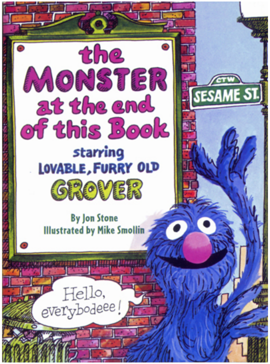
String Universality & 3D Quantum Gravity



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Belin, Keller & A. M.
Belin, Keller & A. M. (to appear)
P. de Lange, A. M. & E. Verlinde (to appear)
A. M. & S. Ross (to appear)
S. Kachru, A. M. & G. Moore (might possibly appear someday, maybe)

The Goal:

Question: Which classical theories of gravity can be quantized?

Strategy: Every CFT_2 can be interpreted as theory of quantum gravity in AdS_3.

The AdS/CFT dictionary:

AdS_3	CFT_2
$\frac{3\ell}{2G} \sim \frac{1}{\hbar}$	c
$E\ \ell$	$\Delta = h + \bar{h}$
Black Holes ($E \geq 8G^{-1}$)	$\Delta \gtrsim \frac{c}{12}$
Perturbative States	$0 \leq \Delta \lesssim \frac{c}{12}$

Bulk Approach:

We could try to quantize AdS_3 gravity+matter directly:

- Question: Does $Z_{T^2}(q,\bar{q}) = \mathrm{Tr}_{\mathcal{H}} q^{L_0} \bar{q}^{\bar{L}_0}$ for some nice $\mathcal{H}$?
- Answer: Only for sporadic values of $c \sim \mathcal{O}(1)$.

In these cases $Z_{T^2}(q)$ matches known theories (minimal models, Monster CFT , . . .).

Perhaps "pure gravity" exists only for discrete values of $c \sim \mathcal{O}(1)$.

New Result:

- To correctly reproduce $Z(\mathbb{RP}^2)$ and $Z(\textit{Klein Bottle})$ we must include non-smooth saddles in the GR path integral.

Lesson: Even in the simplest cases, new particle-like degrees of freedom must be included to reproduce CFT results.

A. M. & S. Ross (to appear)

CFT Approach

We want to understand semi-classical gravity, so consider a family of CFTs, labelled by N, with a $c \to \infty$ limit.

Let $\rho_N(\Delta) =$ number of states in theory N with energy Δ.

For $\Delta \gg c$ we have Cardy growth $\rho_N(\Delta) \sim e^{\sqrt{c\Delta}}$.

Let's characterize the semi-classical theory by $\rho_N(\Delta)$ with $\Delta \ll c$:

$$\log \rho_N(\Delta) \sim \begin{cases} (\Delta/\Delta_0)^{1/2} & \implies \quad \text{general relativity} \\ (\Delta/\Delta_0)^{2/3} & \implies \quad \mathrm{QFT}_3 \\ (\Delta/\Delta_0)^{(d-1)/d} & \implies \quad \mathrm{QFT_d} \text{ on } AdS_3 \times M_{d-3} \\ \Delta/\Delta_0 & \implies \quad \text{String Theory} \end{cases}$$

String Universality

Idea: We must impose constraints on the CFTs in order to have a well defined $c \to \infty$ limit. What are the implications for $\rho_N(\Delta)$?

A basic requirement is that the perturbative density of states

$$\rho_\infty(\Delta) = \lim_{N \to \infty} \rho_N(\Delta) < \infty$$

exists and is finite.

For a large (but simple) family of CFTs, this is enough to imply a Hagedorn density $\rho_\infty(\Delta) \sim e^{\Delta/\Delta_0}$.

Is this a general lesson? Maybe:

- Theories with a finite number of local degrees of freedom exist only for sporadic, $\mathcal{O}(1)$ values of the couplings.
- Every theory of quantum gravity with a semi-classical limit is a theory of extended objects.

Orbifolds

How can we construct generic large c CFTs?

Take a seed CFT $\mathcal{C}$ with $c = c_o$ and

$$Z(q) = 1 + \rho(1)q + \rho(2)q^2 + \ldots$$

The tensor product CFT $\mathcal{C}^{\otimes N}$ with $c = c_o N$ has too many states:

$$Z_N = 1 + N\rho(1)q + \left(N\rho(2) + \frac{N(N-1)}{2}\rho(1)^2\right) q^2 + \ldots$$

We must project onto the singlet sector of some $G_N \subseteq S_N$.

- This is why holographic CFTs are gauge theories.

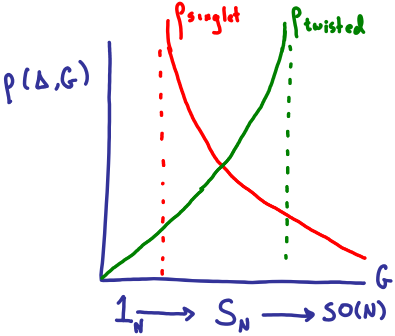
For CFT_2, this means we orbifold by some $G_N \subseteq S_N$.

- Landscape of CFTs labelled by permutation groups G_N.

Twisted Sector States

This introduces new states:

- We need G_N to be large to keep $\rho_N^{singlet}(\Delta)$ finite as $N \to \infty$.
- Larger $G_N \implies$ more twisted states.



Result: $\rho_\infty^{singlet}(\Delta) < \infty \implies \rho_\infty^{twisted}(\Delta) \gtrsim e^{2\pi\Delta}$

Correlation Functions

The correlation functions will also factorize

$$\langle O_1 O_2 O_3 O_4 \rangle = \sum_{contractions} \langle O_i O_j \rangle \langle O_k O_l \rangle + \mathcal{O}(N^{-1/2})$$

provided $G_N \subseteq S_N$ acts "democratically" in the sense that all orbits of G_N become large:

$$\left| \mathrm{Orbit\ on\ k-tuples} \right| \sim \binom{N}{k}$$

as $N \to \infty$.

Correlation functions look like generalized free fields as $N \to \infty$.

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Idea of Proof

A $g \in G_N$ has a cycle decomposition

$$g = (\mathbf{1})^{j_1} \dots (\mathbf{N})^{j_N}, \qquad \sum_{i=1}^{N} i\, j_i = N\ .$$

The untwisted partition function is determined by the statistics of cycle decompositions:

$$Z^u_{G_N} = \langle Z(\tau)^{j_1} \dots Z(N\tau)^{j_N} \rangle$$

where $\langle O \rangle = \frac{1}{|G_N|} \sum_{g \in G_N} O(g)$ is the average over G_N. Expanding to order q^i we find

$$\rho_N(i) = \rho(1)\langle j_i \rangle + \dots < \infty \implies \langle j_i \rangle < \infty$$

So there are cycles of arbitrarily long length.

Hagedorn Density

The i-twisted sector looks like a CFT with $c_{eff} = ic_o$, so

$$\rho^{i-twisted}(\Delta) \sim e^{2\pi\sqrt{\frac{ic_o}{3}(\Delta - \frac{ic_o}{12})}}$$

This is maximized when $\Delta = \frac{ic_o}{6}$.

The total density of states is found by summing over twist sectors:

$$\rho_N(\Delta) \sim \begin{cases} \exp\left(2\pi\Delta\right) & \Delta < \frac{c}{12} \\ \exp\left(2\pi\sqrt{\frac{c}{3}\Delta}\right) & \Delta > \frac{c}{6} \end{cases}$$

The transition between Hagedorn and Cardy regimes is the Hawking-Page phase transition:

- Canonical: at temperature $T = \frac{1}{2\pi}$.
- Microcanonical: in the "enigmatic" regime $\frac{c}{12} < \Delta < \frac{c}{6}$.

Example: S_N Orbifold

The grand canonical partition function

$$\mathcal{Z}(p,q) = \sum_N p^N Z_N(q) = \prod_{n\geq 1, m\geq -1} (1 - p^n q^m)^{-\rho(nm)}$$

is invariant under $\tau \to -1/\tau$, where $q = e^{2\pi i \tau}$

$$\implies \rho_N(\Delta) \sim e^{\sqrt{c(\Delta - c)}} \text{ for } c \text{ fixed and } \Delta - c \text{ large.}$$

It is (almost) invariant under $\mu \to -1/\mu$, where $p = e^{2\pi i \mu}$

$$\implies \rho_N(\Delta) \sim e^{\sqrt{c(\Delta - c)}} \text{ for } c \text{ large and } \Delta - c \text{ fixed.}$$

Lesson: The second ($\mu \to -1/\mu$) Cardy formula is the semi-classical Bekenstein-Hawking entropy, not the first.

Question: What is AdS/CFT in the grand canonical ensemble?

Future Directions

Question: What fraction of CFT_2's are holographic?

Consider lattice theories (N free scalars on $\mathbb{R}^N/\Lambda$):

$$Z = q^{-N}\left(\prod_{n=1}^{\infty}\frac{1}{1-q^n}\right)^N \Theta_\Lambda(q)$$

and their orbifolds by $G \subseteq Aut(\Lambda)$ (like the Monster CFT ).

- To keep $\rho_\infty(\Delta)$ finite we need $|G| \to \infty$ as $N \to \infty$.

The number of lattices grows like $N^{N^2/4}$, but (Bannai)

$$Prob(Aut(\Lambda) \geq \mathbb{Z}_2) \sim N^{-N}$$

using a measure on the space of lattices where Λ is an orbifold point of measure $1/Aut(\Lambda)$.

Lesson: Holographic theories are rare.

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