Operator growth, Toda chain flow, and chaos

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Outline

- Euclidean operator growth as a probe of chaos
  arXiv:1911.09672, with Alexander Avdoshkin

  - generic non-integrable system: time-correlation function continued to imaginary time develops a singularity

- time-correlation function = tau-function of Toda
  arXiv:1912.12227, with Alexander Gorsky

  - Euclidean time evolution = Toda chain dynamics
  - singularity in Euclidean time = delocalization in Krylov space

Euclidean Operator Growth

Upper bound on infinity-norm

- Euclidean time evolution

$$A(t) \equiv e^{tH} A \, e^{-tH} = \sum_k \underbrace{[H, \ldots [H, A]]}_{k \text{ times}} \frac{t^k}{k!}$$

- locality of interaction

$$H = \sum_I h_I, \qquad [H, \ldots [H, A]] = \sum_{I_1, \ldots, I_k} [h_{I_k}, \ldots [h_{I_1}, A]]$$

- the bound

$$|A(t)| \leq |A| f(t), \quad f(t) = \sum_{\text{clusters}} \sum_k n(k) \frac{(2J|t|)^k}{k!}$$

$n(k)$ – number of sets $I_1, \ldots, I_k$, which satisfy adjacency condition, associated with a given cluster (lattice animal)

Counting the sets $I_1, \dots, I_k$

- Each set $I_1, \dots, I_k$ defines lattice animal *history*

$$\{I\} \equiv I_1, \dots, I_k \to \{J\} \equiv J_1, \dots, J_j, \qquad j \leq k$$

- the map $\{I\} \to \{J\}$ defines a partition of k objects into j groups, and vice versa

$$n(k) = S(k, j)\phi(j)$$

$n(k)$ – number of sets $\{I\}$ associated with a given cluster
$\phi(j)$ – number this cluster's histories $\{J\}$

$$N(k) = \sum_{\text{clusters}} n(k) = \sum_j S(k, j)\phi(j)$$

Summing over histories

- Stirling transform

$$N(k) = \sum_j S(k,j)\phi(j), \quad \phi(j) = \sum_k s(k,j)N(k)$$

- Stirling transform

$$f(t) \equiv \sum_k N(k)\frac{t^k}{k!} = \sum_j \phi(j)\frac{q^j}{j!}, \quad q \equiv e^t - 1.$$

- summing over histories – new expansion parameter

$$|A(t)| \le |A| f(t), \qquad f = \sum_j \phi(j)\frac{q^j}{j!}, \quad q \equiv e^{2J|t|} - 1.$$

Bound for Bethe lattices

- Bethe lattice of coordination number $z \geq 2$
- exact number of lattice animal histories

$$\phi(j) = (z-2)^j \frac{\Gamma(j + z/(z-2))}{\Gamma(z/(z-2))}$$

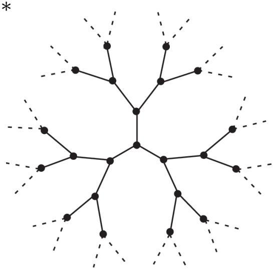
- the bound

  $f = (1-(z-2)q)^{-z/(z-2)}$

  for $z > 2$ there is a pole at some $t = \beta^*$

  for $z = 2$, i.e. 1D lattices, $f = e^{2q}$

- for arbitrary lattices Bethe lattices provide an upper bound



Euclidean operator growth and chaos

- generic non-integrable quantum lattice models
  $D \geq 2$, singularity at finite $t = \beta^*$

$$|A(t)| \lesssim \frac{|A|}{(1 - q/q_0)}$$

  $D = 1$, double-exponential growth

$$|A(t)| \lesssim |A| e^{2q}$$

- Euclidean Lieb-Robinson
  $D \geq 2$, operators spread to spatial infinity at finite $t = \beta^*$
  $D = 1$, operators spread exponentially, $t \sim \ln(\ell)$

$$|[A(t), B]| \leq 2|A||B| e^{q} \frac{q^{\ell}}{\ell!}$$

(Equivalent) signatures of "quantum chaos"

- singularity of time-correlation function in Euclidean time

$$C(t) = \frac{1}{N}\mathrm{Tr}(A(t)A(0)) \equiv ||A(t/2)||^2 \leq |A(t/2)|^2$$

Avdoshkin, AD'19

- maximal growth of Lanczos coefficients

orthogonal Krylov basis A_n

$$A_{n+1} = [H, A_n] - b_{n-1}^2 A_{n-1}, \qquad b_n \propto n$$

Parker, Cao, Avdoshkin, Scaffidi, Altman'18

- exponential decay of power spectrum

$$C(t) = \int d\omega\, \Phi(\omega)\, e^{i\omega t}, \qquad \Phi(\omega) \sim e^{-\omega/\omega_0}$$

Elsayed, Hess, Fine'14

Euclidean operator growth and OTOC

- location of the singularity of $C(\beta^* = \pi/(2\alpha))$ – slope of Lanczos coefficients growth $b_n \propto \alpha\, n$ bounds $\lambda_{\rm OTOC}$

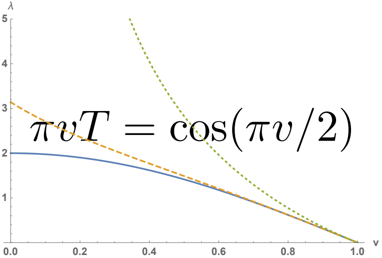
$$\lambda_{\rm OTOC} \le 2\alpha$$

Parker, Cao, Avdoshkin, Scaffidi, Altman'18
Murthy, Srednicki'19

- improved bound on chaos for large T

$$\lambda_{\rm OTOC} \le \frac{2\pi T}{1+2\beta^* T}$$

  - exact SYK
  - improved bound
  - MSS bound



Singularity of $C(t)$

- scalar product in the space of operators

$$\langle B|A\rangle := \frac{1}{N}\mathrm{Tr}(AB^\dagger), \qquad C(t) = \langle A|A(t)\rangle$$

- adjoint action $[H, \ ]$ is self-adjoint with $\langle \ | \ \rangle$

$$C(t_1+t_2) = \langle A(t_1)|A(t_2)\rangle = \langle A(t_1/2)|e^{t_2[H,\ ]}|A(t_1/2)\rangle$$

- assuming $A(t/2)$ is typical

$$C(t+\beta) = ||A(t)||^2\frac{Z(\beta)Z(-\beta)}{Z(0)^2} = ||A(t)||^2 e^{2F(0)-F(\beta)-F(-\beta)}$$

qualitatively, singularity of $C(t)$ is associated with $A(t)$ spreading within Krylov space and becoming more typical

Toda chain flow in Krylov space

Recursion method $\subset$ Toda chain flow

- scalar product in the space of operators

$$\langle B|A\rangle := \frac{1}{N}\mathrm{Tr}(\rho_1 A \rho_2 B^\dagger), \qquad C(t) = \langle A|A(t)\rangle$$

$[H, \ ]$ is self-adjoint with $\langle \ | \ \rangle$

- orthogonal basis in Krylov space

$$A_{n+1} = [H, A_n] - a_n A_n - b_{n-1}^2 A_{n-1}$$

uniquely determined by the choice of the initial operator A_0

- a family of t-dependent Krylov bases A_n^t

$$A_0^t := A(t/2), \qquad q_n(t) = \ln\langle A_n^t|A_n^t\rangle,$$
$$a_n(t) = \dot{q}_n, \qquad b_n^2(t) = e^{q_{n+1}-q_n}$$

Recursion method ⊂ Toda chain flow

- Euclidean time evolution of $A(t)$ – Toda chain dynamics

$$\ddot{q}_n = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}}$$

- time-correlation function = tau-function of Toda

$$C(t) = \langle A|A(t)\rangle = \langle A(t/2)|A(t/2)\rangle = e^{q_0} = \tau_0$$

Toda EOMs in Hirota's bilinear form, $q_n = \ln \tau_n/\tau_{n-1}$

$$\ddot{\tau}_n \tau_n - \dot{\tau}_n^2 = \tau_{n+1}\tau_{n-1}$$

- Toda chain flow in Krylov space

$$G_{nm}(t) = \langle A_n|A_m(t)\rangle, \qquad \frac{d}{dt}(G^{-1}\dot{G}) = 0$$

Exact solutions

- Toda EOMs in Flaschka form

$$\frac{d^2}{dt^2}\ln(b_n^2) = b_{n+1}^2 - 2b_n^2 + b_{n-1}^2$$

- anzats $b_n^2 = b^2(t)p(n)$, asymptotic behavior $b_n \propto n$

$$\begin{aligned} p(n) &= (n+c)(n+1), \qquad & b^2(t) = J^2/\sin^2(J(t_0-t)), \\ C(t) &= \sin(J(t_0-t))^{-c}, \qquad & a_n = (2n+c)J\cot(J(t_0-t)) \end{aligned}$$

  in general non-integrable case both $a_n, b_n \propto n$; slope of a_n, b_n determines the location of singularity

- asymptotic behavior $b_n \propto n^{1/2}$

$$C(t) \sim e^{ae^{mt}}, \qquad C(t) \sim e^{at^2/2}$$

Dynamics of $A(t)$ in Krylov space

- "wave-function" of $A(t)$

$$A(t) = ||A(t)|| \sum_n c_n(t)(A_n/b_n), \qquad ||A_n/b_n|| = 1$$

Inverse Participating Ratio $I = 1/(\sum c_n^4)$

- relation to QR decomposition

$$e^{tM} = Q(t)R(t), \qquad |A(t)| = R_{00}(t), \quad c_n(t) = Q_{n0}(t)$$

- assuming $C(t) = ||A(t/2)||^2$ diverges at $t = t^*$

$$\begin{aligned} ||A(t \to t^*/2)|| &\to \infty, & c_n(t \to t^*/2) \to 0, \\ \langle A_n | A(t \to t^*/2) \rangle &\to \text{regular}, & I(t \to t^*/2) \to \infty \end{aligned}$$

operator delocalizes in Krylov space at $t = t^*/2$

Chaos vs localization in Krylov space

- when the system is chaotic and $C(t)$ has a singularity at $t = t^*$, $A(t)$ delocalizes in Krylov space at $t = t^*/2$
- when the system is integrable and $C(t)$ is analytic, IPR is finite and the operator is Localized

$$C(t) \propto e^{at^2/2}, \qquad I \propto t,$$
$$C(t) \propto e^{ae^{mt}}, \qquad I \propto e^{mt}$$

qualitatively similar to: localization/ergodicity in physical space

= localization / delocalization in Fock space

Altshuler, Gefen, Kamenev, Levitov'97

Basko, Aleiner, Altshuler'06

Main results

- universal bounds on the operator norm growth in lattice models, Euclidean Lieb-Robinson bound
- Toda chain interpretation of the recursion method, time-correlation function
- chaos in the underlying quantum many-body system as delocalization in Krylov space

Outlook

- Connection between Euclidean and Minkowski dynamics

- Can Toda help connect different manifestations of chaos?

  - connection with OTOC
  - connection with spectral properties

- Chaos as delocalization? Connection to BH physics?