horizon, resulting in the area operator.
- However, this defines the entropy to all orders in perturbation theory, because this operator is defined in the split Hilbert space, and the splitting is defined intrinsically – the location of the cut is a dynamical variable.

With exact microstates

- In the exact theory including the microstates, there would be a system with a very large number of degrees of freedom on the boundary. Then one could define an approximate time operator, which would allow a local boundary condition that fixes the total cone angle.
- Because that's the only input of the exact theory in the EE calculation, this explains its universality.

Generalizations

- It should be possible to include matter coupled to gravity, by using whatever are the appropriate boundary conditions and background actions for the matter EE definition. The same logic will imply that the Lorentzian analysis including the defect will exactly reproduce the euclidean Renyi calculation.
- The same applies to higher dimensions. In these cases, it would be interesting to see the results at higher orders in gravitational perturbation theory.

Induced gravity

- An intriguing possibility is a theory in which no curvature terms appear in the UV action, but are all generated by integrating out various matter degrees of freedom.
- Then there would be constraint on the cone angle, and no defect operator in the UV theory. Such a system would have a simple boundary condition that would lead (approximately) to the correct entropy including the area term by a direct count of states.