Uncomplexity and the power of one-clean-qubit.

Leonard Susskind, Stanford

KITP QInfo17 Conference, Oct 12, 2017

Adam Brown, LS, **arXiv:1701.01107**
AB, Ying Zhao, LS, To appear

Which of the following are correct?

1. By adding a single out-of-equilibrium molecule to a mole of gas in thermal equilibrium you can do enough work to raise a kilogram 10 meters.

2. By adding a single bit to a classical computer in a random state you can restore its computing power to that of a computer in an initial state (000000……0).

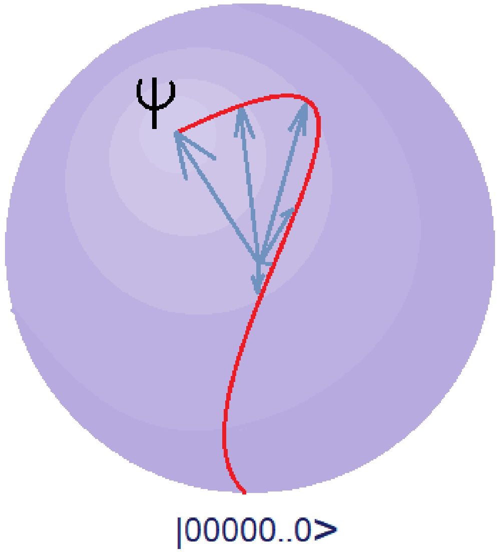
3. By throwing a single thermal photon into a black hole with a super-energetic firewall, the black hole interior can be completely restored.

4. By adding a single qubit to a quantum computer in a random state, the computing power can be restored so that exponentially hard problems can be solved.

Uncomplexity

Quantum computations are dynamical processes which begin with an initial state of N qubits, and progress as quantum gates (or Hamiltonians) are applied. The state of the computer unfolds unitarily, until some goal is reached*.

*I will not count the final measurement processes as part of the computation.hen



Uncomplexity

As the computation proceeds the complexity of the state evolves. If the initial state is simple—unentangled for example—then there is lots of room for the complexity to increase, and during the period of increase the computer can do useful computational work. But after an exponential time the state will reach "complexity equilibrium", a kind of "heat-death" in which the complexity attains the maximum possible value.

Once the computer is in a state of complexity equilibrium, it cannot do further useful directed computation. All it can do is wander around among the enormous (doubly-exponential) number of maximally complex, featureless states.

Uncomplexity is the room for complexity to increase. It is a valuable resource.

We begin with the question: What is the distance between two normalized quantum states?

Standard Metric: $\cos d_{AB} = |\langle A | B \rangle|$

The maximum distance between any two states is $\pi/2$ when they are orthogonal.

$$| A \rangle = (00000000000000)$$

$$| B \rangle = (00000000000001)$$

$$d_{AB} = \pi / 2$$

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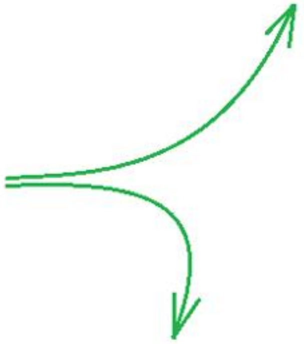
$$| B \rangle = (00000000000001)$$

$d_{AB} = \pi / 2$



$$| A(t) \rangle = e^{-iHt} | A \rangle = \sum_{1}^{2^N} a_n(t) \, | n \rangle$$

$d_{AB} = \pi / 2$

$$| B(t) \rangle = e^{-iHt} | A \rangle = \sum_{1}^{2^N} b_n(t) \, | n \rangle$$

For large time these states are exponentially difficult to superpose or to make transitions between.
And yet according to the standard metric they are no more distant than $|0\rangle$ and $|1\rangle$.

Is there another metric that captures the difficulty of making a transition or interfering states?

Nielsen, Dowling 2007

Relative Complexity* :

the computational circuit complexity of making a transition from |A> to |B>.

C(A, B) = minimum number of gates to get from |A> to |B>.

|B> = g g g g g..... |A>

C(A, B) = C(B, A)

C(A, B) = 0 iff A=B

C(A, B) ≤ C(A, D) + C(D, B) (triangle inequality)

* There are many choices to be made in defining complexity. A choice of allowable gates must be made, a tolerance specified, should we allow ancilla? Does the gate set include only one and two qubit gates or might it allow three, and four qubit gates?Should all gates be weighed equally or should higher number of qubit gates be regarded as more complex?

C(A,B) is a metric on the space of normalized states.

Define the absolute complexity C(A) of a state by the minimum distance to a simple state: i.e., an unentangled state.

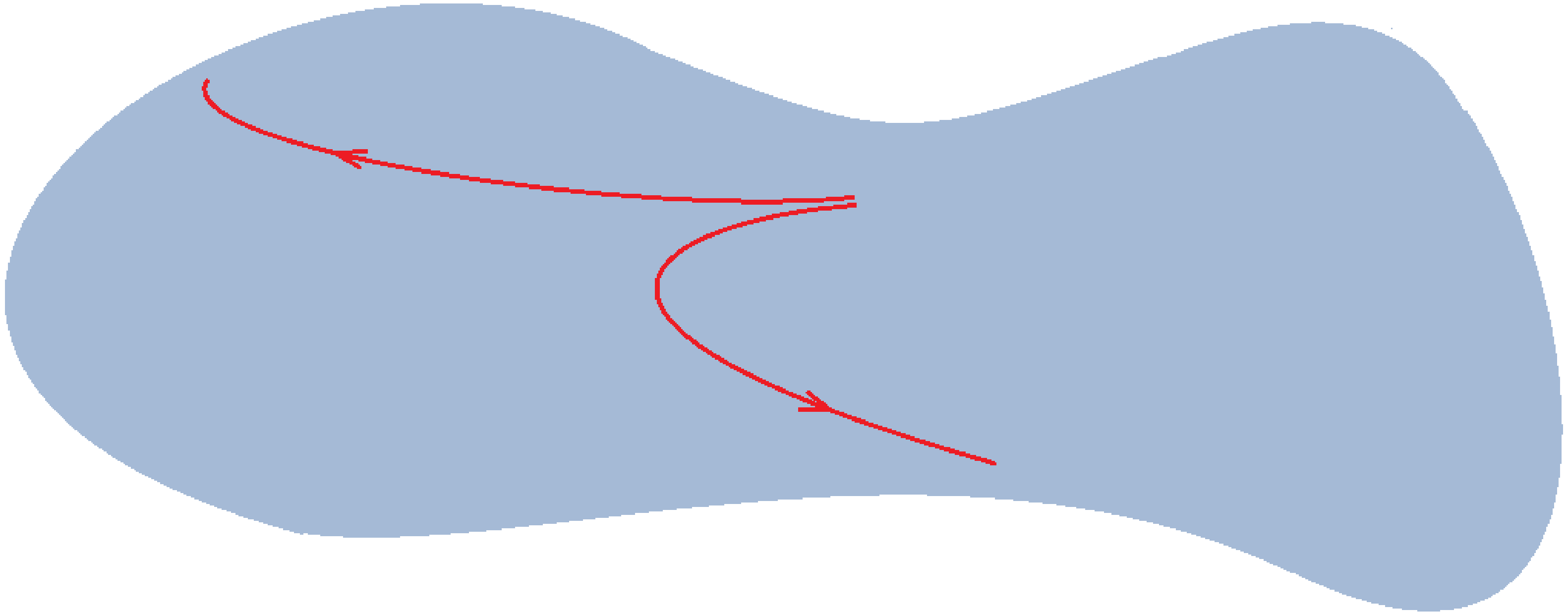
For a system of N qubits the maximum possible complexity is,

$$C_{max} = 2^N \qquad (\text{Compare with } S_{max} = N)$$

Almost all states are close to maximally complex.

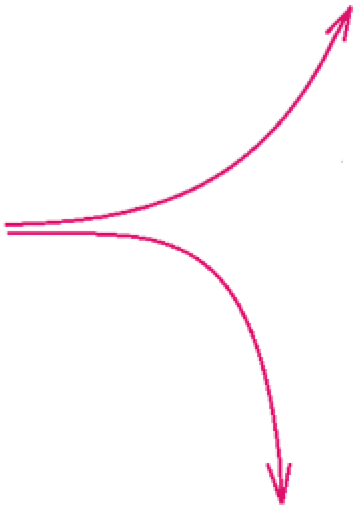
Motions generated by the ordinary Schrodinger equation can appear very complicated on the relative complexity geometry.

$$i \frac{d}{dt} |\Psi\rangle = H |\Psi\rangle$$



These motions define a classical mechanical system.

Sectional curvatures in the relative complexity geometry are generally negative and there are many positive Lyapunov exponents.



This suggests that motion on "complexity" geometry is chaotic.

See
Nielsen-Dowling 2007
Brown-Susskind 2017

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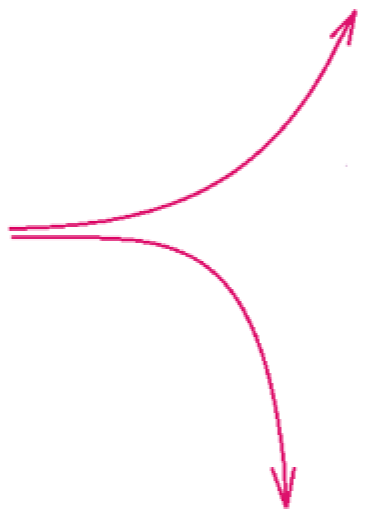


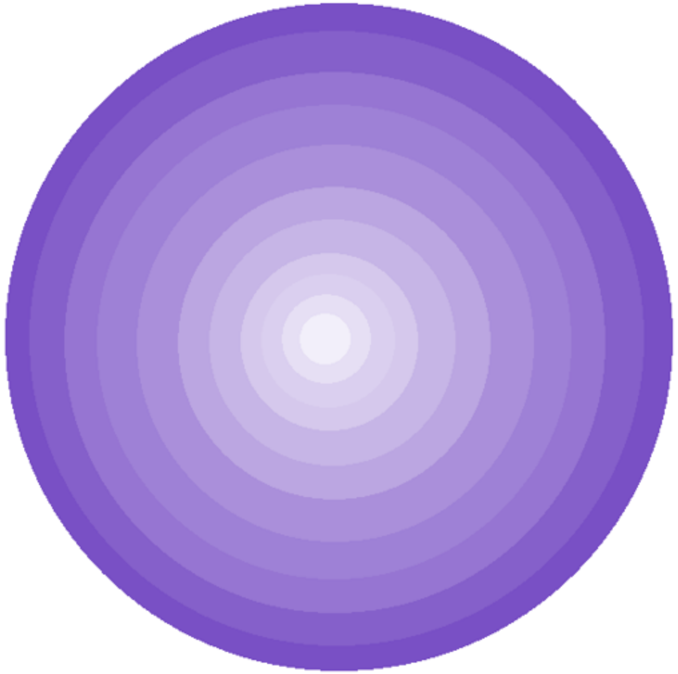
This suggests that motion on "complexity" geometry is chaotic.

Dimension of geometry is very large,

$D = 2^N$.

Classical chaos and large dimension $\longrightarrow$ thermodynamics (of complexity)

Complexity geometry ala Nielsen



The space of states organized so that simple states are near the center and increasingly complex states are further out.

The number of states with complexity C grows exponentially,

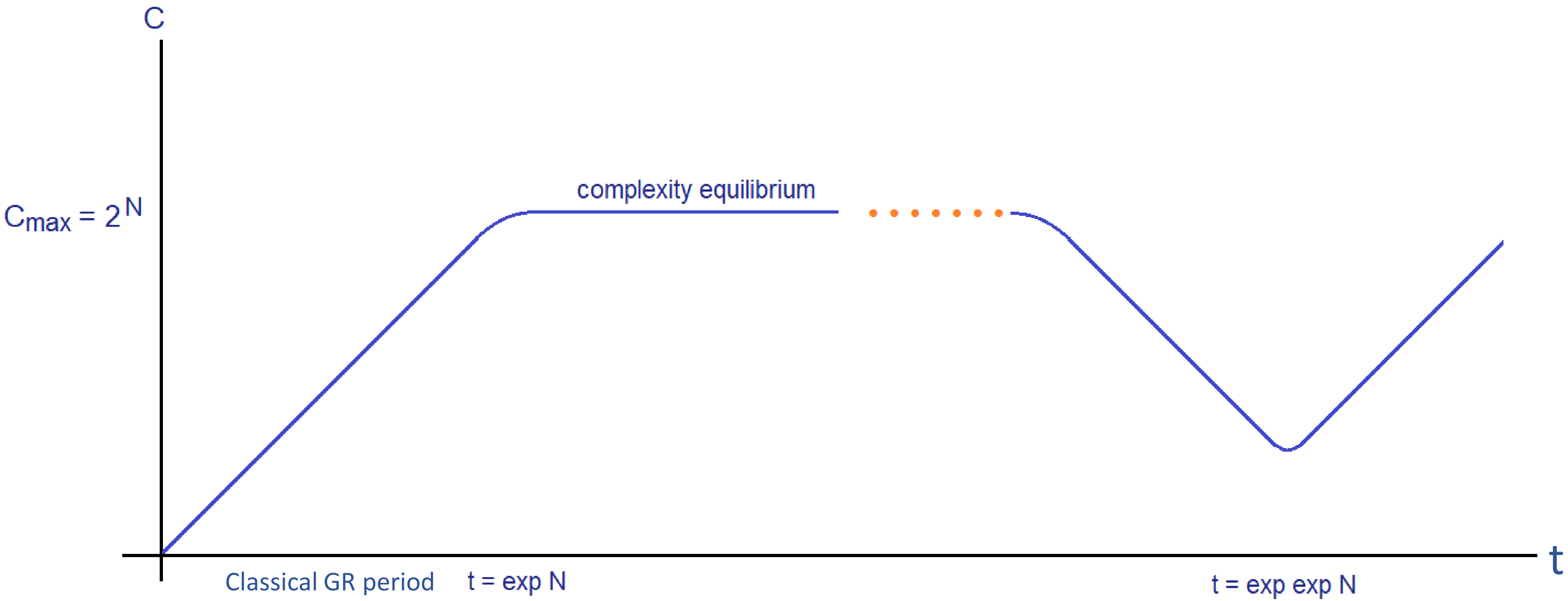
$\mathcal{N} \sim \exp C$

$C = \log \mathcal{N}$

This suggests that complexity is the analog of classical entropy — the log of the volume of phase space.

Entropy ⟶ complexity

Second law of quantum complexity (Brown, Susskind 2017)

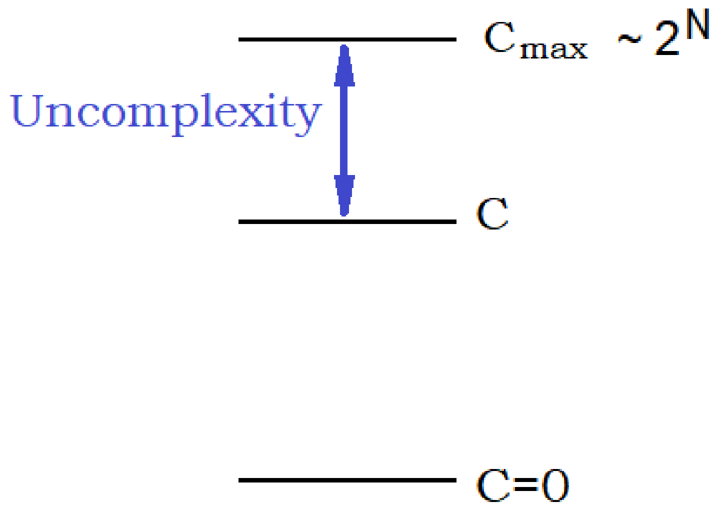


Aaronson, Susskind

The quantum complexity of a system of N qubits behaves like the classical entropy of a system of 2^N degrees of freedom.

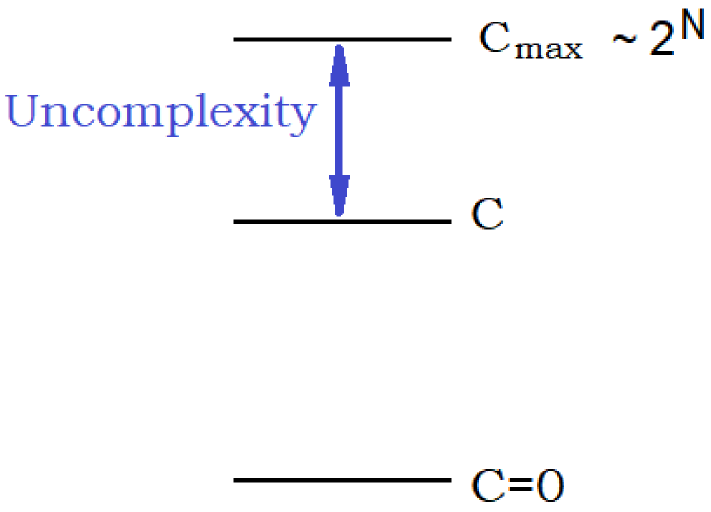
Uncomplexity

Uncomplexity = (Maximum Complexity – Complexity)



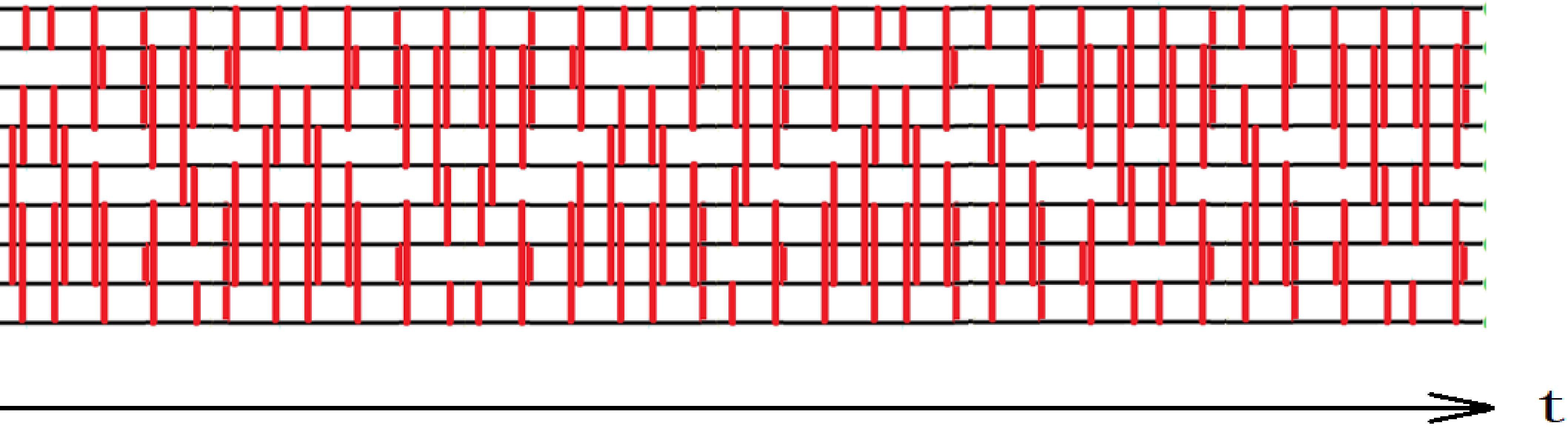
Uncomplexity: a necessary resource for doing "computational work".

Uncomplexity = (Maximum Complexity – Complexity)

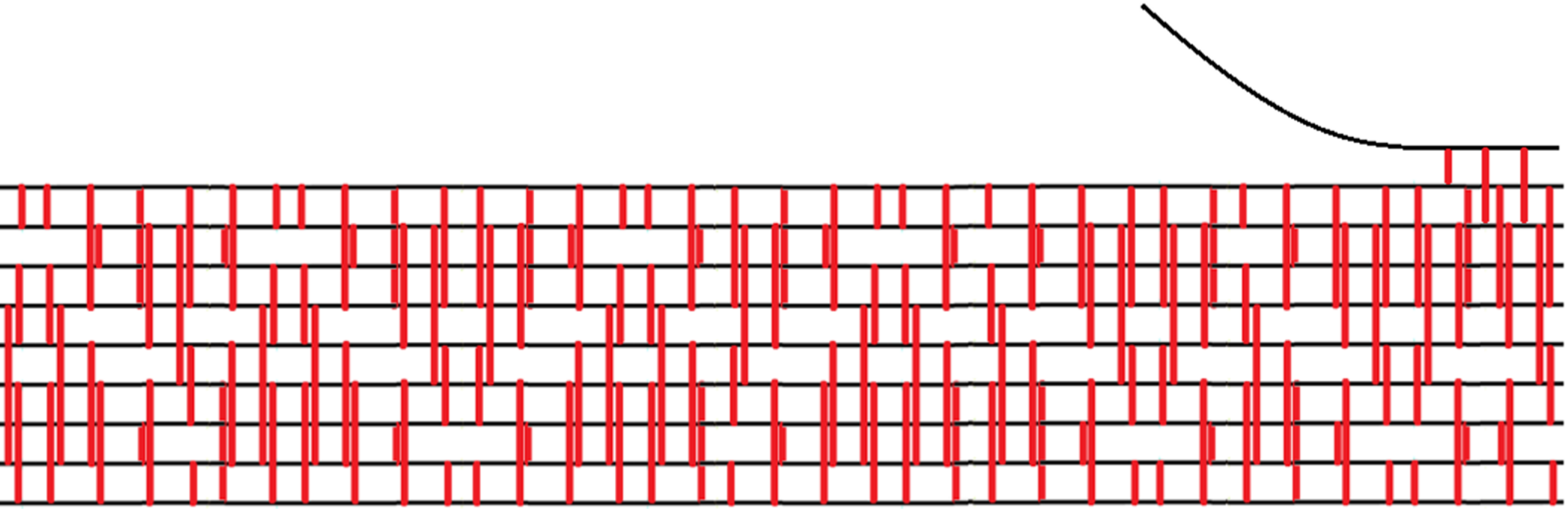


A quantum computer that has run for an exponential time will reach complexity-equilibrium. It will not be able to do any useful computation.

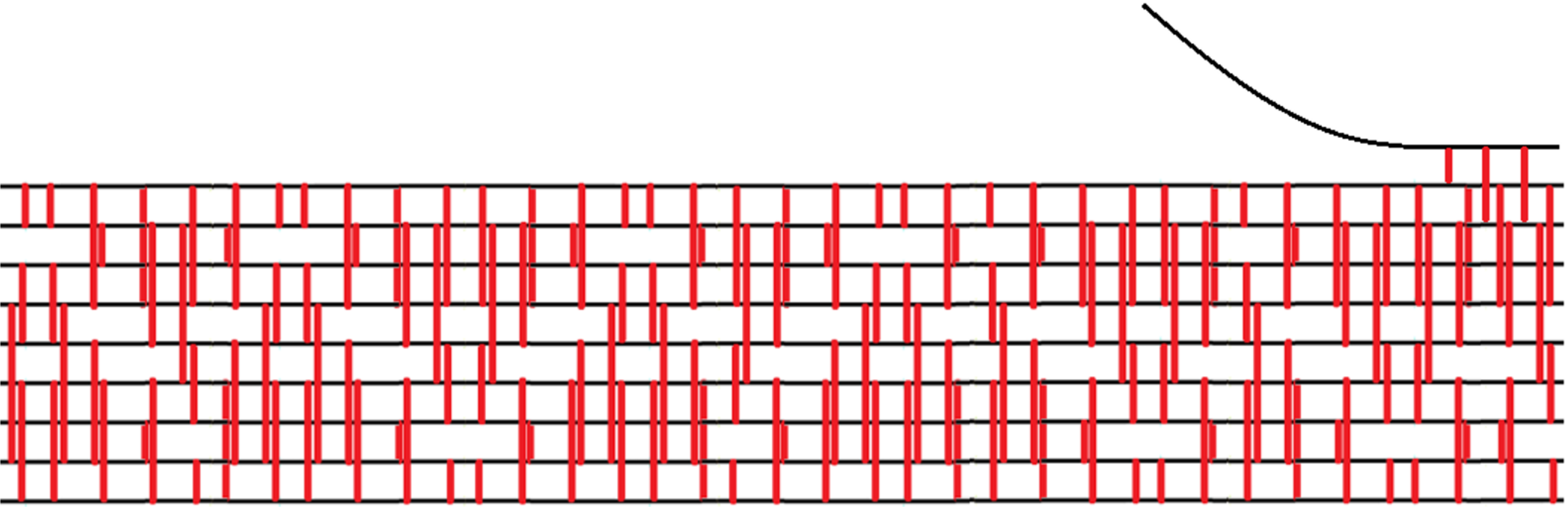
Uncomplexity = 0



Let's be extreme and suppose the circuit has some huge number of qubits, say Avogadro's number, and we add one more qubit. How much computing power does that buy us?



$$C_{max} = 2^N \longrightarrow 2^{(N+1)} = 2 \times 2^N$$



Adding just a single clean qubit buys an exponentially large uncomplexity!

If uncomplexity really is a resource for directed computation, the addition of a single qubit is enough to restore the life of an "exhausted" computer, and allow it to solve exponentially hard problems.

This is very counterintuitive: it sounds like the computational equivalent of:
Adding a single out-of-equilibrium molecule to a mole of gas in thermal equilibrium will allow a heavy weight to be lifted.

This is very counterintuitive: it sounds like the computational equivalent of:
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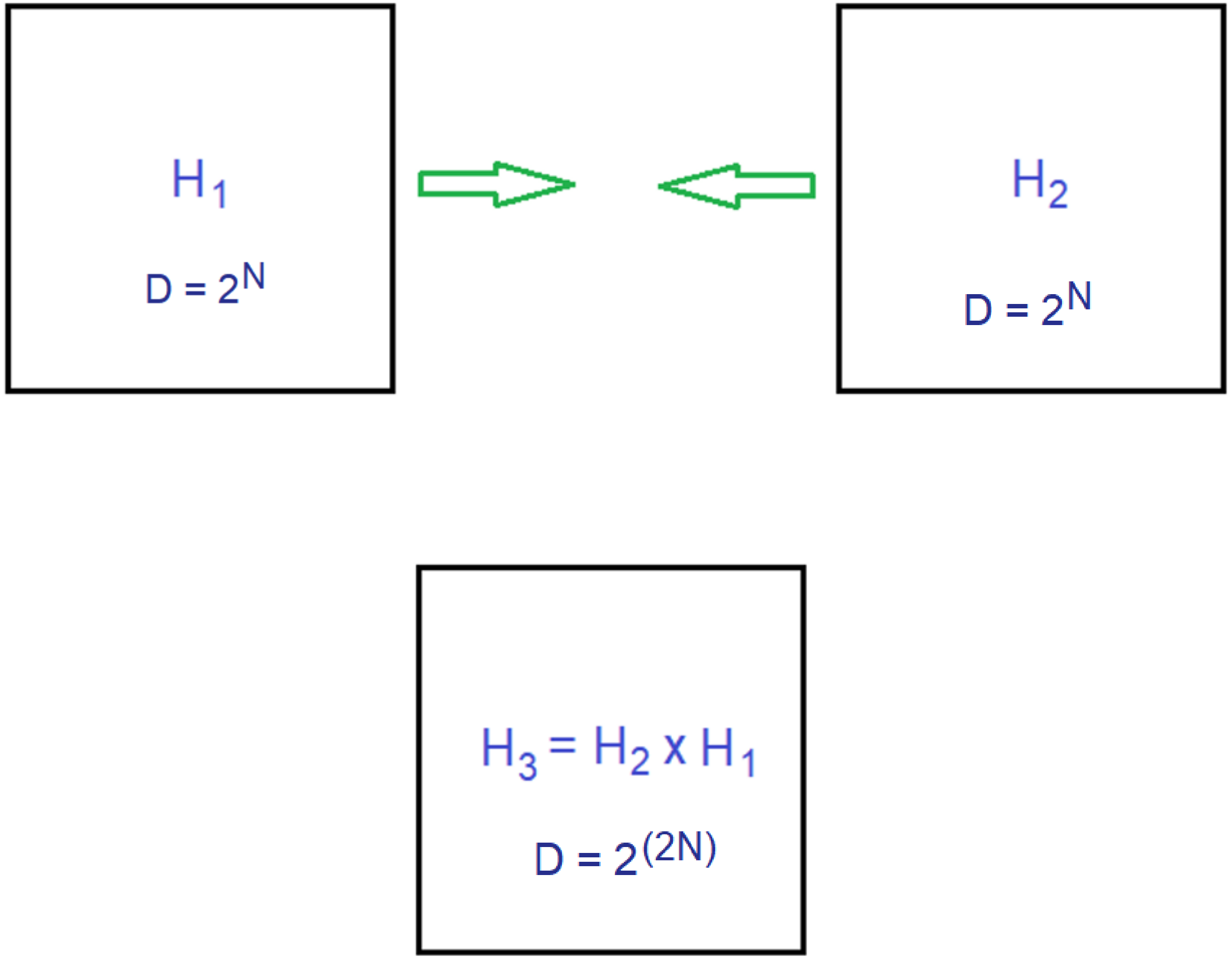
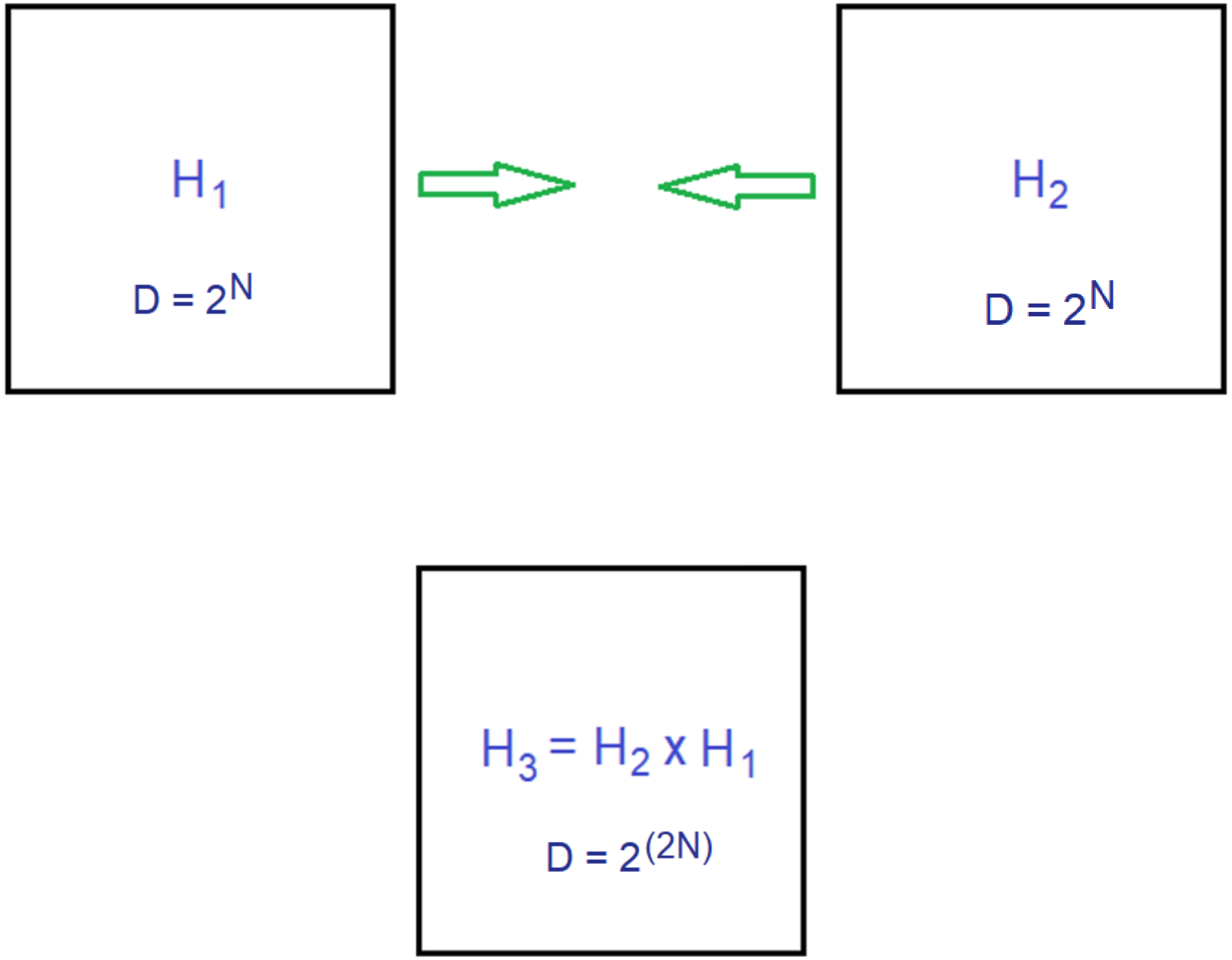
We'll return to this point shortly, but first some thermodynamics.

Thermodynamics: Combining systems to form bigger systems in and out of equilibrium.



$$S_{max\ 3} = S_{max\ 1} + S_{max\ 2}$$

Thermodynamics: Combining systems to form bigger systems in and out of equilibrium.



$$C_{max\ 3} = C_{max\ 1} \ \mathrm{X} \ C_{max\ 2}$$

$$2^{2N} = 2^N \times 2^N$$

It would appear that Complexity does not behave like entropy under composition.
Does that mean that the thermodynamic analogy for complexity breaks down?

No, it does not. We assumed the wrong idea about what it means to combine systems.

Doubling a system in the present context means adding one single qubit!

To see why, consider two quantum states,

$$|A\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \end{bmatrix} \qquad |B\rangle = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{bmatrix}$$

$$C_{\max A} = 2^N$$

$$C_{\max B} = 2^N$$

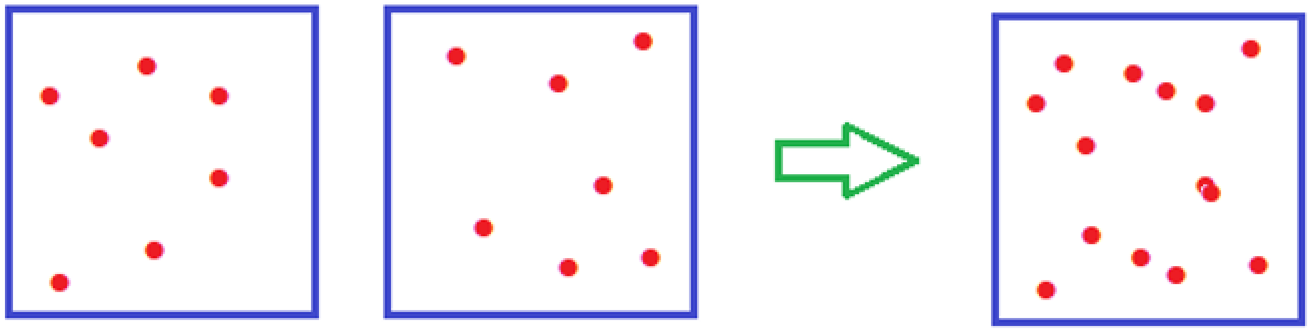
Add one qubit and put the states in a “controlled” superposition.

$$|A\rangle|0\rangle + |B\rangle|1\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ \vdots \end{bmatrix}$$

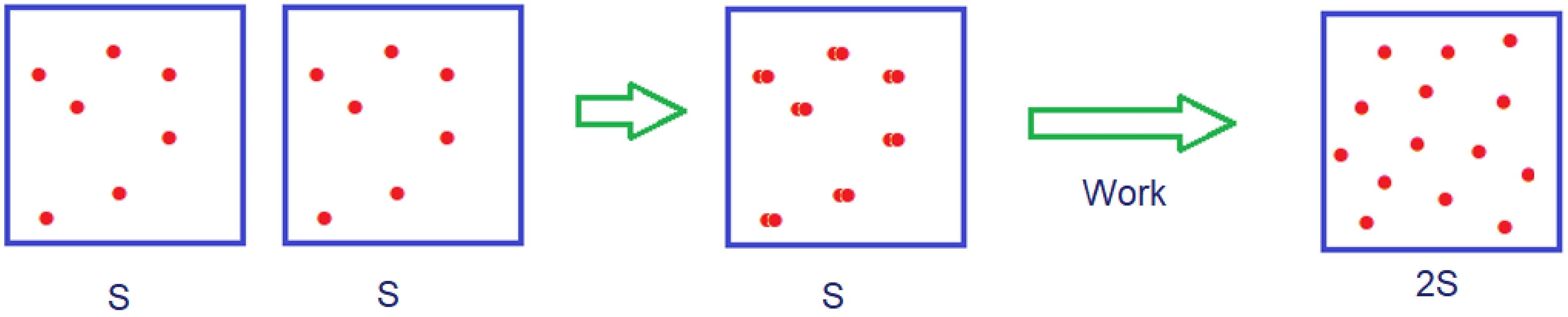
$$C_{max\ (A+B)} = 2^{N+1}$$

If |A> and |B> are maximally complex (in complexity equilibrium) and have maximal relative complexity (uncorrelated) then |A> |0> + |B> |1> is maximally complex with complexity 2^{N+1}.

The result is that there is no uncomplexity.



But if |A> = |B> the extra qubit is "clean." Adding a clean qubit does not increase the complexity so initially it is 2^N. But the maximum complexity is 2^{N+1}. Therefore there is an uncomplexity = 2^N.



Adding a single clean qubit to an “exhausted” computer restores the uncomplexity to its original exponential value.

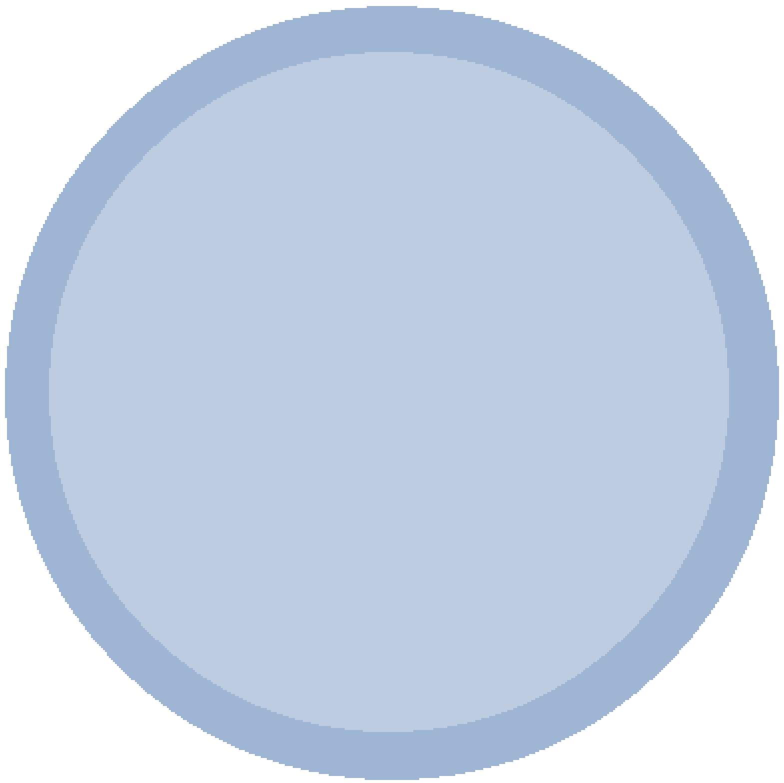
Does this mean that adding one clean qubit allows the computer to solve at least some exponentially hard problems?

Evidently adding a single clean qubit to an “exhausted” computer restores the Uncomplexity to an exponentially large resource.

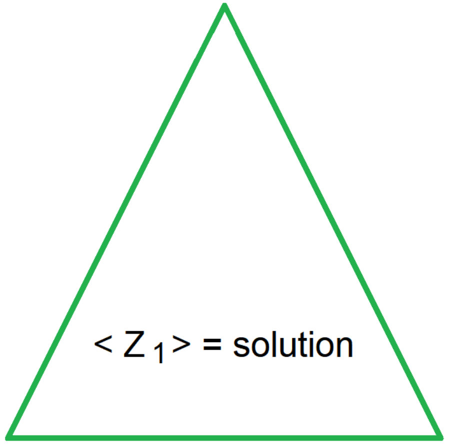
Does this mean that adding one clean qubit allows the computer to solve at least some exponentially hard problems?

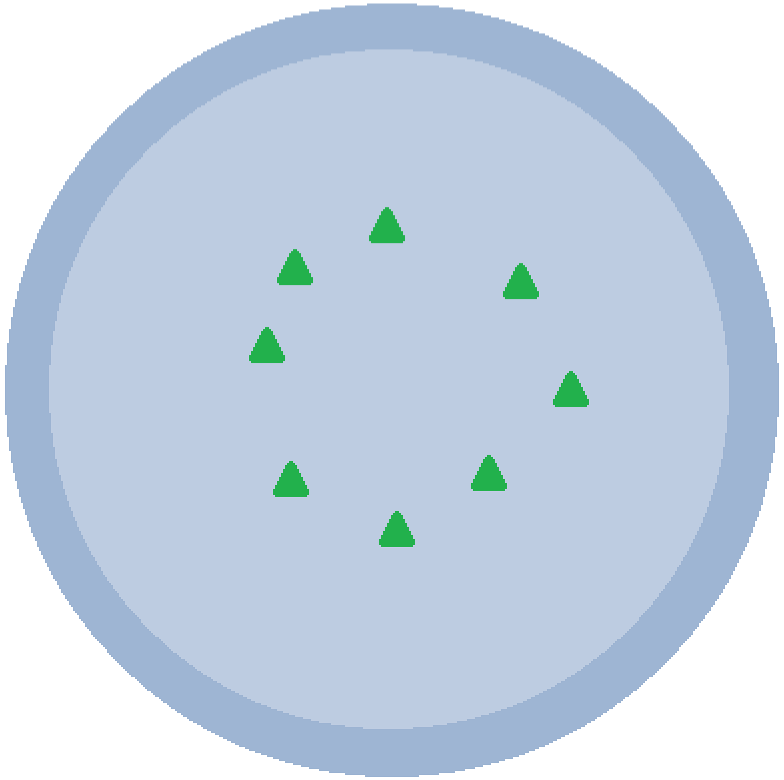
Yes, it does.

One clean qubit computation (DQC1) (E. Knill, and R. Laflamme 1998)

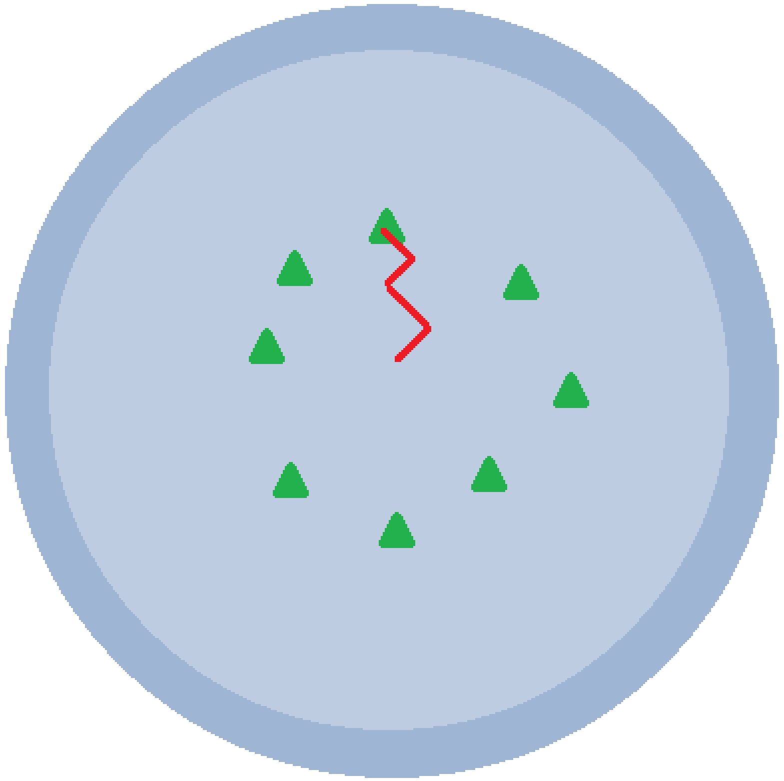


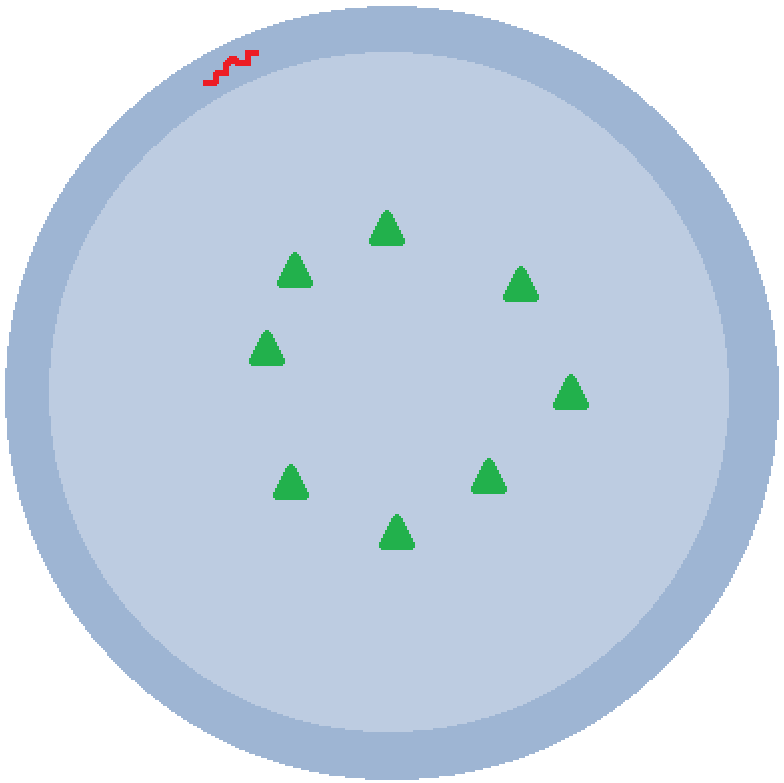
The space of states with the simple state at the center and increasingly complex states at larger distance. The maximally complex states are out near the boundary and they are the overwhelming majority of states.

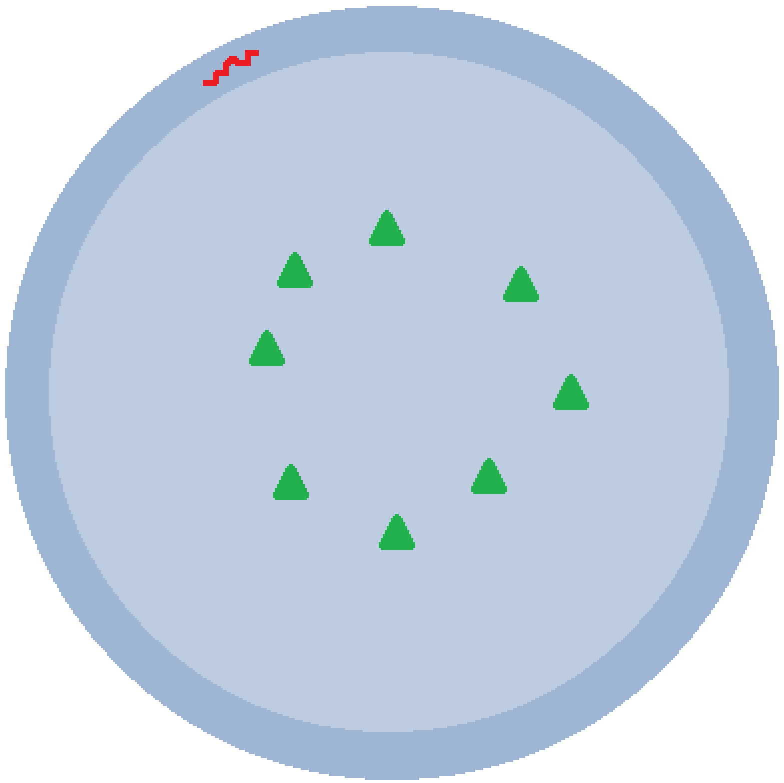




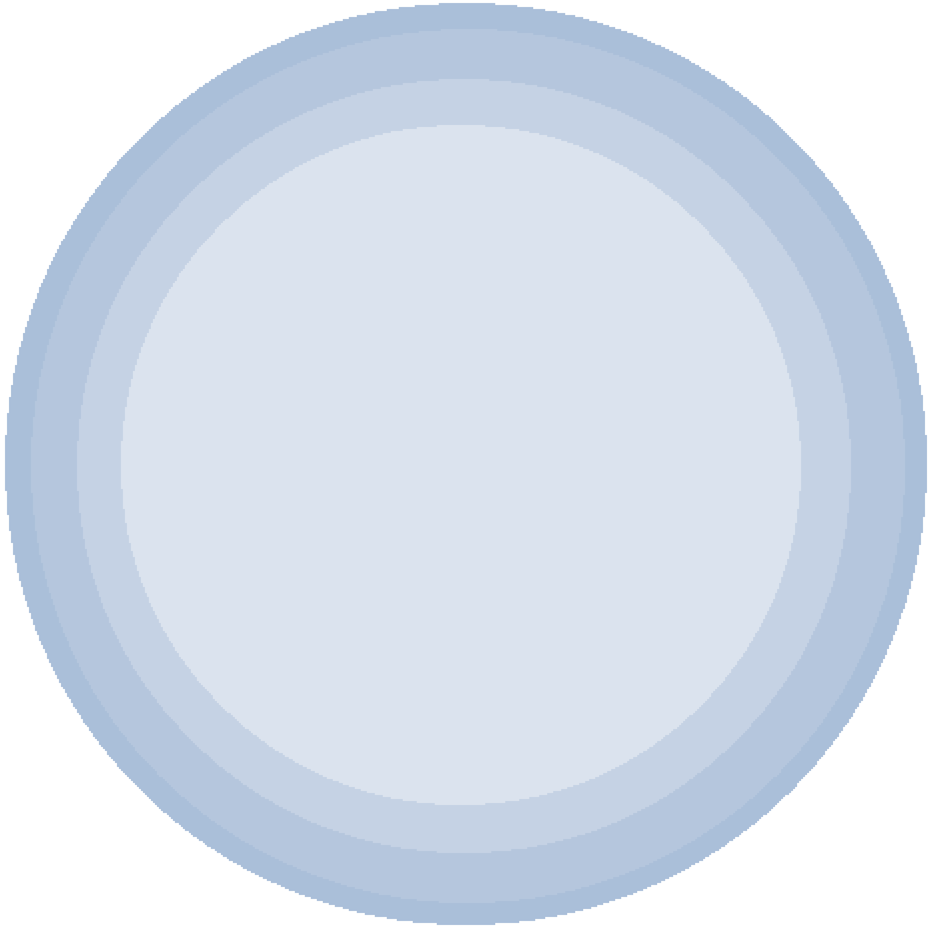
The goal of the computation is to guide the system to a “target region”.

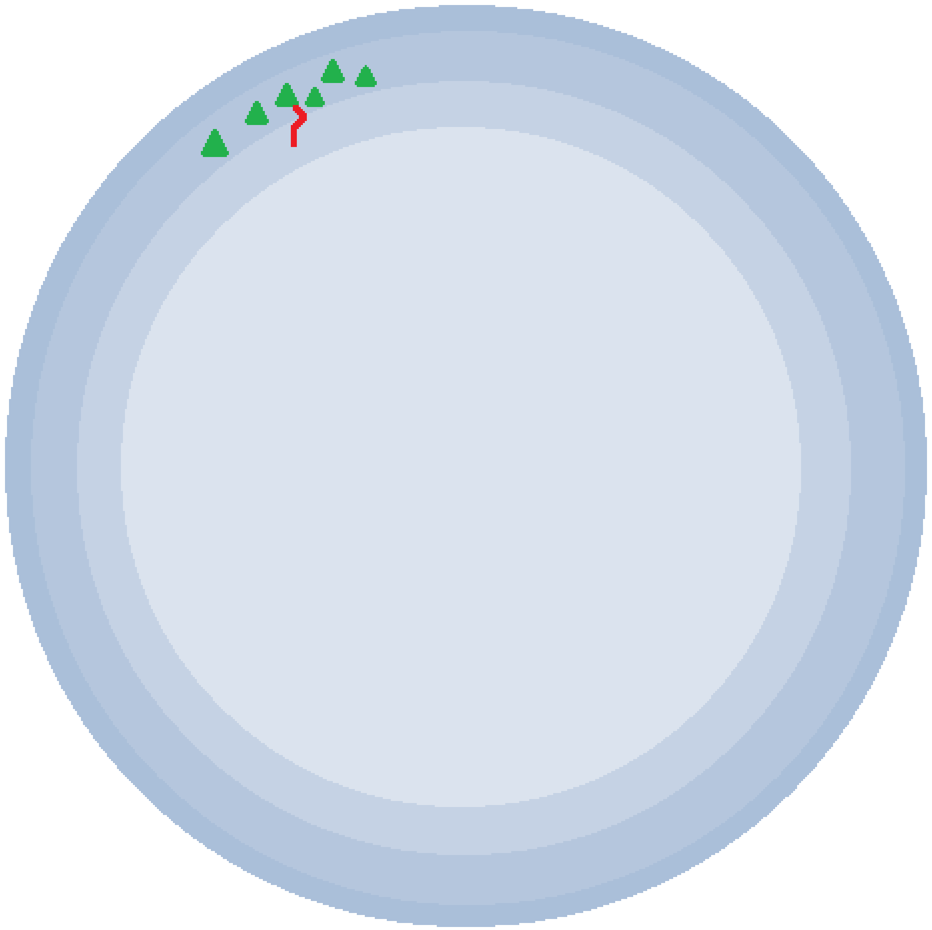




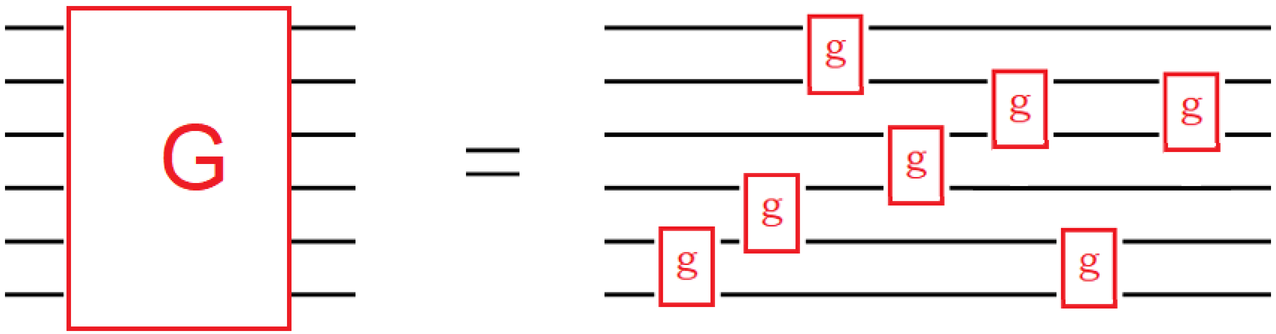


Now let’s add one clean qubit.

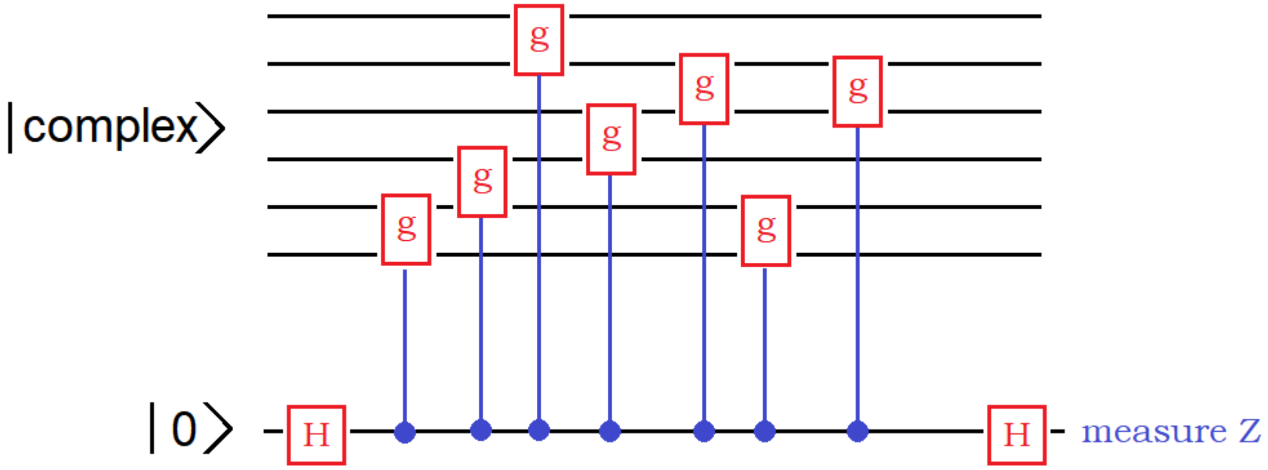




Calculating the trace of a unitary matrix G is hard.



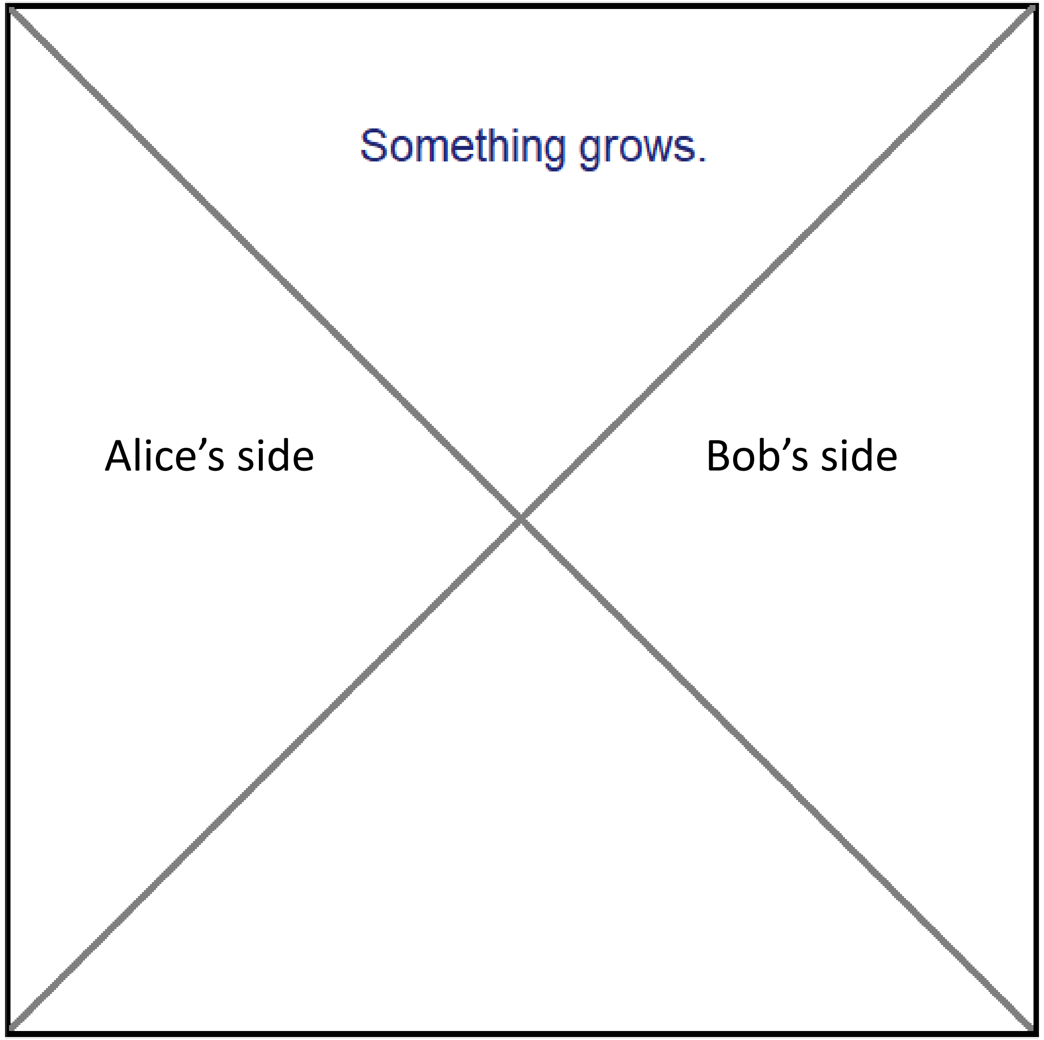
One clean qubit computation to compute Tr G = Tr g g g g g g..

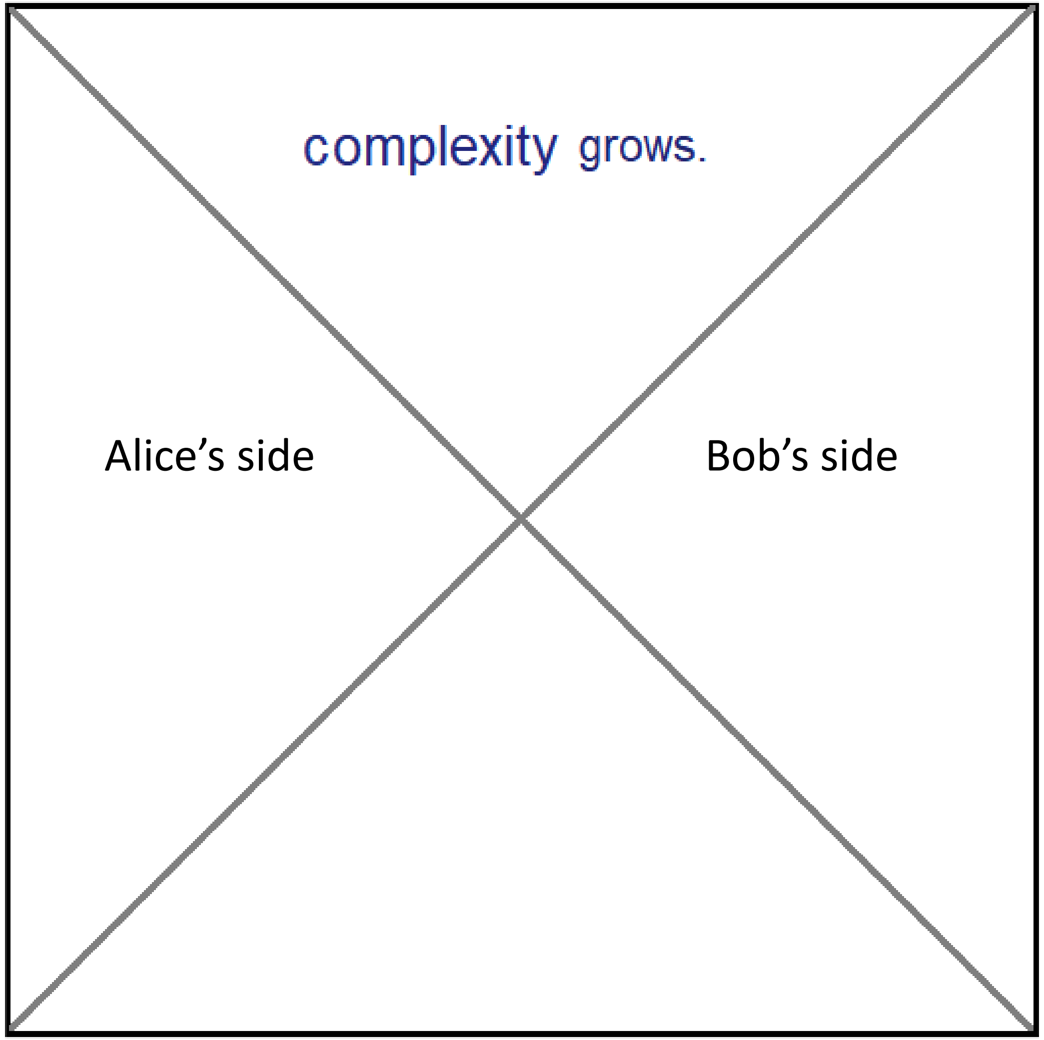


The Uncomplexity resource decreases by the complexity of G.

We have done computational work and spent a resource.

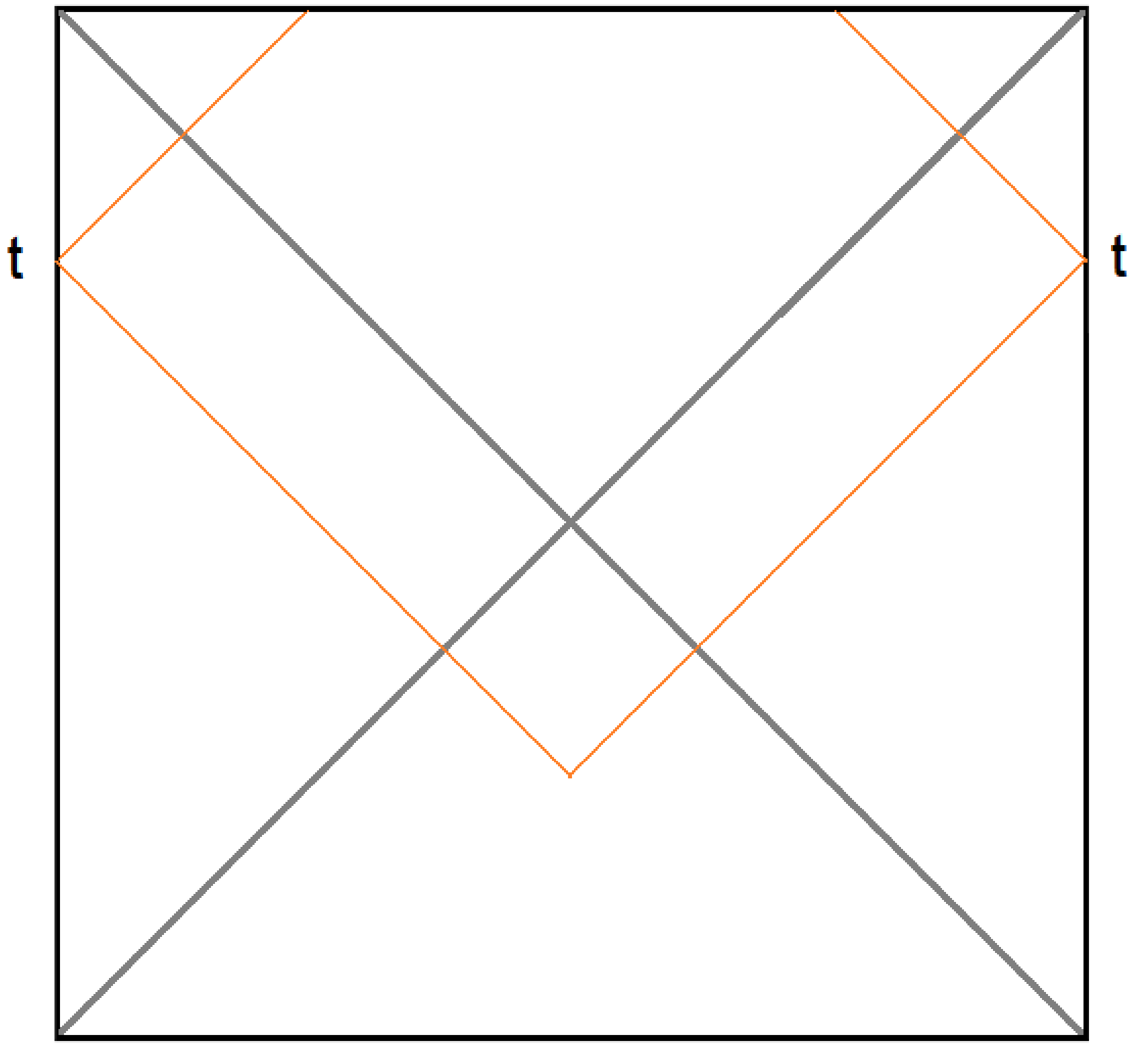
(Un)complexity, gravity, and one clean qubit


Something grows.
Alice’s side
Bob’s side

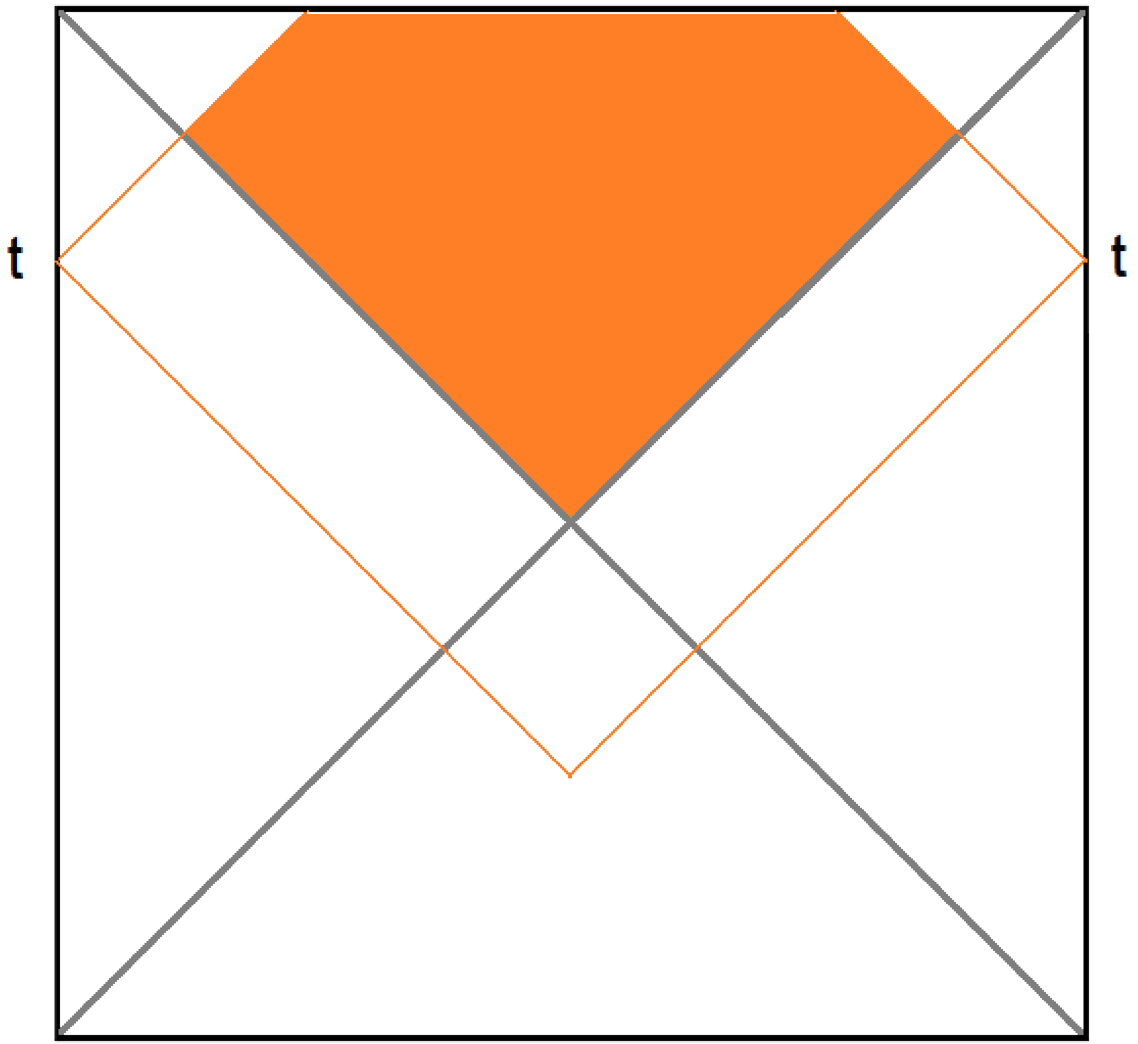


L.S., Computational Complexity and Black Hole Horizons, (2014)

D. Stanford and L.S., Complexity and Shock Wave Geometries, (2014) (2014)

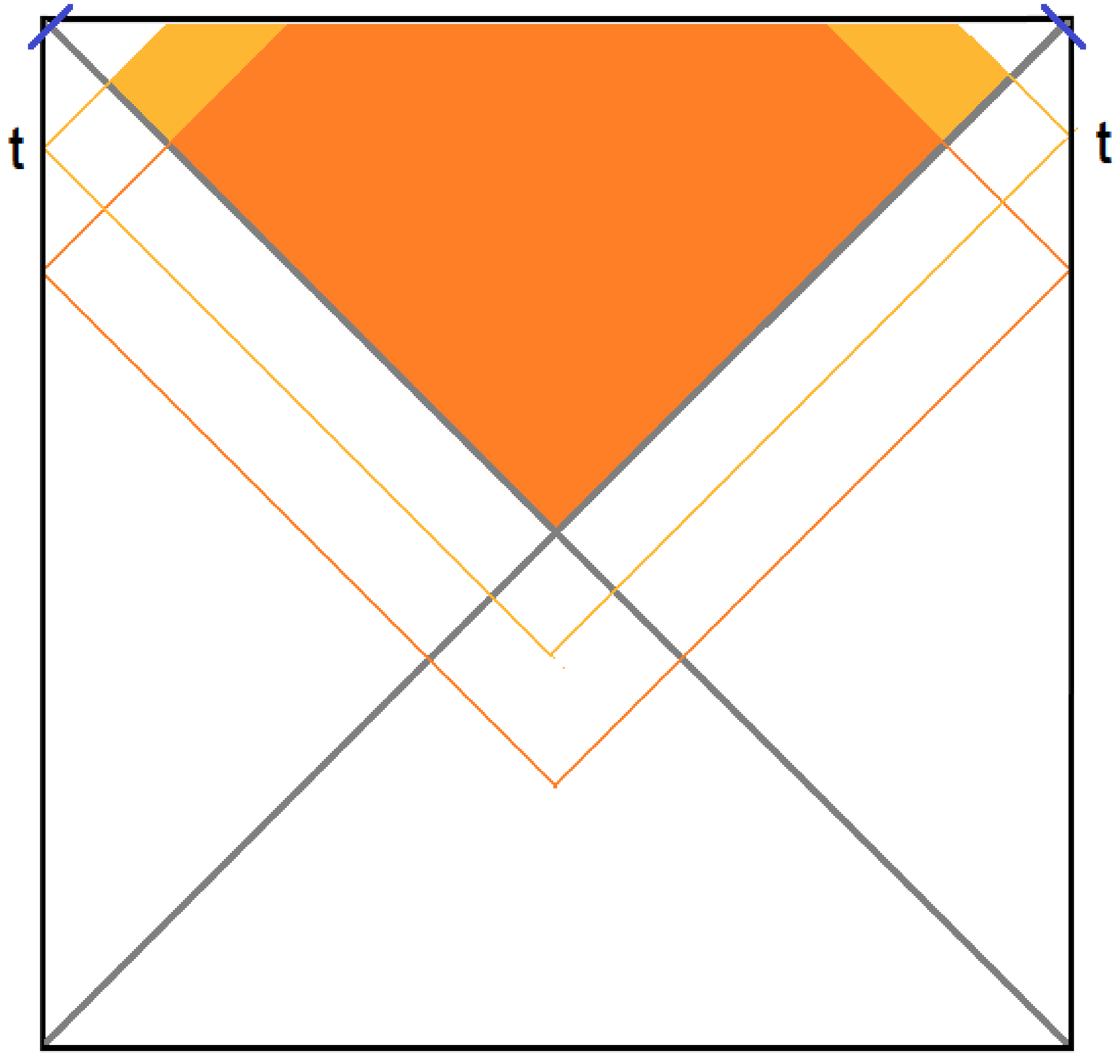


WDW patch

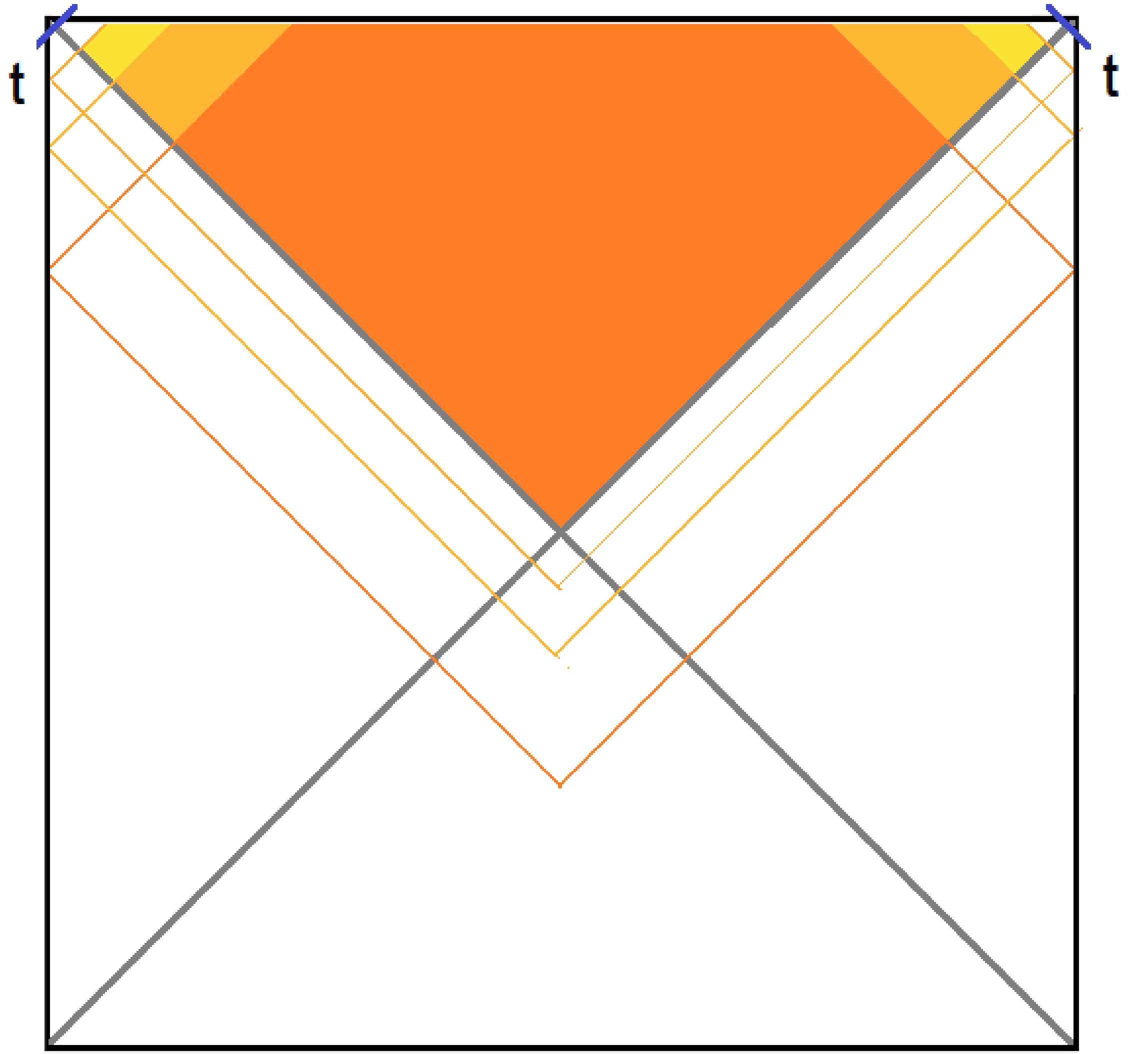


Complexity = Action

arXiv:1512.04993
Complexity, action, and black holes
Adam R. Brown, Daniel A. Roberts, Leonard Susskind, Brian Swingle, Ying Zhao

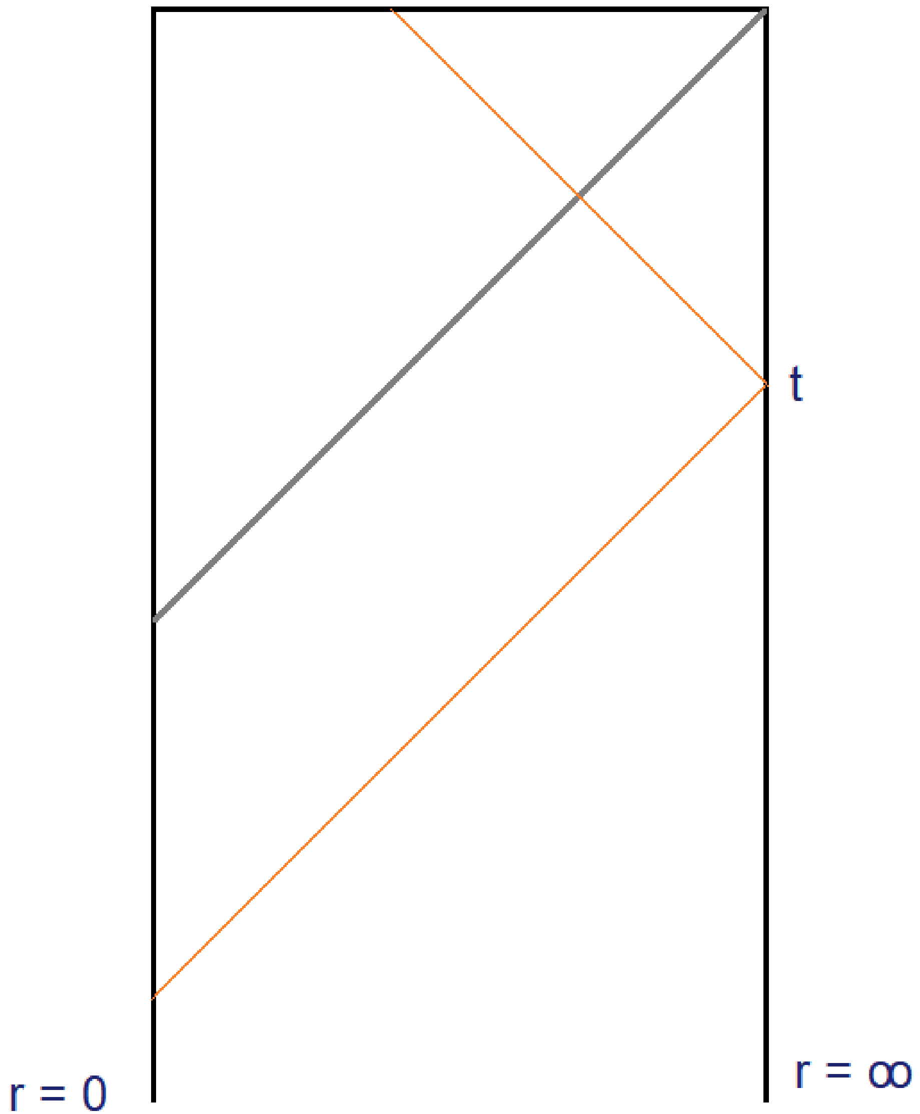


Complexity = Action

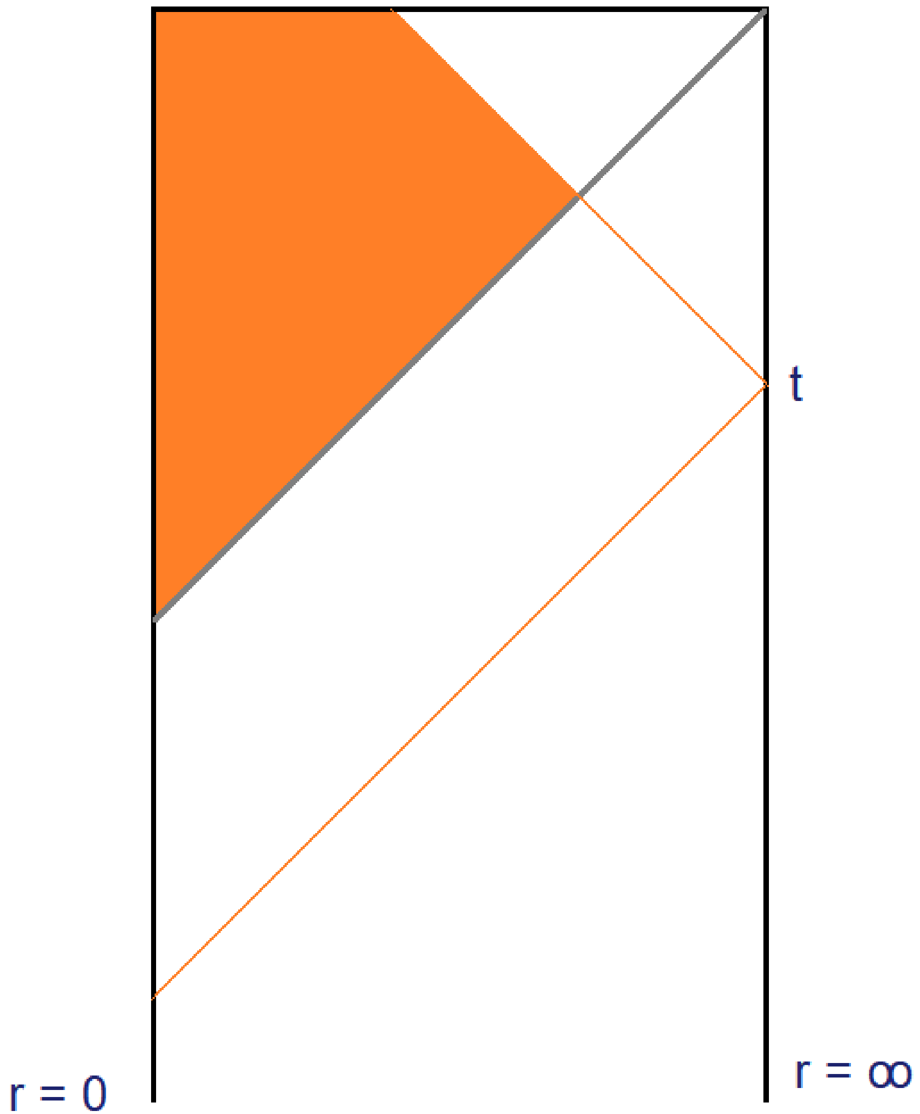


Complexity = Action

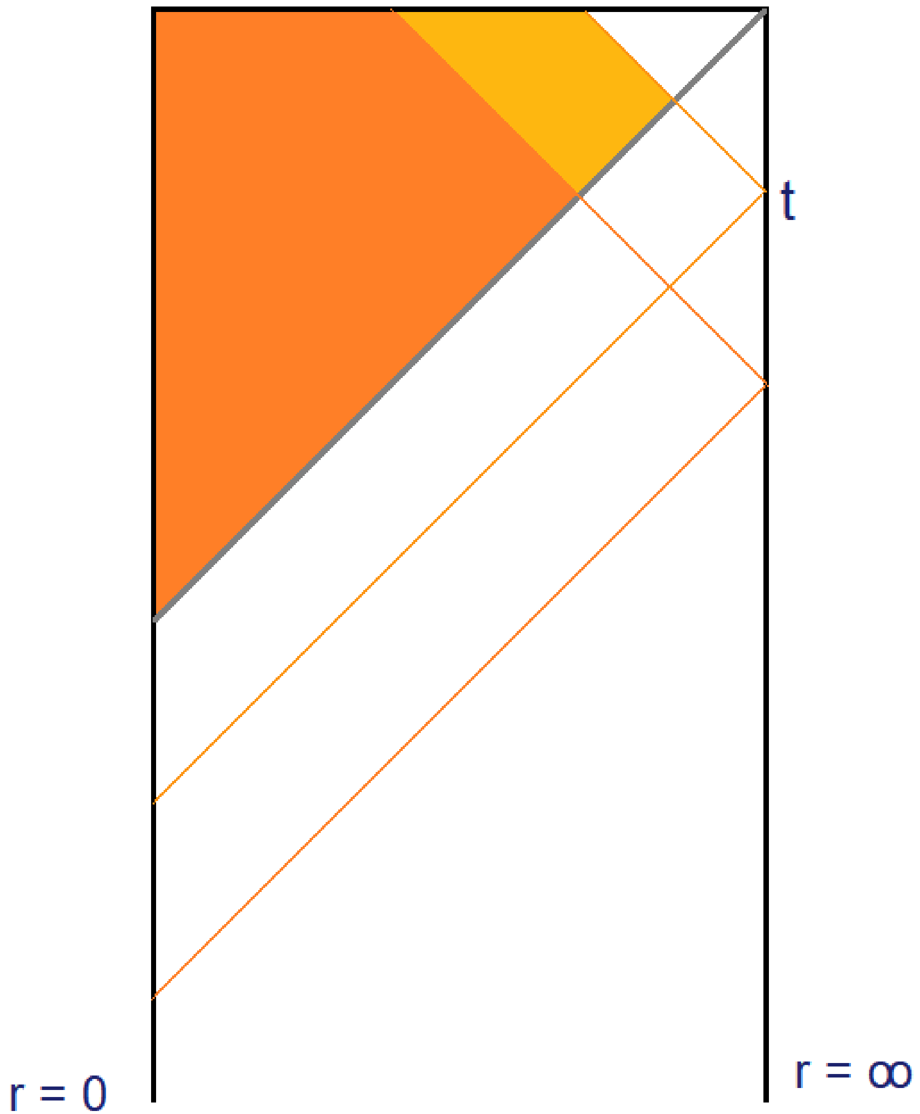
A single unentangled black hole



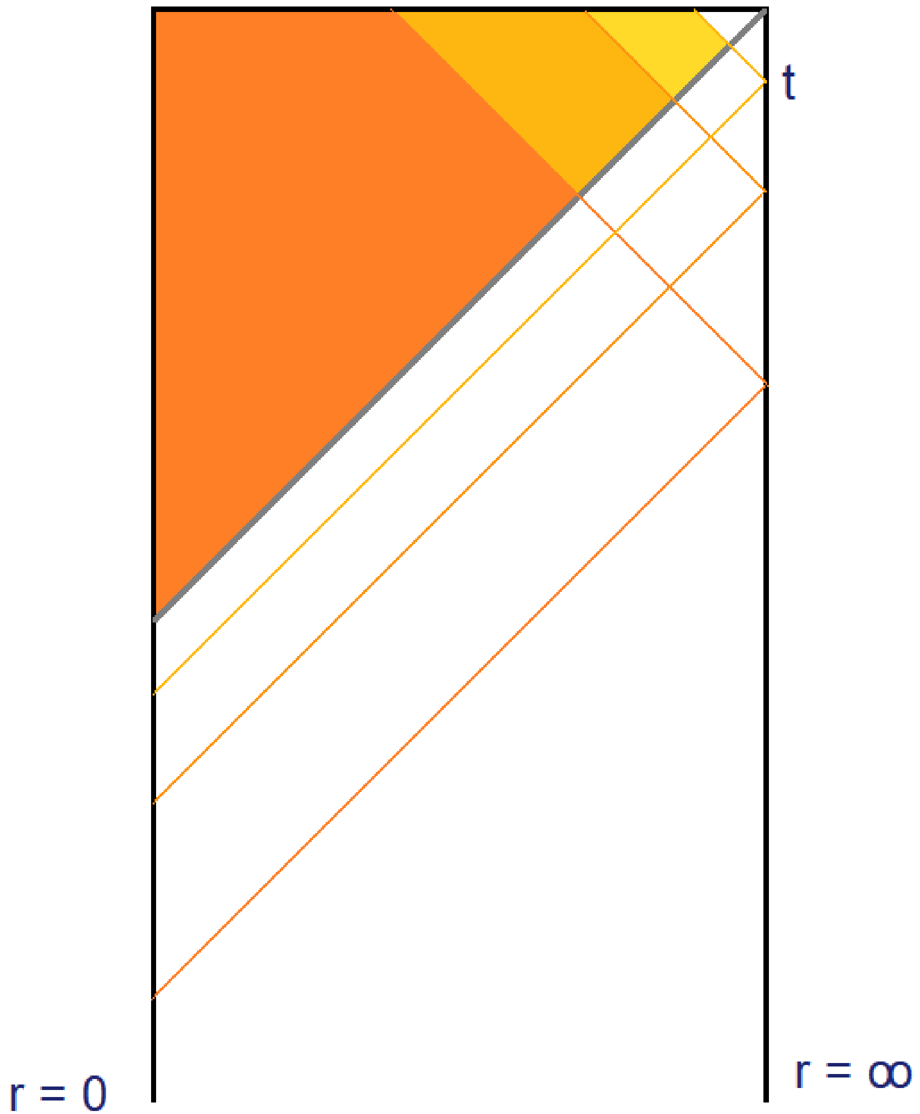
Complexity at time t



Complexity at time t

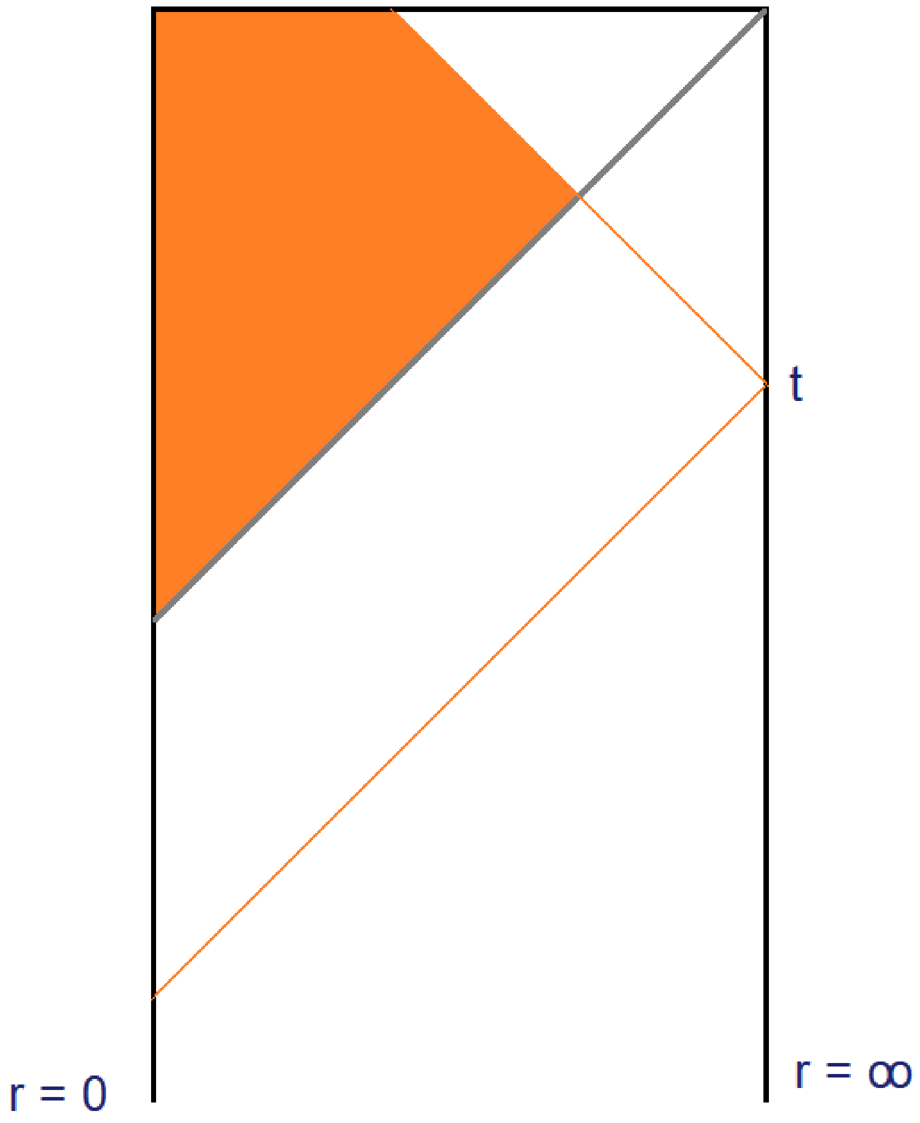


Complexity at time t

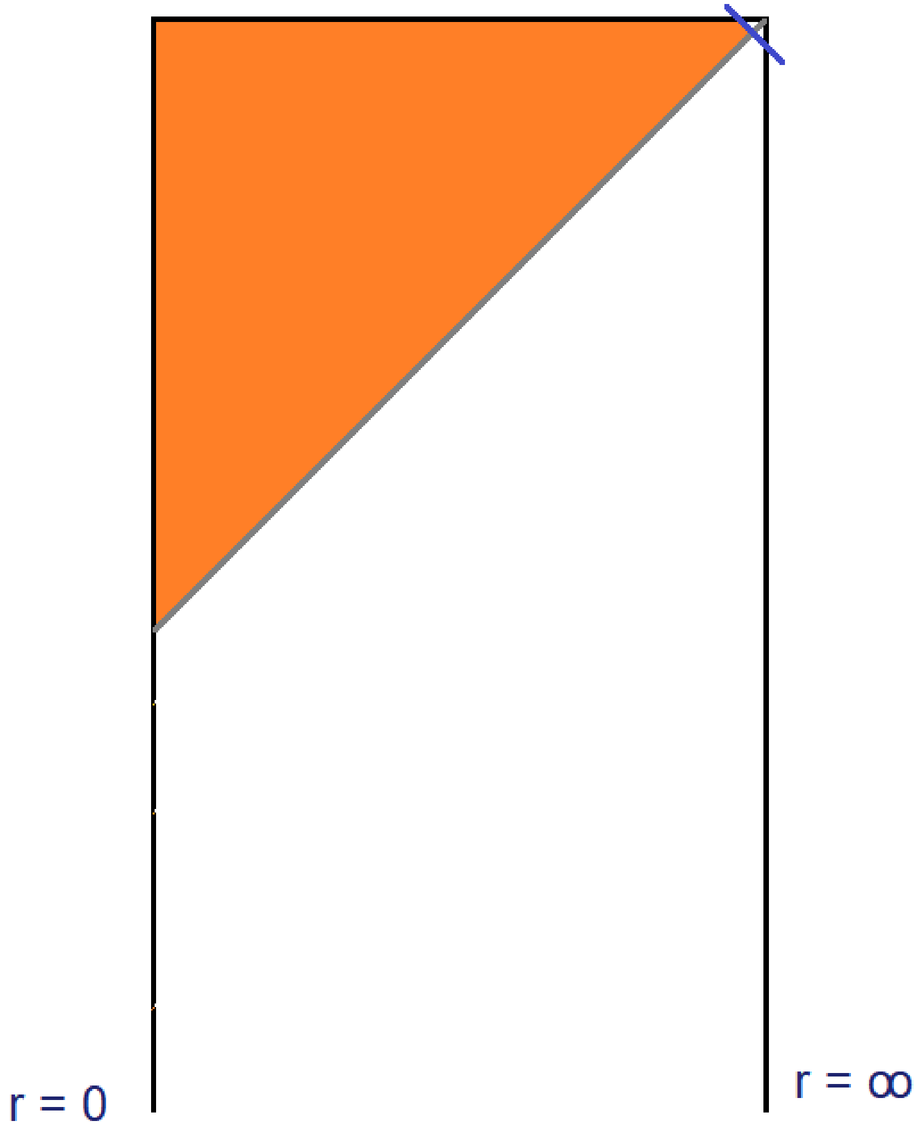


Complexity at time t

Everything Alice cannot experience behind the horizon.

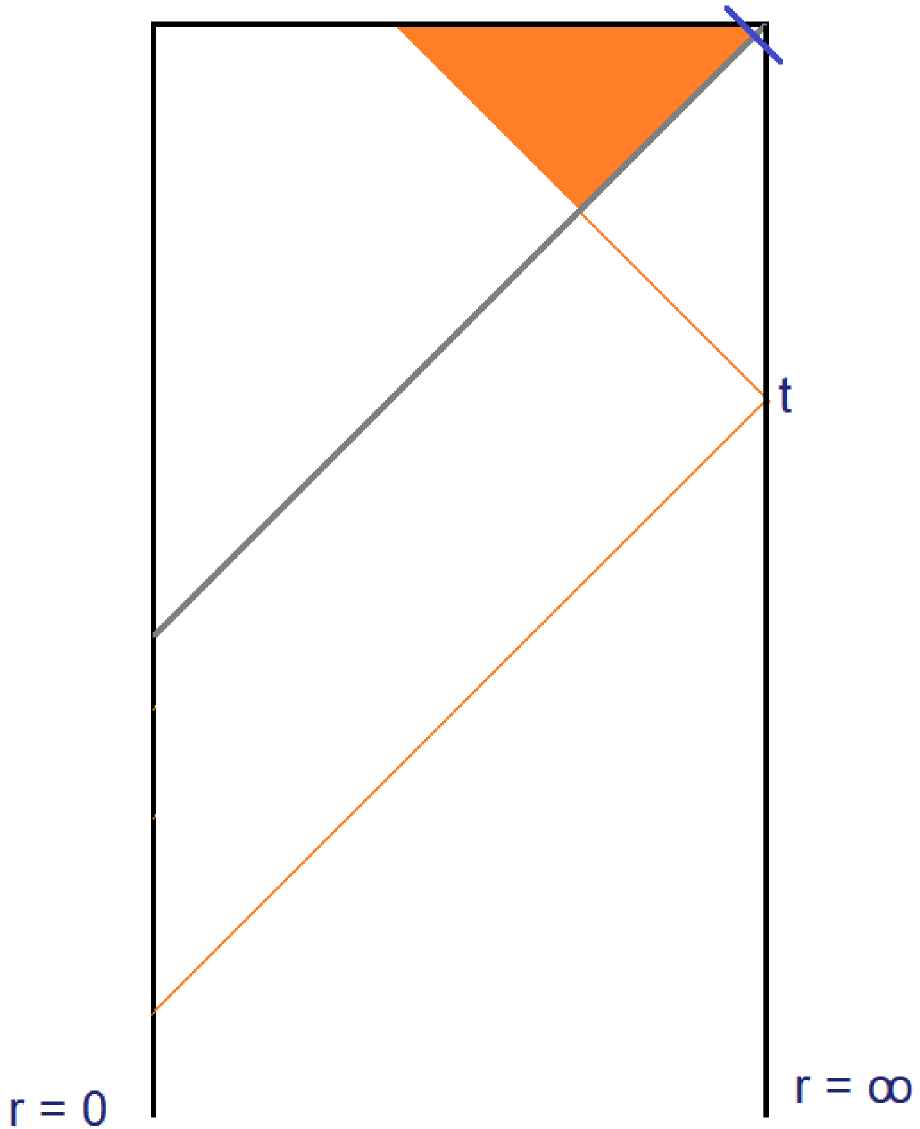


Maximum complexity



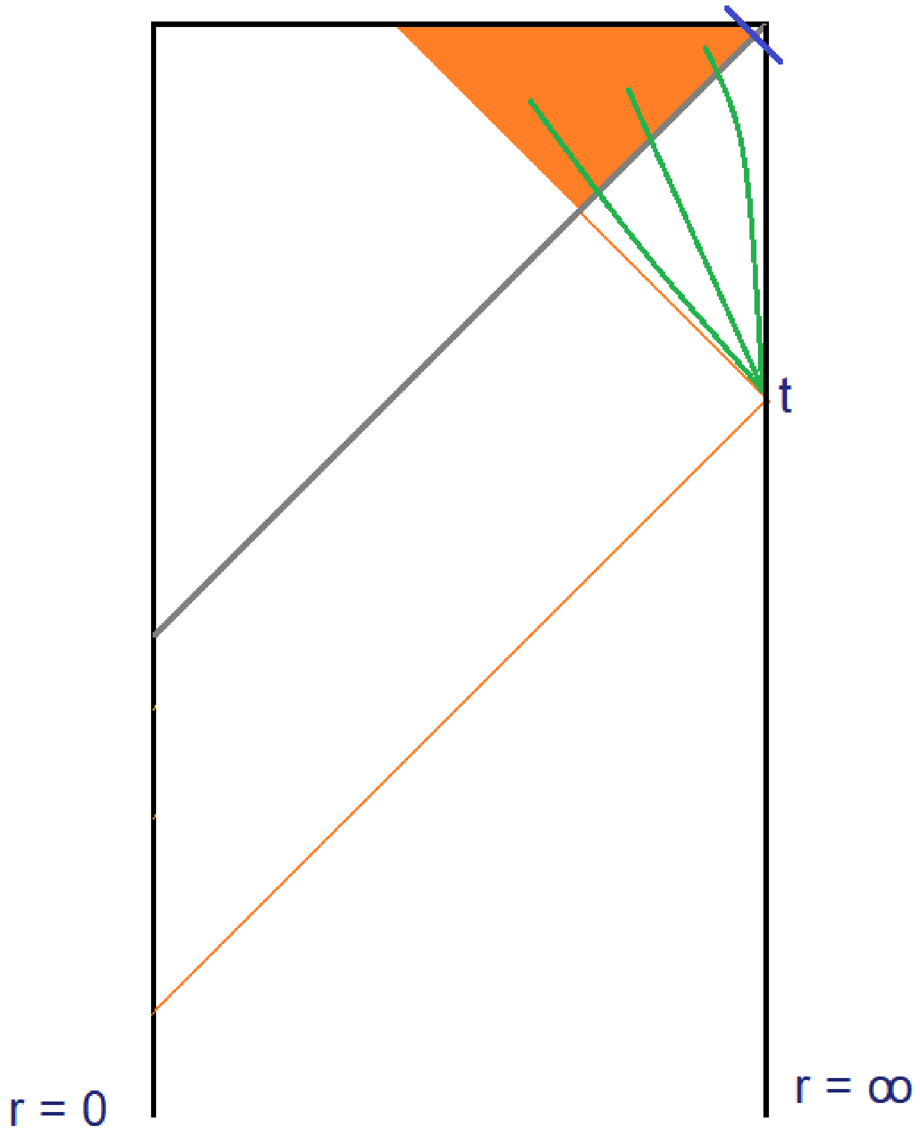
Uncomplexity at time t

$(C_{max} - C)$



Uncomplexity at time t

Everything Alice can experience behind the horizon.



The Uncomplexity available to Alice at time t is the total amount of space-time behind the horizon that she can visit.

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Let’s suppose the black hole is exponentially old so that the resource is completely depleted. It follows that there should be no accessible space-time behind the horizon.

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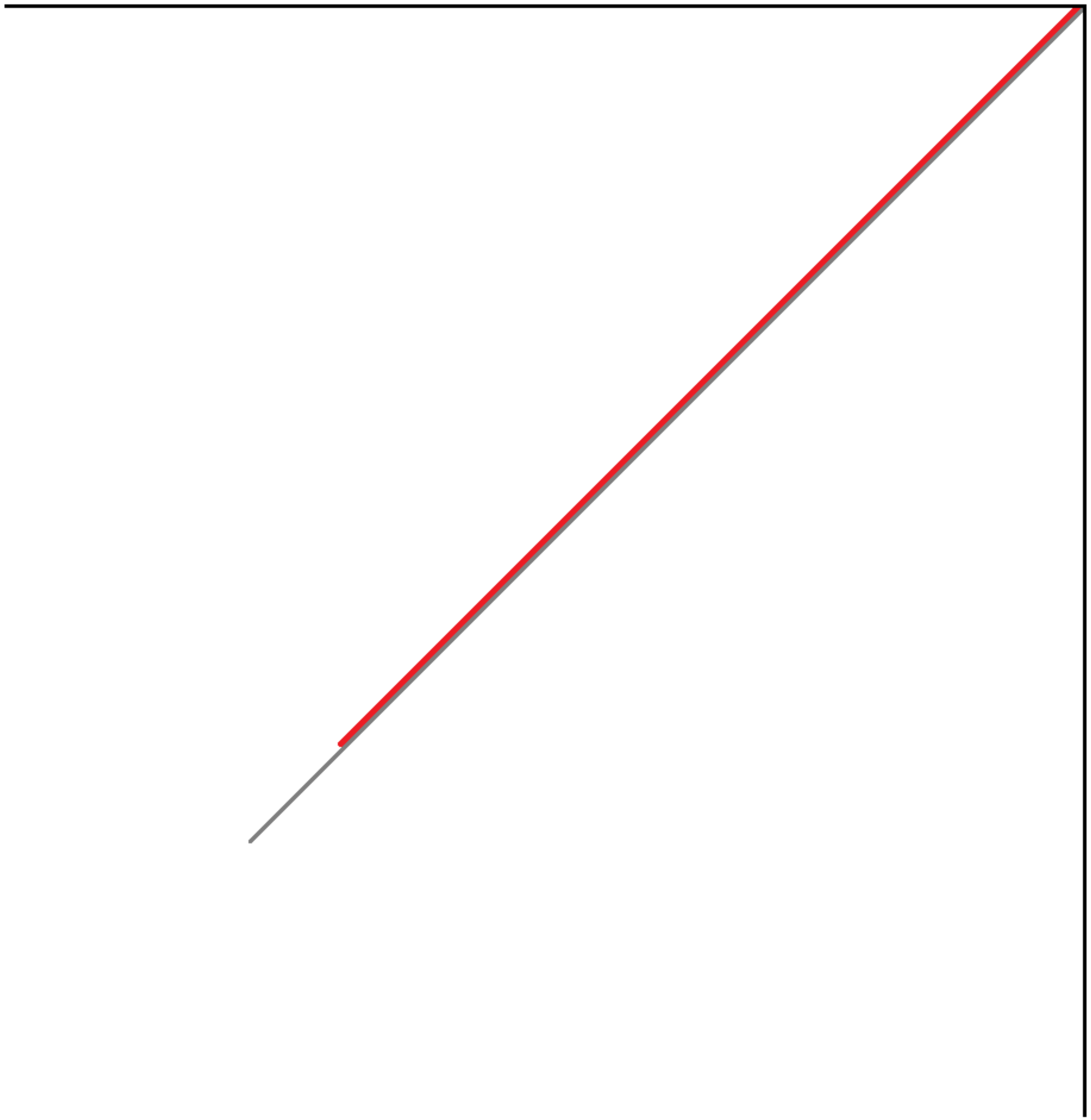
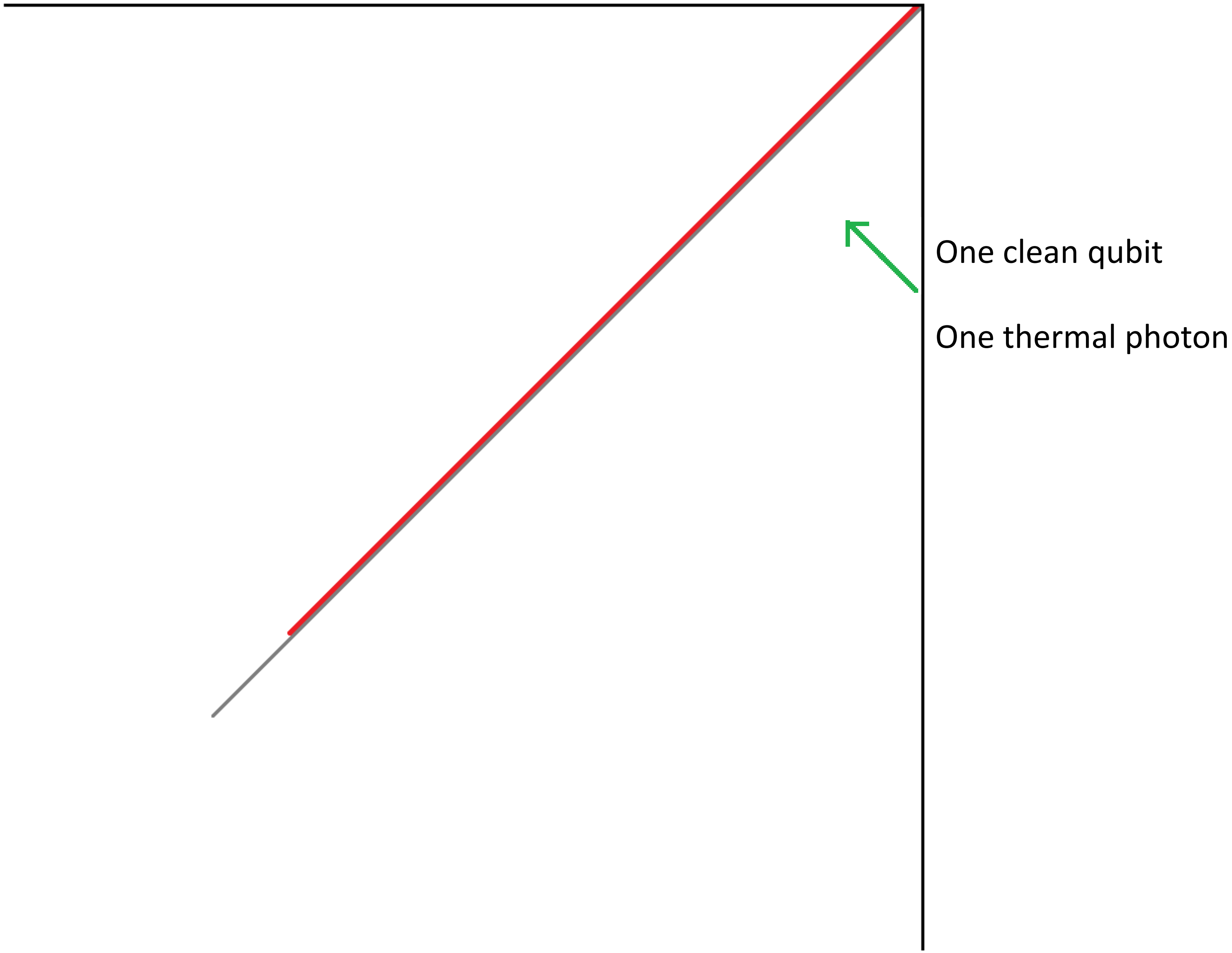
AMPS*: A generic (maximally complex**) black hole state has a firewall,

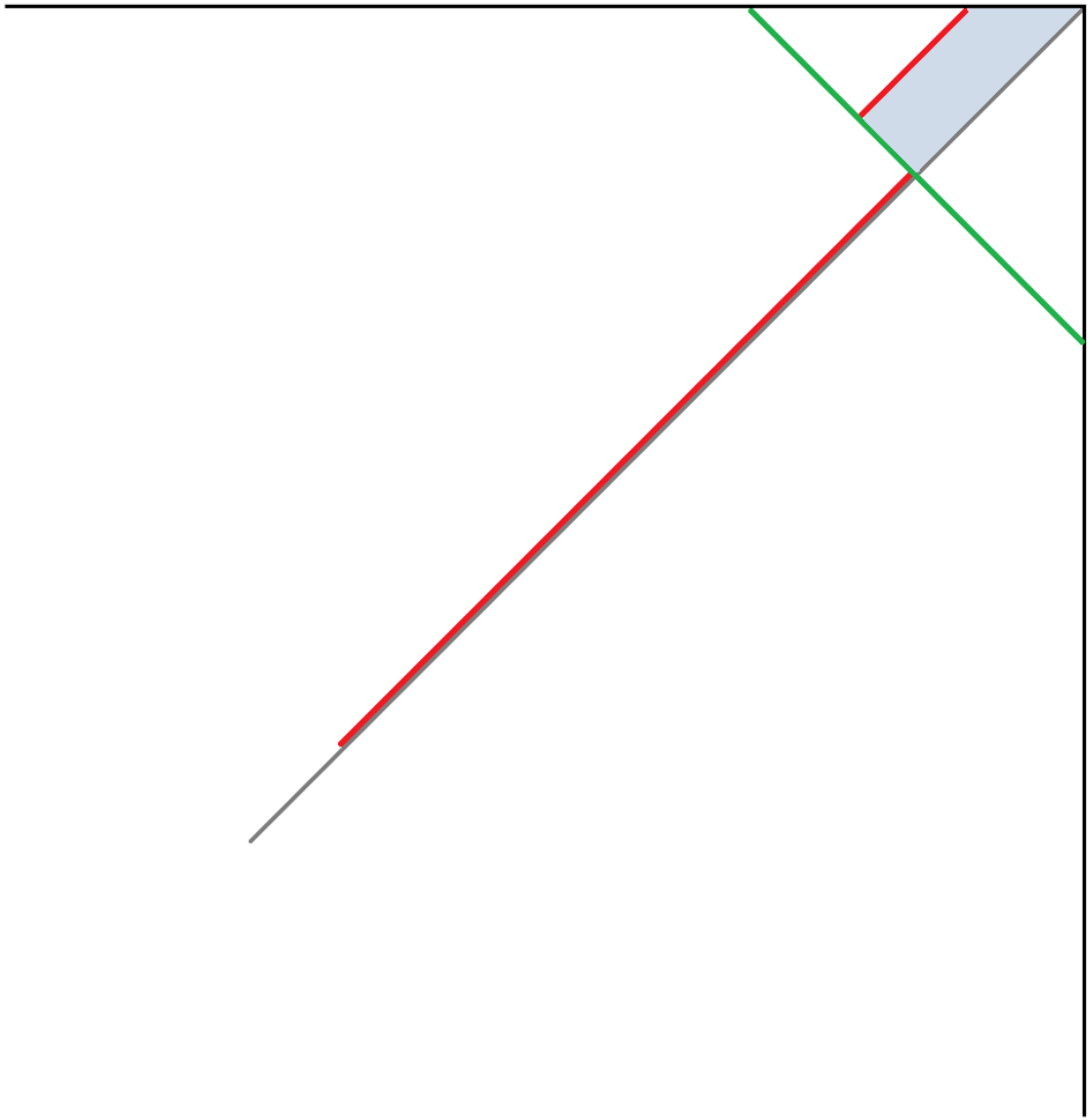
Almheiri, Marolf, Polchinski, Sully, arXiv1207.3123

LS arXiv:1507.02287



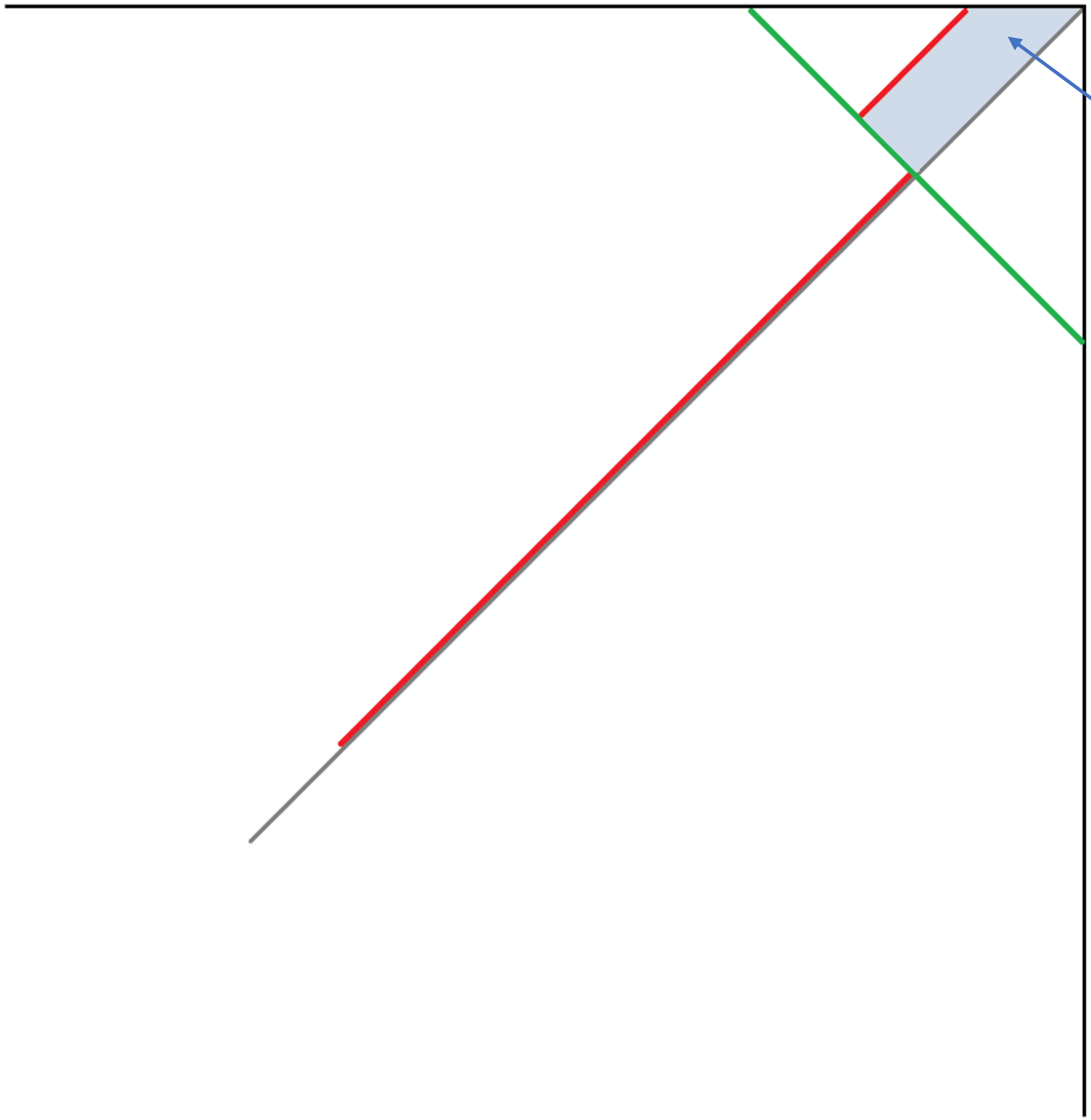
S. Shenker, D. Stanford
arXiv:1306.0622


One clean qubit
One thermal photon

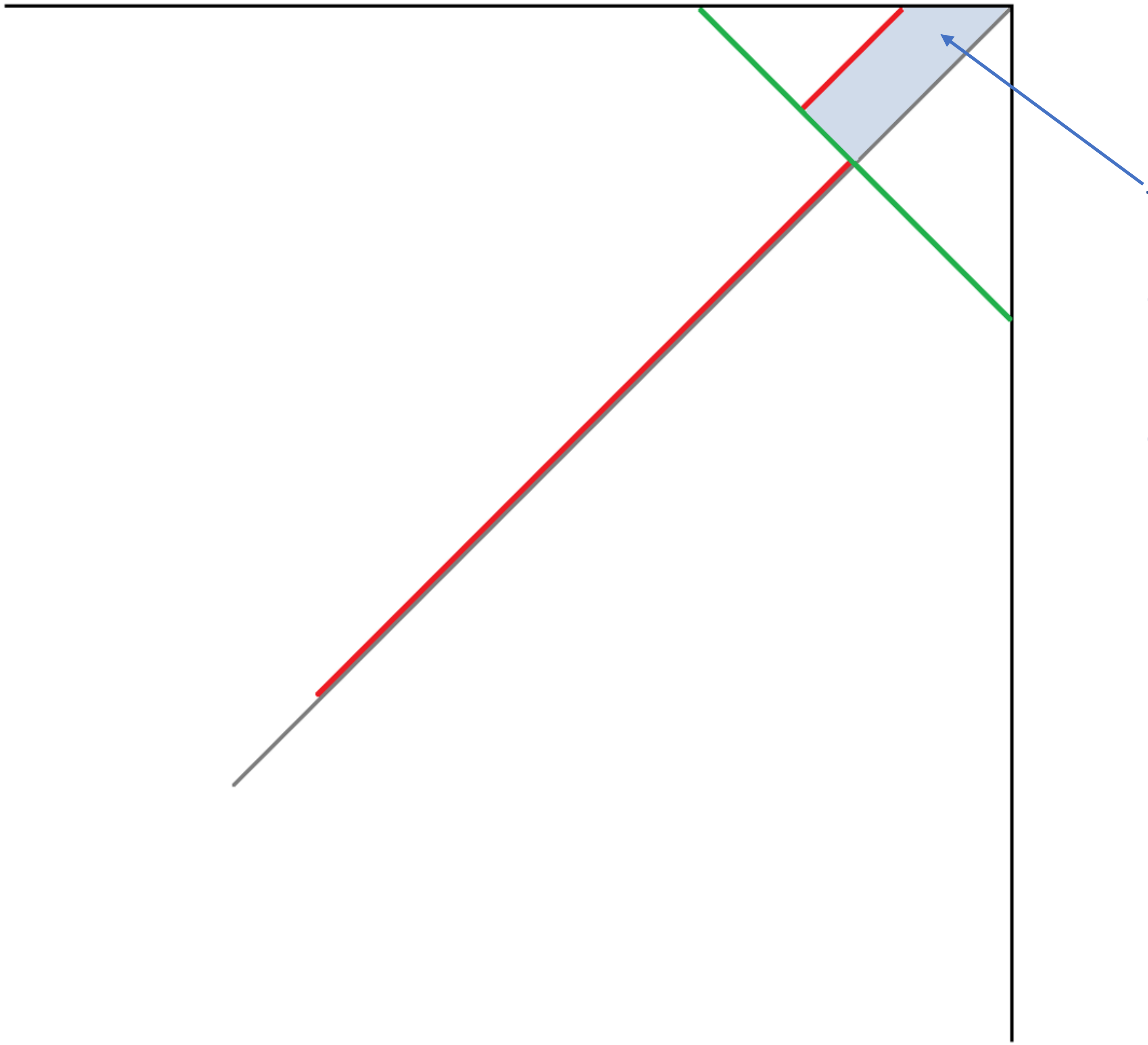


Dray - ‘t Hooft
(pure GR) effect

Also related to
Shenker-Stanford
shock waves.



The black hole interior is
Is rejuvenated for another
exponential time.



The black hole interior is
Is rejuvenated for another
exponential time.

In precise parallel to the
one-clean-qubit effect.

Three tools for thinking about complexity?

1. Thermodynamics of complexity

2. Uncomplexity as a necessary resource for directed computation.

3. Duality between (un)complexity and space-time volume inside black holes.

Three tools for thinking about complexity

1. Thermodynamics of complexity

2. Uncomplexity as a resource for directed computation.

3. Duality between (un)complexity and space-time volume inside black holes.

There is a whole new subject here at the intersection of quantum computation, complexity theory, thermodynamics, and quantum gravity.

Adam Brown, L.S.
The Second Law of Quantum Complexity arXiv:1701.01107

Which of the following are correct?

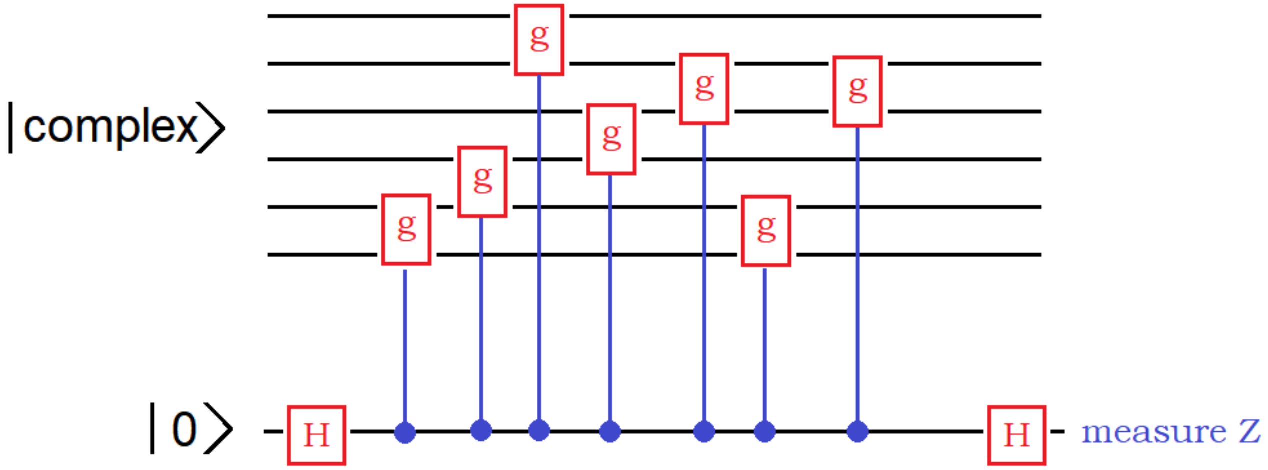
1. By adding a single out-of-equilibrium molecule to a mole of gas in thermal equilibrium you can do enough work to raise a kilogram 10 meters.

2. By adding a single bit to a classical computer in a random state you can restore its computing power to that of a computer in an initial state (000000……0).

3. By throwing a single thermal photon into a black hole with a super-energetic firewall, the black hole interior can be completely restored.

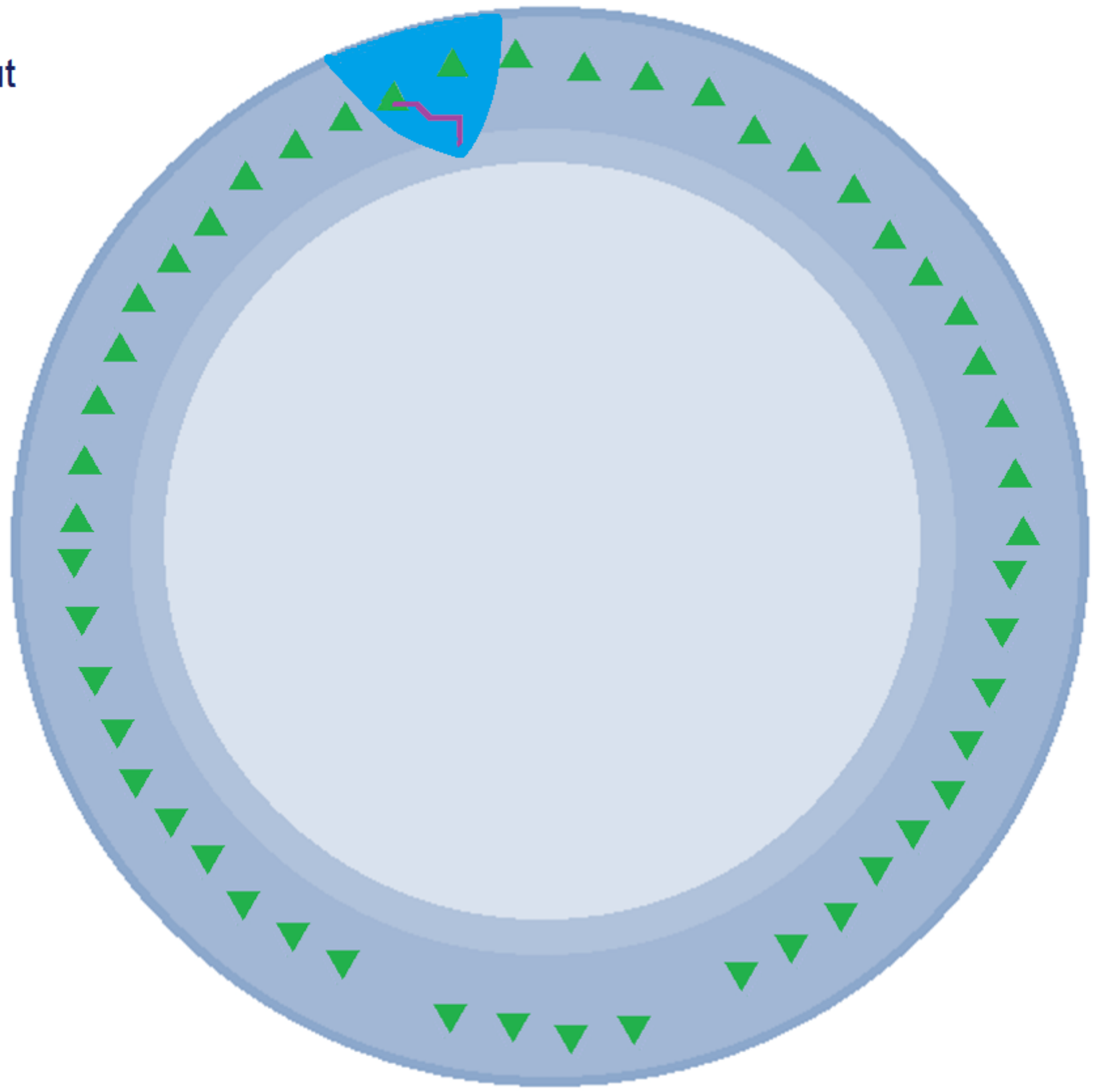
4. By adding a single qubit to a quantum computer in a random state, the computing power can be restored so that exponentially hard problems can be solved.

Uncomplexity is necessary but not sufficient.



One clean qubit is not universal.

It can do some very hard problems but fails at some easy problems.



Free operations

• one can implement any permutation on a system’s physical state space,

• one can prepare any system in the uniform statistical state (i.e. the uniform probability distribution) [and add it to the system],

• one can marginalize over any subsystem of a system.

Resource = neg-entropy

• one can implement any permutation on a system’s physical state space,

• one can prepare any system in the uniform statistical state (maximally complex), [and add it to the system in the controlled additive sense].

• one can average over any subsystem of a system [again in the controlled additive sense].

Resource = neg-complexity = uncomplexity