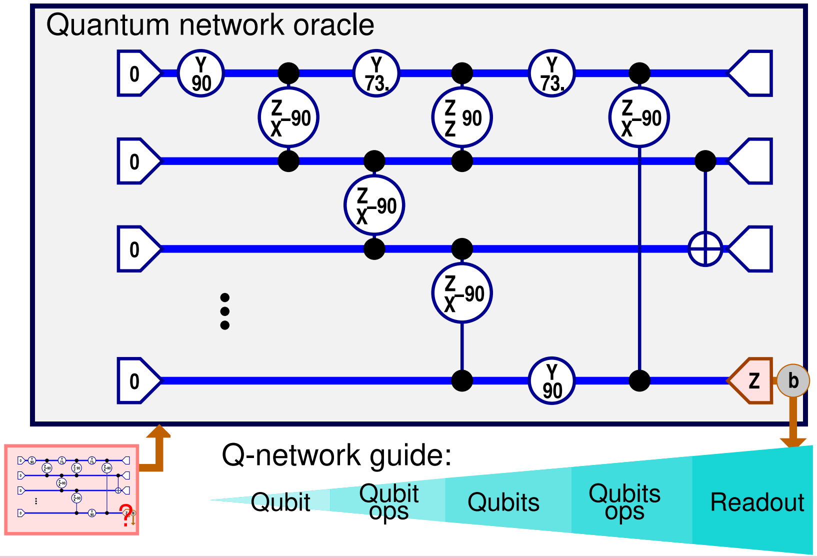
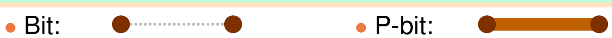
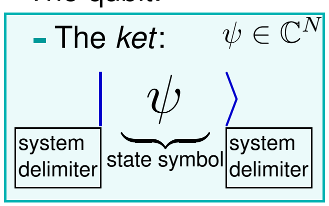
Quantum Networks

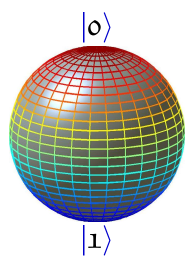


The Qubit



- The qubit:
  - The *ket*: $\psi \in \mathbb{C}^N$





State space: *Superpositions* of *logical* states $|0\rangle$, $|1\rangle$.

$\{\alpha|0\rangle + \beta|1\rangle \mid \alpha, \beta \in \mathbb{C}; |\alpha|^2 + |\beta|^2 = 1\}$ ... *pure* states.

- Kets and vectors.

$\alpha|0\rangle + \beta|1\rangle \leftrightarrow \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ $\quad (\alpha|0\rangle + \beta|1\rangle)^\dagger = \bar{\alpha}\langle 0| + \bar{\beta}\langle 1| \leftrightarrow (\bar{\alpha}, \bar{\beta})$

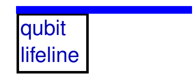
- Projective equivalence: $\alpha|0\rangle + \beta|1\rangle \cong e^{i\phi}(\alpha|0\rangle + \beta|1\rangle)$

Back to: Q. guide

Qubit Operations I

- **noop**:

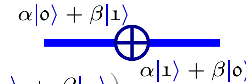
$\mathbb{I} = |0\rangle\langle 0| + |1\rangle\langle 1| \leftrightarrow \begin{pmatrix} |0\rangle 1 \langle 0| & |0\rangle 0 \langle 1| \\ |1\rangle 0 \langle 0| & |1\rangle 1 \langle 1| \end{pmatrix}$



- State preparation:

- The qubit flip (**not**):

$\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$



Ex.: $\sigma_x(\alpha|0\rangle + \beta|1\rangle) = (|1\rangle\langle 0| + |0\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$
$= \alpha|1\rangle + \beta|0\rangle$

- Bra-ket rules: Based on interpretation as vectors.

E.g. $\langle \mathfrak{a}||\mathfrak{b}\rangle = \langle \mathfrak{a}|\mathfrak{b}\rangle = \begin{cases} 0 & \text{if } \mathfrak{a} \neq \mathfrak{b} \\ 1 & \text{if } \mathfrak{a} = \mathfrak{b} \end{cases}$

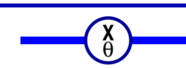
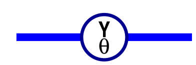
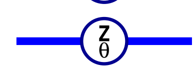
Qubit Operations II

- Pauli operators:

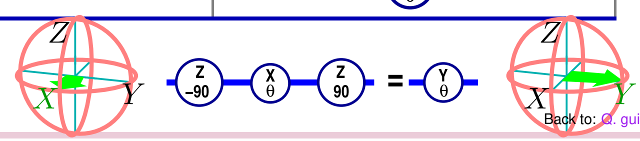
$\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1| \quad \sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1| \quad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$\leftrightarrow \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \leftrightarrow \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \leftrightarrow \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$\sigma_u^2 = \mathbb{I}, \quad \sigma_x\sigma_y = i\sigma_z, \quad \sigma_y\sigma_z = i\sigma_x, \quad \sigma_z\sigma_x = i\sigma_y$

Rotation	Gate
$e^{-i\sigma_x\theta/2} = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2)\sigma_x$	
$e^{-i\sigma_y\theta/2} = \begin{pmatrix}\cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{pmatrix}$	
$e^{-i\sigma_z\theta/2} = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2)\sigma_z$	

- $\mathbb{R}^3$ rotation rules:



Back to: Q. guide

Qubits

- Two qubits, A and B, form system AB:

A's logical states: $|0\rangle_{\mathsf{A}}, |1\rangle_{\mathsf{A}}$ B's logical states: $|0\rangle_{\mathsf{B}}, |1\rangle_{\mathsf{B}}$

AB's logical states:

$|0\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}}, |0\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}}, |1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}}, |1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}}$

AB's state space (pure):

$$\left\{\alpha_{00}|0\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}} + \alpha_{01}|0\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}} + \alpha_{10}|1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}} + \alpha_{11}|1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}} \mid \sum_{\mathfrak{b}} |\alpha_{\mathfrak{b}}|^2 = 1\right\}$$

- Rules: $|\mathfrak{a}\mathfrak{b}\rangle_{\mathsf{AB}} = |\mathfrak{a}\rangle_{\mathsf{A}}|\mathfrak{b}\rangle_{\mathsf{B}} = |\mathfrak{b}\rangle_{\mathsf{B}}|\mathfrak{a}\rangle_{\mathsf{A}}$

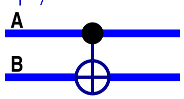
$\left(|\mathfrak{a}\mathfrak{b}\rangle_{\mathsf{AB}}\right)^\dagger = {}^{\mathsf{AB}}\langle\mathfrak{a}\mathfrak{b}| = {}^{\mathsf{A}}\langle\mathfrak{a}|{}^{\mathsf{B}}\langle\mathfrak{b}| = {}^{\mathsf{B}}\langle\mathfrak{b}|{}^{\mathsf{A}}\langle\mathfrak{a}|$

- Many qubits: Multiply kets and superpose . . .

Back to: Q. guide

Two Qubit Gates

- The *controlled-not* ($\mathbf{cnot}$). Flip B if A is in state $|1\rangle$:

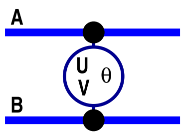


$\mathbf{cnot}_{(\mathsf{AB})} = |0\rangle_{\mathsf{A}}^{\mathsf{A}}\langle 0| + |1\rangle_{\mathsf{A}}^{\mathsf{A}}\langle 1|\,\sigma_{x(\mathsf{B})}$

Ex.: $\mathbf{cnot}_{(\mathsf{AB})}\left(\alpha|0\rangle_{\mathsf{A}} + \beta|1\rangle_{\mathsf{A}}\right)|0\rangle_{\mathsf{B}}$

$= \alpha|0\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}}$

- A "ket" algebra is:
Complex †-algebra generated by the $|\mathfrak{a}\rangle_{\mathsf{A}}$ satisfying
  - Ket/bras with different labels commute.
  - ${}^{\mathsf{N}}\langle\mathfrak{a}|\mathfrak{b}\rangle_{\mathsf{N}} = \delta_{\mathfrak{a},\mathfrak{b}}$
  - $|\mathfrak{a}\rangle_{\mathsf{N}}^\dagger = {}^{\mathsf{N}}\langle\mathfrak{a}|$
- Two-qubit Pauli product rotations.



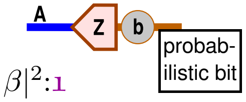
$\left(U_{(\mathsf{A})}V_{(\mathsf{B})}\right)_\theta = e^{-i\sigma_{u(\mathsf{A})}\sigma_{v(\mathsf{B})}\theta/2}$

Ex.: $\left(Z_{(\mathsf{A})}Y_{(\mathsf{B})}\right)_\theta = \cos(\theta/2) - i\sin(\theta/2)\sigma_{z(\mathsf{A})}\sigma_{y(\mathsf{B})}$

Back to: Q. guide

Readout

- Destructively reading a qubit:



$$\alpha|0\rangle + \beta|1\rangle \rightarrow \left\{\begin{array}{ll} 0 & \text{prob. } |\alpha|^2 \\ 1 & \text{prob. } |\beta|^2 \end{array}\right\} \doteq |\alpha|^2{:}0 +\!\!\!\!+ |\beta|^2{:}1$$

Notation with amplitudes:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow \alpha 0 +\!\!\!\!+ \beta 1 = \langle 0|\psi\rangle 0 +\!\!\!\!+ \langle 1|\psi\rangle 1$$
$$= \left(0\langle 0| +\!\!\!\!+ 1\langle 1|\right)|\psi\rangle$$

- Reading qubit A in system AX.
Rewrite $|\psi\rangle_{\mathsf{AX}} = \alpha|0\rangle_{\mathsf{A}}|\psi'_0\rangle_{\mathsf{X}} + \beta|1\rangle_{\mathsf{A}}|\psi'_1\rangle_{\mathsf{X}}$ With: Normalized kets

$$|\psi\rangle_{\mathsf{AX}} \rightarrow \left\{\begin{array}{ll} |\psi'_0\rangle_{\mathsf{X}}0 & \text{prob. } |\alpha|^2 \\ |\psi'_1\rangle_{\mathsf{X}}1 & \text{prob. } |\beta|^2 \end{array}\right\} = \left(0\,{}^{\mathsf{A}}\langle 0| +\!\!\!\!+ 1\,{}^{\mathsf{A}}\langle 1|\right)|\psi\rangle_{\mathsf{AX}}$$

Coefficients: Unnormalized pure states

- Calculating probabilities.

$\mathrm{prob}(0, |\psi\rangle_{\mathsf{AX}}) = |{}^{\mathsf{A}}\langle 0||\psi\rangle_{\mathsf{AX}}|^2 = {}^{\mathsf{AX}}\langle\psi||0\rangle_{\mathsf{A}}^{\mathsf{A}}\langle 0||\psi\rangle_{\mathsf{AX}}$

$\mathrm{prob}(1, |\psi\rangle_{\mathsf{AX}}) = |{}^{\mathsf{A}}\langle 1||\psi\rangle_{\mathsf{AX}}|^2 = {}^{\mathsf{AX}}\langle\psi||1\rangle_{\mathsf{A}}^{\mathsf{A}}\langle 1||\psi\rangle_{\mathsf{AX}}$

Back to: Q. guide

Contents

References

[1] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. *Proc. R. Soc. Lond. A*, 400:97–117, 1985.

[2] D. Deutsch. Quantum computational networks. *Proc. R. Soc. Lond. A*, 425:73–90, 1989.

[3] E. Bernstein and U. Vazirani. Quantum complexity theory. In ?, editor, *Proceedings of the 25th Annual ACM Symposium on the Theory of Computation*, pages 11–20, New York, New York, 1993. ACM press.

[4] A. Yao. Quantum circuit complexity. In ?, editor, *Proceedings of the 34th Annual Symposium on Foundations of Computer Science*, pages 352–360, Los Alamitos, California, 1993. IEEE Press.

[5] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin, and H. Weinfurter. Elementary gates for quantum computation. *Phys. Rev. A*, 52:3457–3467, 1995.

[6] D. Aharonov. Quantum computation. In Dietrich Stauffer, editor, *Annual Reviews of Computational Physics VI*. World Scientific, Singapore, 1999. quant-ph/9812037.

[7] R. Cleve. An introduction to quantum complexity theory. quant-ph/9906111, 1999.

[8] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2001.

[9] E. Knill and R. Laflamme. Introduction to quantum information processing. Technical Report LAUR-01-4761, Los Alamos National Laboratory, 2001. To appear in LA Science.