Operator Product Expansion

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based on joint work with J. Holland and Ch. Kopper

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Introduction

Operator Product Expansion [Wilson '69]

Products of composite fields can be expanded as

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \underbrace{\ldots}_{\text{Spectators}} \rangle \sim \sum_B \underbrace{\mathcal{C}^B_{A_1 \ldots A_N}(x_1, \ldots, x_N)}_{\text{OPE coefficients}} \langle \mathcal{O}_B(x_N) \ldots \rangle$$

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- Asymptotic short distance expansion:
  Difference vanishes in the limit $x_i \to x_N$ for all $i \leq N$

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- Asymptotic short distance expansion:
  Difference vanishes in the limit $x_i \to x_N$ for all $i \leq N$
- Practical application e.g. in deep-inelastic scattering
- Plays fundamental role in conformal field theory
  (Conformal bootstrap, "Vertex operator algebras", ...)
- Plays fundamental role in QFTCST
  (State-independent definition of QFT!)

Topics of today's talk:

1. In what sense does the OPE converge? N-point functions $\leftrightarrow$ 1-point functions & OPE coefficients
2. What are algebraic relations between OPE coefficients? Vertex algebras in d-dims.
3. A novel recursion scheme for OPE coefficients New self-consistent construction method

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   New self-consistent construction method

Model: Perturbative, Euclidean φ_4^4-theory

The model: Euclidean φ^4-theory

- Correlation functions are defined via the path integral

$$\langle \mathcal{O}_{A_1}(x_1)\dots\mathcal{O}_{A_N}(x_N)\rangle := \mathcal{N}\int \mathcal{D}\varphi\, \exp\left[-S\right]\, \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N)\,,$$

where the action is given by

$$S(\varphi) := \int \mathrm{d}^4x\left(\frac{1}{2}(\partial_\mu\varphi)^2(x) + \frac{m^2}{2}\varphi^2(x) + g\varphi(x)^4 - \text{counterterms}\right)$$

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- We use a "renormalization group flow equation" approach [Wilson, Polchinski, Kopper-Keller-Salmhofer]

Outline

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The OPE factorises

Theorem (Holland-SH)

At any arbitrary but fixed loop order:

$$\mathcal{C}_{A_1\dots A_N}^{B}(x_1,\dots,x_N)=\sum_C \mathcal{C}_{A_1\dots A_M}^{C}(x_1,\dots,x_M)\mathcal{C}_{C A_{M+1}\dots A_N}^{B}(x_M,\dots,x_N)$$

holds on the domain $\frac{\max_{1\le i\le M}|x_i-x_M|}{\min_{M<j\le N}|x_j-x_M|}<1$. *(Sum over C absolutely convergent !)*

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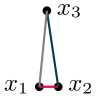
For $N=3$: $\varepsilon = \dfrac{|x_1-x_2|}{|x_2-x_3|} < 1$



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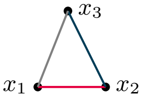

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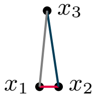

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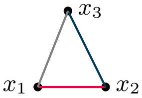

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This shows associativity really holds!

- Vertex Algebras (Borcherds property) also in $4d$.
- $\mathcal{C}^{B}_{A_1\dots A_N}$ uniquely determined in terms of $\mathcal{C}^{B}_{A_1A_2}$
- "Bootstrap construction" of OPE coefficients possible

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Bound on OPE remainder I

Theorem (Holland-Kopper-SH)

At any perturbation order r and for any $D \in \mathbb{N}$,

$$\overbrace{\left|\left\langle \Big(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\le D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\Big) \underbrace{\hat{\varphi}(p_1)\cdots\hat{\varphi}(p_n)}_{\text{Spectator fields}} \right\rangle\right|}^{\text{OPE-Remainder}}$$

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$$\leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(KM \max_{1\leq i\leq N}|x_i - x_N|\right)^{D+1}}{\min_{1\leq i<j\leq N}|x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

- $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale

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$$\leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(K M \max_{1 \leq i \leq N} |x_i - x_N| \right)^{D+1}}{\min_{1 \leq i < j \leq N} |x_i - x_j|^{\sum_i \dim[A_i] + 1}} \cdot \sup \left(1, \frac{|P|}{\sup(m, \kappa)} \right)^{(D+2)(r+5)}$$

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- $|P| = \sup_i |p_i|$: maximal momentum of spectators
- $\kappa := \inf(\mu,\varepsilon)$, where $\varepsilon = \min_{I\subset\{1,\dots,n\}} |\sum_I p_i|$
  ε: distance of $(p_1,\dots,p_n)$ to “exceptional” configurations

Conclusions from bound on OPE remainder

$$\text{“OPE remainder”} \leq \frac{M^{n-1}}{\sqrt{D!}} \; \frac{\left(KM \max\limits_{1\leq i\leq N} |x_i - x_N|\right)^{D+1}}{\min\limits_{1\leq i<j\leq N} |x_i - x_j|^{\sum_i \dim[A_i]+1}} \cdot \sup\left(1, \frac{|P|}{\sup(m,\kappa)}\right)^{(D+2)(r+5)}$$

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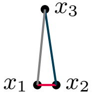
   - ratio of max. and min. distances is large, e.g. for $N = 3$



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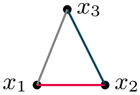

Fast convergence

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At any perturbation order r *and for any* $D \in \mathbb{N}$*, there exists a* $K > 0$ *such that*

$$\left|\left\langle\left(\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_N}(x_N) - \sum_{\dim[B]\le D} \mathcal{C}^B_{A_1\ldots A_N}(x_1,\ldots,x_N)\,\mathcal{O}_B(x_N)\right)\varphi(f_1)\cdots\varphi(f_n)\right\rangle\right|$$

$$\le \frac{M^{n-1}}{\sqrt{D!}}\,\frac{\left(KM\max\limits_{1\le i\le N}|x_i - x_N|\right)^{D+1}}{\min\limits_{1\le i<j\le N}|x_i - x_j|^{\sum_i \dim[A_i]+1}}\ \sup\left(1, \frac{|P|}{M}\right)^{(D+2)(r+5)}$$

M: mass for $m > 0$ or renormalization scale μ for massless fields
$\|\hat{f}\|_s := \sup_{p\in\mathbb{R}^4}|(p^2 + M^2)^s\hat{f}(p)|$ (Schwartz norm)

1. Bound is finite for any $f_i \in \mathcal{S}(\mathbb{R}^4)$ (*Schwartz space*)
   OPE remainder is a tempered distribution
2. Let $\hat{f}_i(p) = 0$ for $|p| > |P|$: Bound vanishes as $D \to \infty$
   $\Rightarrow$ OPE converges at any finite distances!

Outline

Motivation for a new construction method

Textbook method (roughly):

- Write down correlation function with operator insertions
- Perform short distance/large momentum expansion (in some clever way)
- Argue that the coefficients obtained this way are state independent

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- Argue that the coefficients obtained this way are state independent

Not entirely satisfying:

- Relies on correlation functions $\Rightarrow$ OPE not 'fundamental'
- State independence not obvious
- Hard to study general properties of OPE

Recursion formula (for mass $m > 0$)

Theorem (Hollands-JH)

Coupling constant derivatives of OPE coefficients in $g\varphi^4$-theory can be expressed as

$$\partial_g \, \mathcal{C}^B_{A_1 \dots A_N}(x_1, \dots, x_N) = -\int \mathrm{d}^4 y \Big[\mathcal{C}^B_{\varphi^4 A_1 \dots A_N}(y, x_1, \dots, x_N) - \sum_{i=1}^{N} \sum_{[C] \leq [A_i]} \mathcal{C}^C_{\varphi^4 A_i}(y, x_i) \, \mathcal{C}^B_{A_1 \dots \widehat{A_i} \, C \dots A_N}(x_1, \dots, x_N) - \sum_{[C] < [B]} \mathcal{C}^C_{A_1 \dots A_N}(x_1, \dots, x_N) \, \mathcal{C}^B_{\varphi^4 C}(y, x_N) \Big] .$$

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& - \sum_{[C] < [B]} \mathcal{C}^{C}_{A_1 \dots A_N}(x_1, \dots, x_N) \, \mathcal{C}^{B}_{\varphi^4 C}(y, x_N) \Big] .
\end{aligned}
$$

- Compute OPE coefficients to any perturbation order by iteration.
  Initial data: Coefficients of free theory.

Recursion formula (for mass $m > 0$)

Theorem (Hollands-JH)

OPE coefficients at perturbation order $(r+1)$ can be expressed as

$$
\begin{aligned}
(\mathcal{C}_{r+1})^{B}_{A_1 \dots A_N}(x_1, \dots, x_N) = -\int \mathrm{d}^4 y \Bigg[& (\mathcal{C}_r)^{B}_{\varphi^4 A_1 \dots A_N}(y, x_1, \dots, x_N) \\
& - \sum_{s=0}^{r} \sum_{i=1}^{N} \sum_{[C] \le [A_i]} (\mathcal{C}_s)^{C}_{\varphi^4 A_i}(y, x_i)\, (\mathcal{C}_{r-s})^{B}_{A_1 \dots \widehat{A_i}\, C \dots A_N}(x_1, \dots, x_N) \\
& - \sum_{s=0}^{r} \sum_{[C] < [B]} (\mathcal{C}_s)^{C}_{A_1 \dots A_N}(x_1, \dots, x_N)\, (\mathcal{C}_{r-s})^{B}_{\varphi^4 C}(y, x_N) \Bigg] .
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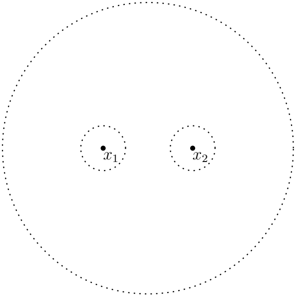
\end{aligned}
$$

- Compute OPE coefficients to any perturbation order by iteration.
  Initial data: Coefficients of free theory.
- State independence obvious.
  No other objects enter the construction.
- The formula depends on the renormalisation conditions.
  (Here BPHZ)

Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\mathcal{C}_{\varphi^4 A_1 A_2}^{B}(y, x_1, x_2) - \sum_{[C] \le [A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C] \le [A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C] < [B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$



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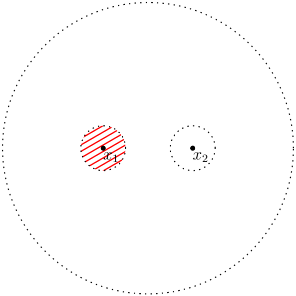
UV-region I ($y \approx x_1$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises



Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\sum_{[C]=0}^{\infty} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C] \leq [A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

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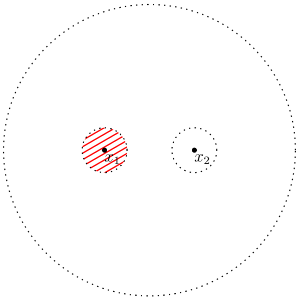

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$$\int \mathrm{d}^4 y \Big[\sum_{[C]>[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

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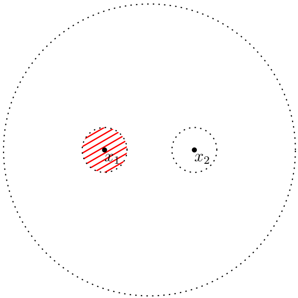
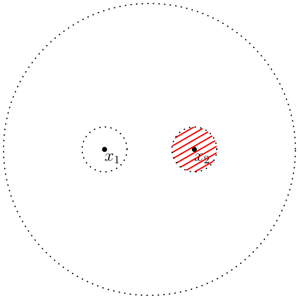


Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\mathcal{C}_{\varphi^4 A_1 A_2}^{B}(y, x_1, x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

UV-region II ($y \approx x_2$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises



Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\sum_{[C]>[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]<[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2) \Big]$$

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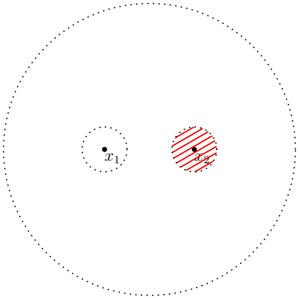
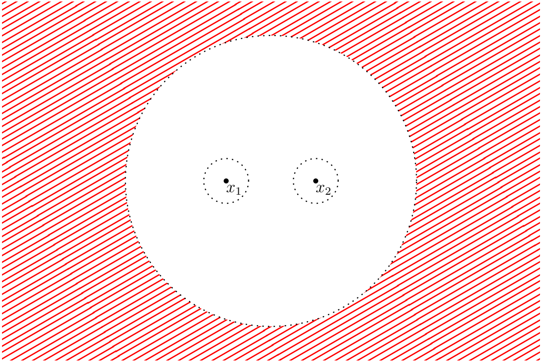


Built-in renormalisation (Example: $N=2$)

$$\int \mathrm{d}^4 y \Big[\sum_{[C]\geq[B]} \mathcal{C}_{A_1A_2}^{C}(x_1,x_2)\,\mathcal{C}_{\varphi^4 C}^{B}(y,x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y,x_1)\,\mathcal{C}_{CA_2}^{B}(x_1,x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y,x_2)\,\mathcal{C}_{A_1 C}^{B}(x_1,x_2) \Big]$$

IR-region ($|y-x_2| \gg |x_1-x_2|$): $\mathcal{C}_{\varphi^4 A_1 A_2}^{B}$ factorises



Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\sum_{[C]\geq[B]} \mathcal{C}_{A_1 A_2}^{C}(x_1, x_2)\, \mathcal{C}_{\varphi^4 C}^{B}(y, x_2)$$

$$- \sum_{[C]\leq[A_1]} \mathcal{C}_{\varphi^4 A_1}^{C}(y, x_1)\, \mathcal{C}_{C A_2}^{B}(x_1, x_2) - \sum_{[C]\leq[A_2]} \mathcal{C}_{\varphi^4 A_2}^{C}(y, x_2)\, \mathcal{C}_{A_1 C}^{B}(x_1, x_2) \Big]$$

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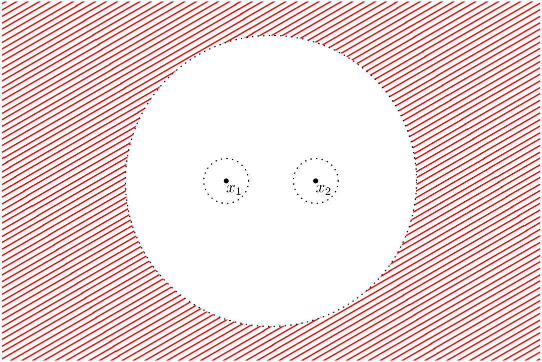

Built-in renormalisation (Example: $N = 2$)

$$\int \mathrm{d}^4 y \Big[\mathcal{C}^{B}_{\varphi^4 A_1 A_2}(y, x_1, x_2) - \sum_{[C]\leq[A_1]} \mathcal{C}^{C}_{\varphi^4 A_1}(y, x_1)\, \mathcal{C}^{B}_{C A_2}(x_1, x_2)$$

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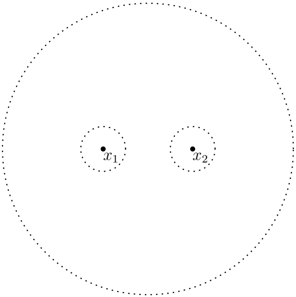
The integral is absolutely convergent due to the factorisation property.



Conclusions & Outlook

In Euclidean perturbation theory, we found that:

1. The OPE converges at finite distances.
2. The OPE factorises (associativity).
3. The OPE satisfies a recursion formula.

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Possible Generalisations

- Gauge theories (in progress)
- Curved manifolds
- Minkowski space
- ...

Applications of the Recursion Formula

- Does the algorithm facilitate computations?
- Does the perturbation series for OPE coefficients converge?