The Analytic Bootstrap, Sphere Packing and Quantum Gravity

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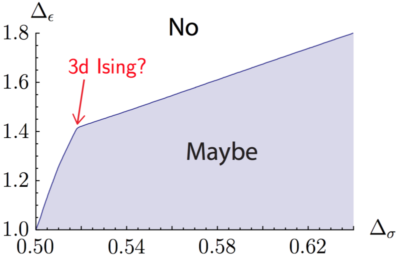
Based on arXiv:1905.01319, with Tom Hartman and Dalimil Mazáč + in progress

Geometry from the Quantum

KITP, January 16 2020

Conformal Bootstrap

The bootstrap has been very successful in constraining the space of CFTs.



El-Showk Paulos Poland Rychkov Simmons-Duffin Vichi

Particularly useful for theories *at the edge*.

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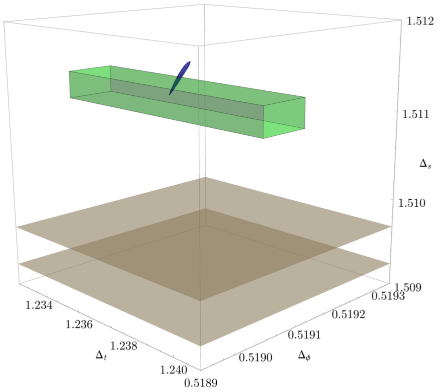


3d $O(2)$ island
[Chester, Landry, Liu, Poland, Simmons-Duffin, Su, Vichi]

Two broad uses of the bootstrap:

- Carving out theory space:
  rule out inconsistent regions of putative CFT data.

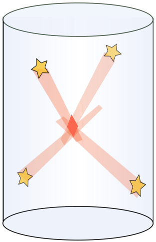
- Solving CFTs of special interest:
  corner them by making a few physical assumptions.

AdS quantum gravity = CFT

Two broad uses of the bootstrap:

- Carving out the space of AdS quantum gravity theories:
  rule out effective field theories with no UV completion ("swampland").

- Solving theories of special interest:
  corner them by making a few physical assumptions.
  E.g., bootstrap string/M-theory models with high supersymmetry.



Some general questions:

- What is the "simplest" theory of AdS quantum gravity in D dimensions?
- Does it require additional states well-below the Planck scale? (Such as strings or KK modes)
- Or does a theory (or theories) of "pure gravity" exist? (Only multigravitons and Plankian microstates)

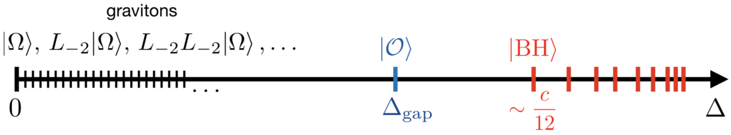
A non-perturbative definition of "pure gravity":
an infinite sequence of CFTs with increasing central charge, such that for $c \to \infty$ the only states with finite Δ are multi-traces of $T_{\mu\nu}$. (Very counterintuitive from QFT perspective.)

Such CFTs would live *at the edge*: prime targets for the bootstrap.

Rapid analytic developments in the bootstrap: we begin to have the right tools.

Question particularly sharp for AdS_3/CFT_2:
multi-graviton states $\sim$ Virasoro module of the identity.

Weakly-coupled AdS bulk $\iff \ell_{\mathrm{AdS}} \gg \ell_{\mathrm{Planck}} \iff c \gg 1$



$\frac{c}{12}$ = BTZ blackhole threshold

Today I will focus on the simplest bootstrap constraint for CFT_2: modular invariance.

Modular bootstrap

Full partition function

$$Z(\tau,\bar\tau) = \sum_{\text{states}} q^{h-c/24}\bar q^{\bar h-\bar c/24}\ , \qquad q = e^{2\pi i\tau}\ , \quad \bar q = e^{-2\pi i\bar\tau}$$

obeys

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar\tau+b}{c\bar\tau+d}\right) = Z(\tau,\bar\tau), \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\ .$$

We specialize to the one-complex-dimensional section $\tau = -\bar\tau = i\beta$,

$$\mathcal{Z}(\tau) = Z(\tau,-\tau) = \sum_{\text{states}} \exp\left[-2\pi\beta\left(\Delta - \frac{c}{12}\right)\right]\ , \qquad \mathcal{Z}(\tau) = \mathcal{Z}(-1/\tau)$$

Angular potential set to zero, discarding spin information.
This includes all rectangular Euclidean tori, for which β is real and positive.

The CFT is *not* chiral, so $\mathcal{Z}$ is not invariant under $\tau \to \tau + 1$ and we cannot use standard results from modular forms.

Simplest bootstrap question:
Given c, what is the largest dimension, Δ_{gap}, of the first primary? Hellerman '09

Write S-invariance as

$$0 = \sum_{\text{primaries}} [\chi_\Delta(\tau) - \chi_\Delta(-1/\tau)] \equiv \sum_{\text{primaries}} \Phi^A_\Delta(\tau)$$

where χ_Δ is the chiral algebra character. Assuming unique vacuum,

$$\Phi^A_{\text{vac}}(\tau) + \sum \mu_\Delta \Phi^A_\Delta(\tau) = 0\,.$$

If $\exists$ linear functional ω such that

$$\omega[\Phi^A_{\text{vac}}] > 0$$
$$\omega[\Phi^A_\Delta] \geqslant 0 \quad \text{for all} \quad \Delta \geqslant \Delta_*\,,$$

then $\Delta_{\text{gap}} < \Delta_*$.
(Inspired by correlator bootstrap *à la* Rattazzi Rychkov Tonni and Vichi).

Bounds on the Virasoro gap at large c

Hellerman '09, Fridan Keller '13, Collier Lin Yin '16, Afkhami-Jeddi Hartman Tajdini '19

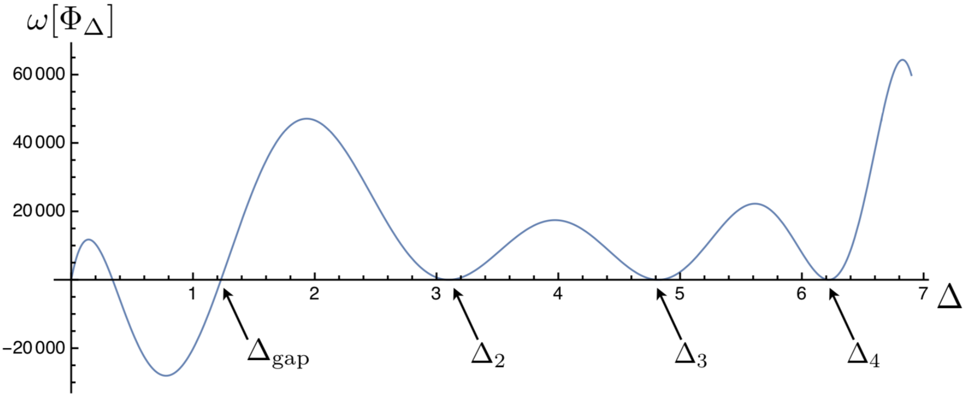
Finding the best bound Virasoro gap $\Delta_V(c)$ is a linear optimization problem in the infinite-dimensional space of functionals.

$$\text{Ansatz}: \quad \omega = \sum_{n=0}^{N} \alpha_n \partial_\tau^{2n+1}|_{\tau=1}, \quad \text{optimize over } \alpha_n$$

- Analytics: $N = 1$, $\Delta_V(c) < \frac{c}{6} + 0.4737$ Hellerman
  No asymptotic improvement for any fixed finite N Friedan Keller

- Numerics indicates that true asymptotic bound is stronger (need $N \to \infty$, then $c \to \infty$).
  $\Delta_V(c) \lesssim \frac{c}{9.08}$: conjectured asymptotics (from $c \lesssim 2000$ numerics) Afkhami-Jeddi et al.

The optimal (aka extremal) functional in non-negative about Δ_{gap} and has zeros at the actual spectrum, in particular it vanishes on the vacuum.



Saturation at $c = 4$, $c = 12$

The numerical bound at $c = 12$ is

$$\Delta_V(12) \leqslant 2 + 10^{-30} \,.$$

Zeros of numerical functional converge to non-negative integers, with single roots at $\Delta = 0, 1, 2$ and double roots beyond. To high accuracy,

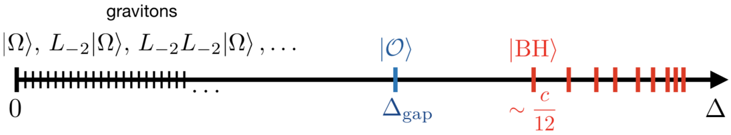
$$\mathcal{Z}_{12}(\tau) = \frac{1}{\eta(\tau)^{24}} \Theta_{\Lambda_{24}}(\tau) - 24 = j(\tau) - 744 \,,$$

which is the partition function of the chiral monster CFT with $c = 24$, $\bar{c} = 0$.
In the present context, this is a complete surprise!

Similarly, for $c = 4$, numerics converge towards $\Delta_V(4) = 1$,
with zeros at nonnegative integers. To high accuracy,

$$\mathcal{Z}_4(\tau) = \frac{1}{\eta(\tau)^8} \Theta_{\Lambda_8}(\tau) = (j(\tau))^{1/3} \;.$$

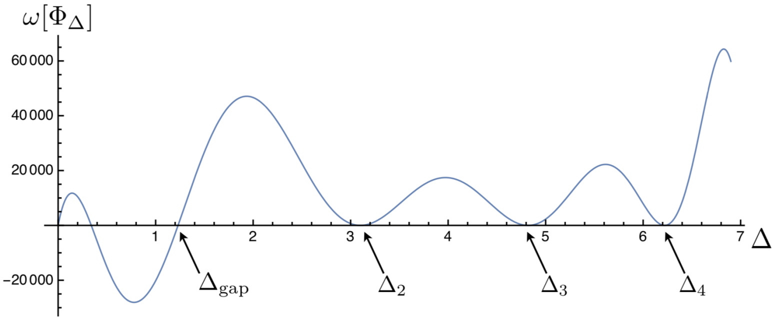
We are going to provide an analytic understanding of the $c = 4$ and $c = 12$ numerics, and improve on the asymptotic analytic bound, $\Delta_V(c) \lesssim \frac{c}{8.503}$,



$\frac{c}{12}$ = BTZ blackhole threshold

In the process, a surprise: connection with sphere packing problem.

The optimal (aka extremal) functional in non-negative about Δ_{gap} and has zeros at the actual spectrum, in particular it vanishes on the vacuum.



Till recently, the only analytic construction of the optimal functional known so far was for the four-point function bootstrap on a line. This will be enough to prove the main results of today.

Optimal Functional for the 1D Correlator Bootstrap

Mazáč, Mazáč Paulos

Consider four identical primaries on a line, $\langle\sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle$. Crossing equation:

$$\sum_{\text{primaries}} f^2_{\sigma\sigma\Delta}\left[G^{(s)}_{\Delta}(z)-G^{(t)}_{\Delta}(z)\right]=0\,,$$

with $G_{\Delta}=\mathfrak{sl}(2,\mathbb{R})$ conformal blocks.

Solution with maximal gap is fermionic mean field theory. Spectrum: $2\Delta_\sigma+1,\ 2\Delta_\sigma+3,\ \ldots$

Theorem: $\Delta_{\text{gap}}=2\Delta_\sigma+1$, i.e. the OPE of two identical primaries always contains a non-identity primary with $\Delta<2\Delta_\sigma+1$.

Proof: construct analytically the optimal functional. Natural ansatz:

$$\omega[G^{(s)}_{\Delta}(z)-G^{(t)}_{\Delta}(z)]=\sin^2\left[\frac{\pi}{2}(\Delta-2\Delta_\sigma-1)\right]\int_0^1 dz Q_{\Delta_\sigma}(z)G^{(s)}_{\Delta}(z)$$

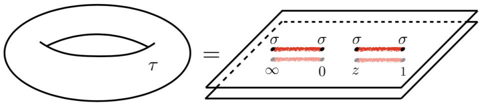
Explicit expression for kernel Q_{Δ_σ} in terms of hypergeometrics.

Back to the Torus with the Pillow Map

A partition function can be viewed as a 4-point function of twist operators

$$\mathcal{Z}_A(\tau) = (2^8 z(1-z))^{c/12} \langle \sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle_{A\times A/\mathbb{Z}_2}$$

$$z = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^4$$



S-transformation of torus $\tau \leftrightarrow -1/\tau$ maps to crossing of four points $z \leftrightarrow 1-z$.

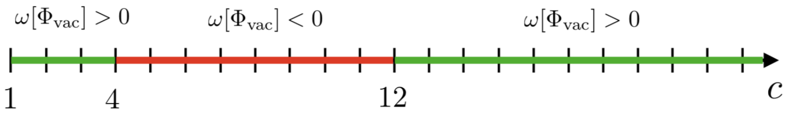
With this map, we can just *recycle* the functionals for the $sl(2,\mathbb{R})$ correlator bootstrap.

Recall that twist operator dimension is $\Delta_\sigma = c/8$.

Naive conclusion:
If we could ignore the difference between the Virasoro character $\chi^V_\Delta(\tau)$ and $sl(2,\mathbb{R})$ block of dimension 2Δ, then

$$\Delta_V(c) = \frac{\Delta_{1D}(c/8)}{2} = \frac{c+4}{8}\,.$$

Too fast, because we need to check action ω on the vacuum.



- For $c \in (4, 12)$, ω is invalid.
- For $c \in (1, 4) \cup (12, \infty)$, ω is valid but suboptimal.
- For $c = 4$ and $c = 12$, ω is optimal!

Optimal functionals at $c = 4$ and $c = 12$:

- $\Delta_V(4) = 1$, spectrum $\Delta = 1, 2, 3, \ldots$ $\quad \mathcal{Z}_4(\tau) = \frac{1}{\eta(\tau)^8}\Theta_{\Lambda_8}(\tau) = (j(\tau))^{1/3}$
- $\Delta_V(12) = 2$, spectrum $\Delta = 2, 3, 4, \ldots$ $\quad \mathcal{Z}_{12}(\tau) = \frac{1}{\eta(\tau)^{24}}\Theta_{\Lambda_{24}}(\tau) - 24 = j(\tau) - 744$

  Numerics

Sphere packing

What is the densest configuration of identical, non-overlapping spheres in $\mathbb{R}^d$?

Deep problem, with connections to number theory, cryptography, etc.

$d = 2$ honeycomb lattice Toth, 1940

$d = 3$: Kepler's conjecture: FCC lattice. Proved by Hales in 1998. Computer-assisted proof.

$d = 8$: solved by Viazovska in 2016, E_8 lattice.

$d = 24$: Leech lattice Cohn Kumar Miller Radchenko Viazovska 2016

A periodic packing is a crystal of spheres, specified by an arbitrary unit cell V, with N spheres repeated by a lattice Λ.
(A lattice packing is the special case where $N = 1$).

The density ρ_d of a packing in $\mathbb{R}^d$ is the filled fraction of the unit cell,

$$\rho_d = \frac{N\mathsf{vol}(B^d)}{2^d|\Lambda|} .$$

For bounding the density, with no loss of generality any packing can be approximated by a periodic one.

Cohn-Elkies approach

Consider first the special case of an isodual lattice (a lattice congruent to its dual).

For an isodual lattice in $\mathbb{R}^d$, consider the partition function

$$\mathcal{Z}(\tau) = \sum_{x\in\Lambda} \chi_{x^2/2}(\tau)\,, \qquad \chi_{x^2/2}(\tau) \equiv \frac{q^{x^2/2}}{\eta(\tau)^d}$$

By Poisson summation,

$$\mathcal{Z}(\tau) = \sum_{x\in\Lambda} \chi_{x^2/2}(\tau) = \sum_{y\in\Lambda^*} \chi_{y^2/2}(-1/\tau) = \mathcal{Z}(-1/\tau)\,.$$

In this special case, direct connection with the modular bootstrap with $U(1)$ chiral algebra. Indeed, the $U(1)^c \times U(1)^c$ characters for $\tau = -\bar{\tau}$ read

$$\chi^U_\Delta(\tau) = \frac{q^\Delta}{\eta(\tau)^{2c}}\,,$$

which coincide characters above if one takes $d = 2c$, $\Delta = x^2/2$.

In this case the linear programming problem has an elegant geometric reformulation.

For a linear functional ω, define

$$\omega[\chi_{x^2/2}(\tau) - \chi_{x^2/2}(-1/\tau)] \equiv g(x)\,.$$

It is easily seen that $g = -\widehat{g}$, where $\hat{g}$ is the Fourier transform of the radial function g in $\mathbb{R}^d$.

Given a Fourier-odd Schwartz function g in $\mathbb{R}^d$ with

$$\begin{aligned} &\text{(i)} \quad g(0) = -\widehat{g}(0) = 0 \\ &\text{(ii)} \quad g(x) \geqslant 0 \quad \text{for all} \quad x \geqslant 2R_*\,, \end{aligned}$$

- the packing density is bounded by $\rho_d \leqslant \text{vol}(B^d) R_*^d$;
- $\Delta_U(c) \leqslant 2R_*^2$, where Δ_U = gap for spinless $U(1)^c \times U(1)^c$ problem, with $d = 2c$.

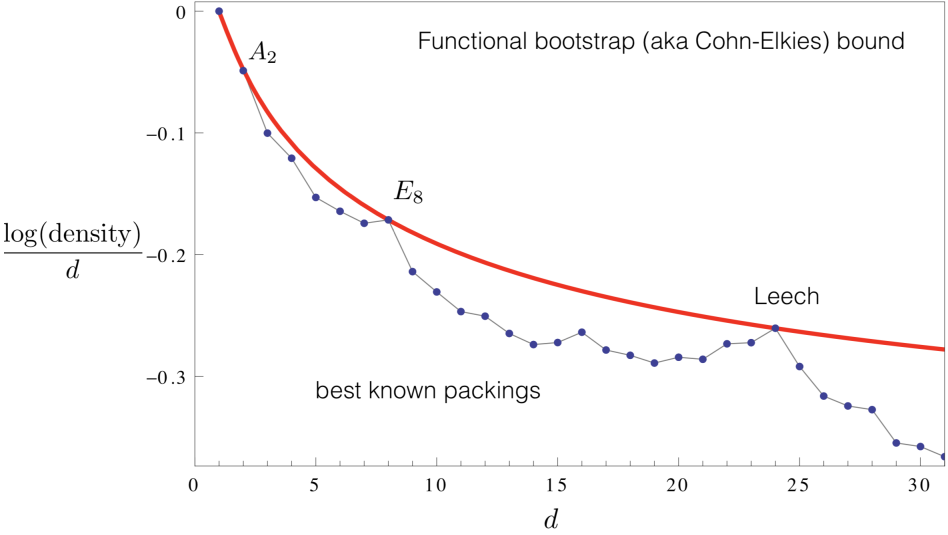
The Cohn-Elkies approach is much in fact general, applying to *any* periodic packing.

In the general case, in addition to g, one needs to also construct a Fourier-*even* radial function h with suitable positivity properties.

It is observed experimentally and checked in all concrete cases that h adds no new information.

Hence, the connection between the Cohn-Eilkies approach to sphere packing and modular bootstrap is completely general.

Cohn-Elkies numerics



Numerical bounds left no real doubt that the E_8 and Leech lattice are the optimal packings in 8 and 24 dimensions.

Theorem [Viazovska 2016]: E_8 is optimal.

Proof: Found the extremal functional ("magic function") analytically.

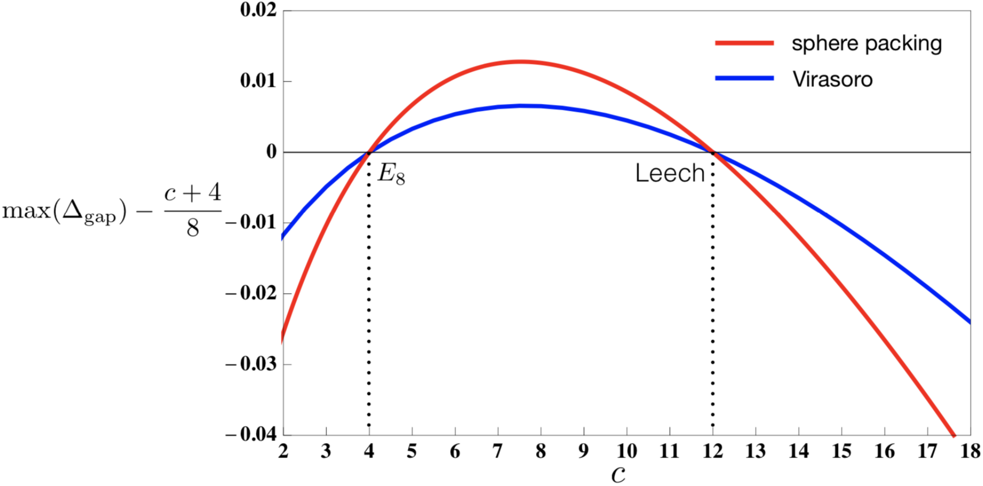
A clever ansatz for g (and h) turns this into a problem of finding quasi-modular forms with certain properties,

$$g(x) = \sin^2(\pi x^2/2) \int_0^{i\infty} d\tau \, e^{i\pi\tau x^2} \, [\text{modular object}]$$

The punchline should be clear.

Viazovska's magic functions *coincide* with Mazáč's analytic functionals for $d = 8$ and $d = 24$.

One just needs to use the pillow map and apply Mazáč's functionals on the $U(1)$ characters.



Upper bounds from linear programming on the gap for $U(1)^c$ ($\equiv$ sphere packing) and Vir_c. $\frac{c+4}{8}$ is the optimal bound in the case of the four-point function bootstrap in 1D.

Summary: asymptotic bounds for spinless modular bootstrap

For $c > 12$, our functional proves the upper bound

$$\Delta_V(c) < \frac{c+4}{8} \ .$$

With some more work, we find an improved (but still sub-optimal) functional that vanishes on the vacuum for large c, leading the asymptotic bounds

$$\Delta_V(c) \lesssim \frac{c}{8.503} \ .$$

Better than Hellerman's bound $c/6$ but weaker than conjectured asymptotics $\Delta_V \sim c/9.08$.

Numerical functionals for the spinless modular bootstrap appear to approach our analytic functionals for fixed c and large Δ, with zeros at $\Delta_k \approx \frac{c-4}{8} + k$.

This does *not* look like the spectrum expected for "pure gravity", which should be non-degenerate and chaotic, with tiny spacing dictated by Cardy formula.

Possible resolutions:

- $Z_{\text{extremal}}(q, \bar{q})$ for the full modular bootstrap will exhibit expected chaotic behavior (and perhaps the $\sim c/12$ gap suggested by BTZ).
  Numerics not encouraging, but convergence may be slow.
- More constraints (such as crossing constraints) needed to reach the *edge* of theory space.
- "Pure gravity" is not a thing. (As we have defined it)
- ...

Outlook: Modular Bootstrap and Sphere Packing

- Spinless modular bootstrap $\Longleftrightarrow$ *linear programming bounds* on sphere packing.
  Both are just *necessary* conditions for their respective problem.
  Is there a deeper connection?
  Further relations between additional constraints on both sides?

- Of great interest to find the best asymptotic bounds for both Δ_U and Δ_V.

  Δ_V constraints the spectrum of black holes in 3D gravity.

  Δ_U constraints the most efficient classical error-correcting codes.

  Black holes are known to saturate many bounds on entropy, chaos, complexity, ...
  Yet another sense in which they live *at the edge*.

Outlook: Bootstrap and Swampland

Many questions for the future.
We now have the tools, e.g. new analytic functionals for the correlator bootstrap.
Mazáč LR Zhou, Carmi Caron-Hout, Paulos, Sleight Taronna

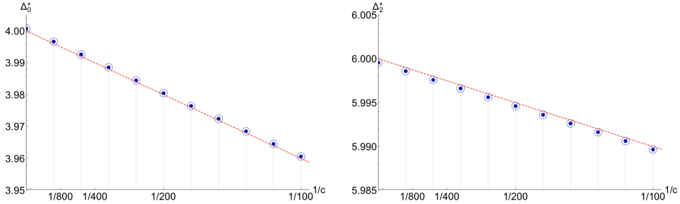
Swampland questions for AdS quantum gravity $\to$ precise questions for the bootstrap

Recent claim by Penedones, Silva, Zhiboedov:
AdS$_{d+1}$ Einstein gravity + one minimally coupled scalar admits *no* UV completion,
if the scalar mass is such that dual operator dimension $\Delta \in \left(\frac{d-2}{2}, \frac{3(d-2))}{4}\right)$

Constraints on $\mathcal{N} = 1$ AdS_4 vacua with large gap?

....

Warm-up problem:
Show that $\mathcal{N} = 8$ AdS$_5$ gauge supergravity does *not* have a UV completion.



Numerical bootstrap bounds saturated by AdS_5 sugra up to order $O(1/c)$.

At order $O(1/c^2)$, KK modes contribute a (negative) anomalous dimension

Analytic (or even numerical) control at $O(1/c^2)$ might rule out pure AdS_5

Mazáč, Perlmutter, LR, in progress