$T\bar{T}$ as a Movie, With and Without 3d Glasses

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based on
1906.05439 with Evan Coleman, Jeremias Aguilera-Damia and Daniel Freedman
1910.06675 with Jeremias Aguilera-Damia, Victor Ivan Giraldo-Rivera, Ignacio Salazar Landea and Edward Mazenc
1912.09179 with Edward Mazenc and Vasudev Shyam

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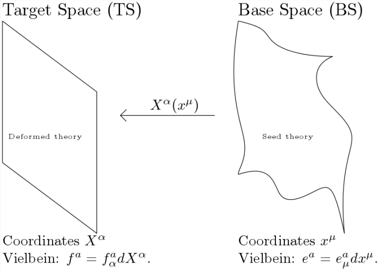
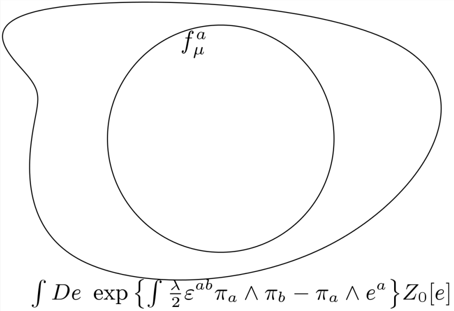
Takeaway


Target Space (TS)
Base Space (BS)
$X^\alpha(x^\mu)$
Deformed theory
Seed theory
Coordinates X^α
Vielbein: $f^a = f^a_\alpha dX^\alpha$.
Coordinates x^μ
Vielbein: $e^a = e^a_\mu dx^\mu$.


f^a_μ
$\int De \ \exp\left\{\int \frac{\lambda}{2}\varepsilon^{ab}\pi_a \wedge \pi_b - \pi_a \wedge e^a\right\} Z_0[e]$

Motivation Slide

$T\bar{T}$ (and related) deformations are cool.

- Universal: exist in every theory with relevant symmetries. $T\bar{T}$ exists in any QFT with a conserved stress tensor.
- New type of deformation: “recursive,” rather than adding a term to the action.
- Solvable!
- Preserve integrability.
- Definition very field-theoretic, but does not always give you a QFT.
- $T\bar{T}$: toy model of quantum gravity.
- $T\bar{T}$: toy version of finite-cutoff AdS/CFT.
- $T\bar{T} + \Lambda_2$: AdS/CFT $\to$ theory bulk dual dS.
- Discovered 5 times.

  Zamolodchikov ‘04 + Smirnov-Zamolodchikov ‘16, Cavaglia-Negro-Szeczenyi-Tateo ‘16, Dubovsky-Flauger-Gorbenko ‘12, Lechner ‘12, Freidel ‘08.

What is $T\bar{T}$?

In 2d flat space,

For any two conserved currents J_A^μ, $A = 1, 2$,

the operator

$$\text{``}J_A\bar{J}_B\text{''} = \varepsilon^{AB}\varepsilon_{\mu\nu}J_A^\mu(x)J_B^\nu(0)$$

has no $x \to 0$ power-law divergences,[1]

and so the limit is easy to define in a theory-independent manner.

Examples:

- $T\bar{T} = T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}T_{\bar{z}z}$
- $J\bar{T} = J_z T_{\bar{z}\bar{z}} - J_{\bar{z}}T_{\bar{z}z}$
- $T\bar{J} = T_{zz}J_{\bar{z}} - T_{z\bar{z}}J_z$

$T\bar{T}$ deformation: a class of theories defined by

$$\partial_\lambda \log Z_\lambda = \int d^2x \langle J_A\bar{J}_B(x)_\lambda\rangle_\lambda$$

[1] Zamolodchikov '04, Smirnov-Zamolodchikov '16.

Solvability

Can find flow of S-matrix and energy levels.

The $T\bar{T}$-deformed S-matrix:

$$S_\lambda(p_i) = S_0(p_i) e^{\frac{i}{4\lambda}\sum_{i<j}\varepsilon_{\mu\nu}p_i^\mu p_j^\nu}.$$

$T\bar{T}$-deformed energy Levels on S^1 of radius r:

$$\begin{aligned} P_n(\lambda, r) &= P_n(0, r) \\ E_n(\lambda, r) &= \frac{r}{\lambda}\left\{1 - \sqrt{1 - \frac{2\lambda E_n(0,r)}{r} + 2\frac{\lambda^2 P_n^2}{r^2}}\right\}. \end{aligned}$$

Similar formulas in other deformations.

Cutoff AdS_3/CFT_2[2]

Write AdS_3 in FG gauge,

$$ds_3^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_2^2.$$

and make the identifications

$$c = \frac{3\ell}{2G_N}, \quad \frac{\lambda}{r^2} \sim \frac{G_N \ell}{\rho_c^2}.$$

(The stress tensor sector of)

a $T\bar{T}$-deformed holographic CFT

is dual to

(the pure GR sector of)

AdS_3 with Dirichlet boundary conditions at $\rho = \rho_c$,

[2] McGough-Mezei-Verlinde '16

Cutoff AdS_3/CFT_2 from Dimensional Analysis

When we deform a CFT,

the dimensionful parameters are $\lambda \sim [L]^2$, $r \sim [L], \epsilon \sim [L]$.

So, dimensional analysis requires

$$(r\partial_r + 2\lambda\partial_\lambda + \epsilon\partial_\epsilon) \log Z_\lambda = 0.$$

(In CFT, last term is conformal anomaly.)

Making assumption that the anomaly is untouched, this becomes

$$\int \sqrt{g} \left(T^\mu_\mu + 2\lambda T\bar{T} - \frac{c}{24\pi} R \right) = 0.$$

After identifications,

This is one of the Einstein equations, $\int E^r_r$.

Before we move on, we need to know what vielbeins are

We're all (I hope) used to describing a geometry by its metric, $g_{\mu\nu}$,

which gives infinitesimal distances, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

Sometimes useful to deal not with the metric but the transformation between real coordinates and a local inertial frame/tangent space:

$$e^a_\mu = \text{``}\frac{\partial y^a}{\partial x^\mu}\text{''}, \quad ds^2 = \text{``}\eta_{ab} dy^a dy^b\text{''} = \eta_{ab} e^a_\mu e^b_\nu dx^\mu dx^\nu.$$

Why quotes? $\because$ ys are only really coordinates for flat manifolds.

Vielbein/frame field[3] $= e^a_\mu$.

Note: rotations of the tangent space index a are a redundancy if you only care about the metric.

[3] Do yourself a favour and refuse to call these by any other name.

The Spin Connection

For covariant derivatives of object with both spacetime (μ, ν) and tangent space indices (a, b), we use the spin connection,

$$\nabla_\mu A_{\nu a}{}^b = \partial_\mu A_{\nu a}{}^b - \Gamma^\rho_{\mu\nu} A_{\rho a}{}^b + \omega_\mu{}^b{}_c A_{\nu a}{}^c - \omega_\mu{}^c{}_a A_{\nu c}{}^b, \quad \omega_{\mu ab} = -\omega_{\mu ba}. \tag{1}$$

In 2d, the only anti-symmetric two-tensor is ε_{ab}, and so

$\omega_{\mu ab} \equiv \omega_\mu \varepsilon_{ab}$.

As you might expect, $\nabla_\mu e^a_\nu = 0$.

Of interest: the antisymmetric part of this ("torsionlessness") is just

$$de^a + \varepsilon^a{}_b \omega \wedge e^b = 0, \quad \omega = (*de_a)\, e^a. \tag{2}$$

Finally,

$$R = *d\omega, \quad d\omega = \frac{1}{2} R\epsilon \equiv \frac{1}{2} R\sqrt{g}\varepsilon_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{3}$$

$T\bar{T}$CU part 2: From $T\bar{T}$ to a Fluctuating Vielbein[4]

Define the stress tensor,

$$\langle T_a^\mu \rangle \equiv -\frac{1}{\det e}\delta_{e_\mu^a} \log Z. \tag{4}$$

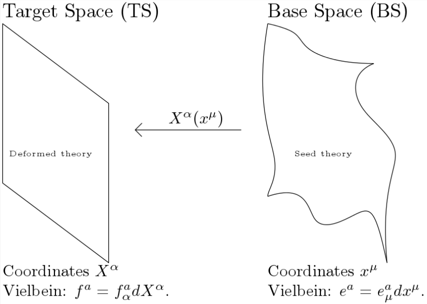
An infinitesimal amount of $T\bar{T}$ deformation is

$$\begin{aligned} Z_{\delta\lambda}[f] &= e^{-\frac{\delta\lambda}{2}\int \varepsilon_{\mu\nu}\varepsilon^{ab}\delta_{f_\mu^a}\delta_{f_\nu^b}} Z_0[f] \\ &\xrightarrow{\text{Hubbard-Stratonovich}} \int D\delta e \ e^{-\frac{1}{2\delta\lambda}\int \varepsilon_{ab}\delta e^a\wedge\delta e^b} Z_0[f-\delta e] \end{aligned} \tag{5}$$

How to exponentiate?

[4] Cardy 1801.06895.

The $T\bar{T}$ deformed theory can be exactly written down as a quantum gravity path integral.



$$df^a = \omega = 0 \text{ (flat TS)} \quad \Rightarrow$$

$$Z_\lambda[f] = \int \frac{DeDY}{\text{vol(diff)}} e^{-\frac{1}{2\lambda}\int \varepsilon_{ab}(dX-e)^a \wedge (dX-e)^b} Z_0[e],$$

$$dX^a \equiv f^a + dY^a = X^* f. \tag{6}$$

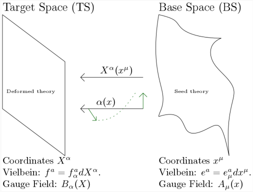
Virtues of the DGH-C Kernel

1. S-matrix:[5] The dressing comes from the coordinate transformation between the two spaces; scattering is happening on the BS, but clocks and rods are on the TS.
   (Kernel reduces to JT gravity in $\mathbb{R}^2$.)
2. Partition Function:[6] The new energy eigenstates are the old energy eigenstes on the BS.
3. Classical Actions:[7] The deformed classical action can be found by setting the gravitational variables to their saddle-point values.

[5] Dubovsky-Gorbenko-Mirbabayi '17.
[6] Dubovsky–Gorbenko–Hernandez-Chifflet '18
[7] Conti–Negro–Tateo '18, Coleman–Aguilera-Damia–Freedman–RMS '19

$T\bar{T}$CU Part 4: A Kernel for $J\bar{T} + T\bar{\bar{J}} + T\bar{T}$ Deformations[8]

Also introduce a dynamical relative "$U(1)$ frame" between the manifolds,



$$df^a = dB = 0 \Rightarrow$$

$$Z_{\ell_1,\ell_2,\lambda}[f, B, \tilde{B}] = \int \frac{D[e, Y, A, \alpha, \tilde{A}, \tilde{\alpha}]}{\mathrm{vol}(\mathrm{diff} \times U(1) \times \tilde{U(1)})} e^{-S_K} Z_0[e, A, \tilde{A}]$$

$$S_K = \frac{1}{\ell_1} \int \tilde{n}_a (f + dY - e)^a \wedge (B + d\alpha - A) + \frac{1}{\ell_2} \int n_a (f + dY - e)^a \wedge (\tilde{B} + d\tilde{\alpha} - \tilde{A}) - \frac{\lambda}{\ell_1 \ell_2} \int (B + d\alpha - A) \wedge (\tilde{B} + d\tilde{\alpha} - \tilde{A})$$

[8] A-DG-RMLS '19, Anous-Guica '19.

Related deformation that is related to bulk dS_3, $Mink_3$:[9]

$$\partial_\lambda \log Z_\lambda = \int \langle T\bar{T} \rangle - \frac{c}{\lambda^2}$$

The torus partition function is[10]

$$Z_\lambda[r,\tau] = \int \frac{DYDe}{\text{vol(diff)}} e^{-\frac{1}{2\lambda}\int \varepsilon_{ab}(dX-e)^a \wedge (dX-e)^b + \frac{\tilde{c}}{2\lambda}\frac{r^2}{r_{BS}^2}\int e^1 \wedge e^2} Z_0[e]$$

Here, $\tilde{c} \propto c$, but have the same sign.

[9] Gorbenko-Silverstein-Torroba '18.
[10] Mazenc-Silverstein-RMS, unpublished.

The main input of flatness into DGH-C kernel is that

the coordinate transformed vielbein $X^* f^a$ can be written as $f^a + dY^a$.

Reason: f^a has no spatial dependence.

Let's gauge-fix this coordinate transformation to trivial, obtaining

$$Z_\lambda[f] = \int De\ e^{-\frac{1}{2\lambda}\int \varepsilon_{ab}(f-e)^a \wedge (f-e)^b} Z_0[e].$$

This satisfies the equation[11]

$$\partial_\lambda \log Z_\lambda = \int d^2x \sqrt{g} \langle T\bar{T}(x) \rangle,$$

where the $T\bar{T}$ operator is stupidly defined as a coincident derivative

$$\langle T\bar{T} \rangle_\lambda = \frac{1}{Z_\lambda} \left\{ \epsilon_{\mu\nu} \varepsilon^{ab} \frac{1}{(\det f)^2} \frac{\delta}{\delta f^a_\mu(x)} \frac{\delta}{\delta f^b_\nu(x)} Z_\lambda - \frac{1}{\det f} \left(\frac{\delta f^a_\mu(x)}{\delta f^a_\mu(x)} \right) Z_\lambda \right\}$$

[11] Tolley '19, Mazenc-Shyam-Soni '19.

$T\bar{T}$CU Part 1: The 3d Glasses[12]

Turns out that this kernel also solves a local equation,

$$\langle T^\mu_\mu(x)\rangle_\lambda + 2\lambda\langle T\bar{T}(x)\rangle_\lambda - \frac{c-24}{24\pi}R(x)$$
$$= \int De\ e^{-S_K} Z_0[e](\dots)\left\{\langle T^\mu_{0\mu}(x)\rangle_0 - \frac{c}{24\pi}R[e]\right\},$$

for the *same* $T\bar{T}$ operator.

RHS vanishes when seed is CFT.

Note similarity to 3d Einstein eqn from earlier:

this *quantum* equation is the Wheeler-de Witt equation of 3d GR,

a *constraint* that is satisfied by any GR path integral with a boundary.

[12]Freidel '08, see also Verlinde '89.

A Clarification

This is *not* a duality.

In general the deformation is *building* a “fake” 3d bulk.

When seed has a bulk dual,

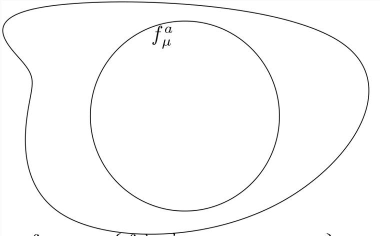
this can be thought of as flowing into the dual bulk

as long as you ignore bulk matter.[13]

[13] See Hartman-Kruthoff-Shaghoulian-Tajdini ‘18 for inclusion of matter fields.

Putting on the 3d Glasses

The picture for general CFTs is that the deformed partition function is the following 3d GR path integral.[14]



$$\int De\ \exp\left\{\int \frac{\lambda}{2}\varepsilon^{ab}\pi_a \wedge \pi_b - \pi_a \wedge e^a\right\} Z_0[e]$$

Note the state at the outer boundary is not quite the CFT partition function; in holographic limit, this convolution transforms to the known mixed boundary conditions.[15]

[14] Mazenc-Shyam-Soni '19.
[15] Guica-Monten '19.

A Classical Limit[17]

The gravitational path integral has a classical limit when

$c \to \infty, \lambda c$ finite.

Taking TS vielbein f to be a vielbein for S^2 of radius r,

classical solution of BS vielbein e is S^2 of radius

$$r_{BS} = \frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{c\lambda}{24\pi}}.$$

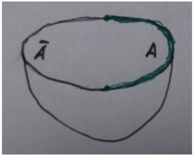
Plugging it back in reproduces holographic answer[16] for S^2 partition function.

[16] Donnelly-Shyam '18.
[17] Mazenc-Shyam-Soni '19.

Known Results: $T\bar{T}$ Smooths out Entropies

Let's take the dS ground state of a deformed holographic CFT

and think about the entanglement of half the S^1.[18]



Example of smoothing out:

On the n-sheeted manifold for an interval in $\mathbb{R}^2$, the stress tensor near the conical singularity behaves as[19]

$$T(z) \sim \delta n \frac{1}{z^2} \frac{1}{\sqrt{1+\delta n \frac{\lambda}{|z|^2}}} \xrightarrow{z \to 0} \frac{|z|}{\lambda z^2}, \quad \delta n \equiv n-1 \ll 1.$$

[18] Donnelly-Shyam '18.
[19] Lewkowycz-Liu-Silverstein-Torroba '19.

A New Perspective

Look at the classical base space

corresponding to a smoothed out *n*-replica manifold.

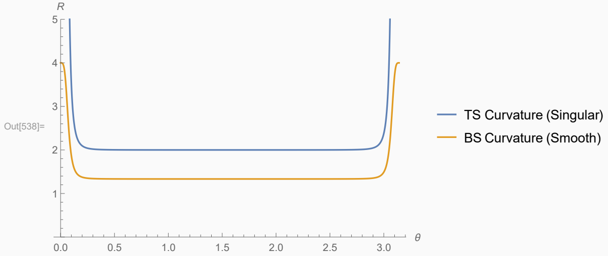


Figure 1: $\lambda c = 12\pi$, $\delta = .01$, $n = .5$

so algebraic smoothing of stress tensor

is

the geometric smoothing of base space.

Conclusions

$T\bar{T}$ (and related) deformations can naturally be thought of as integrating over related gauge fields with topological Gaussian kernels.

Perspective allowed us to

1. Generalise DGH-C kernel to other deformations,
2. Move beyond flat space.

Further, 3d glasses told us that the deformed theory satisfies a *local* equation in which $T\bar{T}$ behaves like a relevant deformation!

Future Directions

1. Can we use WdW equation to find the finite-c deformed S^2 partition function?
2. Polyakov: Flat space $T\bar{T}$ deformations of CFTs can also be related to usual Polyakov action.[20]
   Needs to be understood, because Polyakov naively doesn't allow arbitrary target space.
   Classical calculation in Tolley '19, but quantum calculations need to be done.
3. $T\bar{T} + \Lambda_2$ deformations.
4. Algebraic non-locality of deformed theory should be geometric non-locality of BS,
   like in Renyi entropy case.
   (Ongoing conversations with many people.)
5. "Lorentzian" theory, in Harlow's sense.

[20] Dubovsky-Gorbenko-Mirbabayi '17, Callebaut-Kruthoff-Verlinde '19.