Tutorial: quantum information & quantum optics / atomic physics

Peter Zoller
University of Innsbruck,
Institute for Theoretical Physics
Road Map

• our goal is to implement quantum networks

- nodes: local quantum computing
  - quantum processors with atoms
- channels: communication
  - transmission of photons

• quantum optics toolbox
  - photons
  - atoms
Quantum information processing

- quantum computing
  \[
  |\psi_{\text{out}}\rangle = \hat{U}|\psi_{\text{in}}\rangle
  \]

- quantum communications
  \[
  |\phi\rangle 
  \]

quantum weirdness:
- superposition
- entanglement
- interference
- nonclonability and uncertainty
- no decoherence!

- teleportation
- cryptography
Qubits, Quantum Gates etc.

- quantum bits or qubits
  \[ |0\rangle, |1\rangle, \ldots, |n\rangle \text{ spin-1/2} \]
  \[ |\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle \]

- single qubit gate
  \[ |\psi'\rangle = \hat{U}_1|\psi\rangle \]

- two qubit gate
  \[ |a\rangle \xrightarrow{\text{control}} |a\rangle \]
  \[ |b\rangle \xrightarrow{\text{target}} |a \oplus b\rangle \text{ Controlled-NOT} \]
Examples from quantum optics

- ion traps '95

- neutral atoms:

  \[ V_{\text{dip}}(\mathbf{r}) \]

  interacting Rydberg dipoles

- cavity QED

These systems realize manipulation on the single quantum level.
Quantum computing: check list

- quantum memory
- single qubit gate
- two-qubit gate
  - quantum control
- preparation
- read out
  - "controlled decoherence"
- decoherence

issues
- zero vs. finite temperature
- single system vs. ensemble
- scalability
- robustness
- speed
- simplicity
- ...
Atoms as quantum Memory

single trapped atom:

\[ |0\rangle \quad |1\rangle \]

qubit in long-lived internal states

or: atomic ensembles:
• trapping and cooling of atoms and ions

- linear ion trap
- atom "chips"
- laser traps
- arrays of microtraps: optical lattice
- FORT

- traps: conservative potential
- load the trap
- laser cooling: prepare motional ground state

Remarks: long coherence times demonstrated.
Single qubit gates

\[ \Omega_{\text{eff}} = \frac{1}{4} \frac{\Omega_1 \Omega_2}{\Delta} \]

\[ \Gamma_{\text{eff}} \sim \Gamma \frac{1}{4} \frac{\Omega_{1,2}^2}{\Delta^2} \]

Remarks:

✓ state preparation by optical pumping
✓ decoherence due to spontaneous emission (and collisions)
Two-qubit gates

• implement entanglement of two qubits

\[
\begin{align*}
|00\rangle & \rightarrow |00\rangle \\
|01\rangle & \rightarrow |01\rangle \\
|10\rangle & \rightarrow |10\rangle \\
|11\rangle & \rightarrow e^{i\phi}|11\rangle
\end{align*}
\]

• How?

✓ auxiliary collective mode as data bus
✓ controllable two body interactions
✓ dynamical phases
✓ geometric phases (holonomic quantum computing)
Two-qubit gates via quantum databus

- Entanglement via collective auxiliary quantum degree of freedom

Examples:
Ion traps (Cirac and Zoller PRL '95)
Cavity QED (Pellizzari et al PRL '96)

state vector:
\[ |\psi\rangle = \sum_{\{x\}} c_x |x_{N-1}x_{N-2} \ldots x_0 \rangle \otimes \text{collective mode} \]

gate:

requirement: cooling of the collective mode (= prepare a pure state)
Two-qubit gates via two-body interactions

- Controlled two-body interaction

We must design a Hamiltonian

\[ H = \Delta E(t) |1\rangle_1 \langle 1| \otimes |1\rangle_2 \langle 1| \]

so that

\[ |1\rangle_1 \otimes |1\rangle_2 \rightarrow e^{i\phi} |1\rangle_1 \otimes |1\rangle_2 \]

Examples:
- Cold collisions (Jaksch et al PRL '99)
- Ion trap 2000 (Cirac and Zoller, Nature 2000)
- Rydberg gate (Jaksch et al. PRL 2000)
Ion Trap Quantum Computer '95

J. I. Cirac, P. Zoller  PRL '95

- Cold ions in a linear trap

Qubits: internal atomic states
Quantum gates: entanglement via exchange of phonons of quantized center-of-mass mode

- State vector

\[ |\Psi\rangle = \sum c_x |x_{N-1}, \ldots, x_0\rangle_{\text{atom}} |0\rangle_{\text{phonon}} \]

quantum register \hspace{1cm} \text{databus}

- QC as a time sequence of laser pulses
- Read out by quantum jumps (100 % efficiency)
Physics behind the quantum gate

- Swap the qubit (internal state) to phonon data bus

\[(\alpha |0\rangle_A + \beta |1\rangle_A )|0\rangle_{\text{phonon}} \xrightarrow{\text{laser}} |0\rangle_A (\alpha |0\rangle_{\text{phonon}} + \beta |1\rangle_{\text{phonon}})\]

- Two qubit gate
The Innsbruck Linear Ion Trap

Achievements & expectations:
- storing ~ 10 qubits
- ground state cooling
- addressing single ions
- two qubit gate ... soon!?
- few qubits within the next years?
- decoherence times: ~ 40 gates

Boulder Linear Ion Trap

C. Monroe, D. Wineland et al., Nature 2000

- Lithographic trap
- Maximally entangled state of N=4 ions

\[ |0000\rangle + |1111\rangle \]

Mølmer - Sørensen protocol
(no individual addressing)

- Bell state measurements
- Two atom entangled state interferometry
A fast 2-qubit gate with Rydberg atoms


- Rydberg atom in constant electric field
  - setup
  - linear Stark effect
  - permanent dipole moment

- Large dipole-dipole interaction

\[ \mu \sim n^2 \text{ huge!} \]

\[ E \sim 1 \text{kV/cm} \]
\[ R \sim \lambda_{opt}/2 \sim 300 \text{ nm} \]
\[ \Delta E \sim 60 \text{ GHz} \]
\[ V_{\text{dip}} \sim 4 \text{ GHz} \]

for \( n \sim 15 \)
Two-qubit gate: internal dynamics

- atomic configuration

<table>
<thead>
<tr>
<th>Atom 1</th>
<th>Atom 2</th>
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<tbody>
<tr>
<td>$</td>
<td>r\rangle_1$</td>
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<td>0\rangle_1$</td>
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<td>$</td>
<td>1\rangle_1$</td>
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- two qubit gate

$|11\rangle \rightarrow |11\rangle$

$|10\rangle \rightarrow |10\rangle$

$|01\rangle \rightarrow |01\rangle$

$|00\rangle \rightarrow e^{i\phi}|00\rangle$

- dipole – dipole interaction
  = phase shift

- force !? 😐
**Scheme:** \( \Omega_{1,2} \ll u \)  
gate time \( \Delta t \sim 2\pi/\Omega_{1,2} \ll 1/v_{\text{trap}} \)

- Energy levels of two atom 1 + atom 2

  atom 1

  ![Diagram of atom 1](image)

  atom 2

  ![Diagram of atom 2](image)

- Laser pulse sequence

  ![Diagram of laser pulse sequence](image)

Large dipole-dipole interaction shifts level off resonance:

No double excitation - no force!

"dipole blockade"
• Laser pulse 1

atom 1 atom 2

\[ |rr\rangle \]

\[ V_{\text{dip}} = u \]

\[ \Omega_1 \]

\[ |1I\rangle \quad |1r\rangle \quad |r0\rangle \quad |0r\rangle \]

\[ |11\rangle \quad |01\rangle \quad |10\rangle \quad |00\rangle \]

\[ \Omega_1 \]

\[ \pi \quad 2\pi \quad \pi \]

\[ \text{time} \]
Laser pulse 2

atom 1 atom 2

\[ |rr\rangle \quad V_{\text{dip}} = u \]

\[ \Omega_2 \]

\[ |r1\rangle \quad |1r\rangle \quad |r0\rangle \quad |0r\rangle \]

\[ |11\rangle \quad |01\rangle \quad |10\rangle \quad |00\rangle \]

\[ \Omega_1 \quad \Omega_2 \quad \Omega_1 \]

\[ \pi \quad 2\pi \quad \pi \]

time

No excitation!

"dipole blockade"

minus sign!
• Laser pulse 3

atom 1  atom 2

\[ |rr\rangle \]

\[ V_{\text{dip}} = u \]

<table>
<thead>
<tr>
<th>Fast phase gate</th>
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\( \checkmark \) no force
Atomic ensembles

M Lukin, L.M. Duan et al., PRL 2001

- mesoscopic atomic ensembles (instead of microscopic quantum objects)

- coherent manipulation of collective excitations of atomic ensembles

- underlying physics:
  dipole blockade
Manipulating collective excitations

- ground state
  \[ |g^N\rangle = |g_1\rangle|g_2\rangle \ldots |g_N\rangle \]

- one excitation (Fock state)
  \[ |g^{N-1}q\rangle \sim \sum_i |g_1\rangle \ldots |q_i\rangle \ldots |g_N\rangle \]

- two excitations
  \[ |g^{N-2}q\rangle \sim \sum_{i,j} |g_1\rangle \ldots |q_i\rangle \ldots |q_j\rangle \ldots |g_N\rangle \]

We can store and manipulate qubits.
cont.

- qubits

\[ |\psi\rangle = \alpha |g^N\rangle + \beta |g^{N-1}q\rangle \]

- entanglement of ensembles

superposition
Atomic ensembles: quantum memory for light

• purpose

incoming light pulse \quad \text{Atomic ensemble} \quad \text{outgoing light pulse}

✓ unknown (arbitrary) state
✓ known shape of wave packet

• how? example ...

theory: Lu Ming Duan; M. Lukin and M. Fleischhauer PRL '00
exp: R. Walsworth et al, PRL '01
      L. Hau et al., Nature '01
Teleportation with coherent light + atomic ensembles

- **theory:** Lu Ming Duan, J.I. Cirac, P.Z. and E. Polzik, PRL Dec 2000
- **experiment:** E. Polzik et al., Nature 2001

**features**
- so far: quantum computing and communications requires
  - ✓ single atoms and single photons
  - ✓ high-Q cavities
- now: can we get away with ...
  - ✓ atomic ensembles?
  - ✓ free space?
- continuous variable quantum communications
- we use *measurements* to generate EPR states
Quantum Communication

• the goal of quantum communication is to transmit a quantum state reliably

Alice  |φᵢ⟩  Bob

transmission of a quantum state

• The quantum state can be a qubit

|φ⟩ = α|0⟩ + β|1⟩

or a continuous variable quantum state

|φ⟩ = ∫ₓ⁻∞ dx |x⟩ψ(x)  \[\hat{x}\] … position  \[\hat{p}\] … momentum  \[[\hat{x},\hat{p}] = i\]
Teleportation of Continuous Quantum Variables

- continuous variable quantum states

\[ |\phi\rangle = \int dx |x\rangle \phi(x) \]

\[ \hat{x} \quad \text{... position} \]

\[ \hat{p} \quad \text{... momentum} \]

\[ [\hat{x}, \hat{p}] = i \]

- continuous variable protocol

✓ prepare an EPR state as a resource

\[ |\text{EPR}\rangle \sim \int dP |P\rangle_A - P\rangle_B = \int dX |X\rangle_A |X\rangle_B \]

eigenstate of commuting operators:

\[ \langle \hat{X}_A - \hat{X}_B |\text{EPR}\rangle = X_1 |\text{EPR}\rangle \]

\[ \Delta (\hat{X}_A - \hat{X}_B)^2 \to 0 \]

\[ \langle \hat{P}_A + \hat{P}_B |\text{EPR}\rangle = P_1 |\text{EPR}\rangle \]

\[ \Delta (\hat{P}_A + \hat{P}_B)^2 \to 0 \]
cv teleportation cont.

- The initial state is

\[ |\psi_{\text{in}}\rangle = |\psi\rangle_T \otimes |\text{EPR}\rangle_{AB} \]

- We make a Bell measurement

\[ \hat{X}_T - \hat{X}_A \rightarrow X_2 \]
\[ \hat{P}_T + \hat{P}_A \rightarrow P_2 \]

result of measurement:

\[ |\psi_{\text{out}}\rangle \sim |X_2, P_2\rangle_{TA} \otimes_{TA} \langle X_2, P_2|\psi_{\text{in}}\rangle \]
\[ \sim |X_2, P_2\rangle_{TA} \otimes e^{iX_2\hat{P}_B} e^{-iP_2\hat{X}_B} |\psi\rangle_B \]

apart from a displacement B is in \( |\psi\rangle \)

- Classical communication and displacement

\[ |\psi_{\text{out}}\rangle \sim |X_2, P_2\rangle_{TA} \otimes |\psi\rangle_B \]

We have teleported the state.
Physical realization with squeezed light

Braunstein and Kimble '96, Caltech exp '98

• squeezed vacuum as a cv EPR state

\[ |EPR\rangle = e^{r(a^\dagger b^\dagger - ab)}|0\rangle_A \otimes |0\rangle_B \]

\[ = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n\rangle_A \otimes |n\rangle_B \quad (\lambda = \tanh r) \]

with quadrature components
\[ a = \frac{1}{\sqrt{2}}(\hat{X}_A + i\hat{P}_A) \] with \([a, a^\dagger] = 1\)

– position representation: Gaussian

\[ \langle X_A, X_B|E \rangle \sim \exp(-(X_A + X_B)^2e^{-2r} - (X_A - X_B)^2e^{+2r}) \]

– variances

\[ \Delta(\hat{X}_A - \hat{X}_B)^2 = e^{-2r} \]
\[ \Delta(\hat{P}_A + \hat{P}_B)^2 = e^{-2r} \]
Implementation of cv teleportation

1. EPR: two mode squeezed light
   - parametric downconversion
   - EPR source
   - quantum variables
     - quadrature components
   - EPR correlations
     - light squeezing

1. atomic / spin squeezing
   - coherent light
   - atoms 1
   - atoms 2
   - EPR
   - measure
   - atomic collective "spin"
   - spin squeezing

• Questions:
  interactions? noise?
teleportation cont.

2. Bell measurement

- Questions:
  - efficiency of Bell measurement
  - can we do better than with light?
**Scheme**

- **atomic level scheme**
  
  $$|3\rangle \quad \quad \quad \quad |4\rangle$$
  
  $$|1\rangle \quad |2\rangle$$

  $$\sigma^+ \times \sigma^-$$

- **incident light:**
  - coherent
  - linear light polarization

  $$e_x = \sigma^+ + \sigma^-$$

- **atoms initially:**
  - prepared in superposition

  $$M = -\frac{1}{2}$$

  $$M = +\frac{1}{2}$$
Scheme

- atomic level scheme
  \[ |3\rangle \quad \sigma^+ \quad \sigma^- \quad |4\rangle \]
  \[ |1\rangle \quad |2\rangle \]
  \[ M = -\frac{1}{2} \quad M = +\frac{1}{2} \]

- incident light:
  - coherent
  - linear light polarization
  \[ e_x e^\dag = \sigma^+ + \sigma^- \]

- atoms initially:
  - prepared in superposition

- Remarks:
  - light propagation problem
  - spontaneous emission noise
1. Atoms as a continuous variable quantum system

- continuous atomic operators

\[
\sigma_{\mu
\nu}(z, t) = \lim_{\delta z \to 0} \frac{1}{\rho A \delta z} \sum_{i}^{z \leq z_i < z + \delta z} |\mu\rangle_i \langle \nu|
\]

- collective spin operators for the atomic ground states

\[
\begin{align*}
S_x^a &= \frac{\rho A}{2} \int_0^L \left( \sigma_{12} + \sigma_{12}^\dagger \right) dz \\
S_y^a &= \frac{\rho A}{2i} \int_0^L \left( \sigma_{12} - \sigma_{12}^\dagger \right) dz \\
S_z^a &= \frac{\rho A}{2} \int_0^L \left( \sigma_{11} - \sigma_{22} \right) dz
\end{align*}
\]

number of atoms \(2N_a = \rho AL \gg 1\)
atoms cont.

- superposition of the two ground states: coherent spin state

\[
\left| 1 \right\rangle \left( \frac{1}{\sqrt{2}} \left( \left| 1 \right\rangle + \left| 2 \right\rangle \right) \right)^\otimes N
\]

Bloch vector

\[
\langle \vec{S}^a \rangle = (\langle S_x^a \rangle, \langle S_y^a \rangle, \langle S_z^a \rangle) = \left( \frac{N_a}{2}, 0, 0 \right)
\]

- quantum fluctuations

\[
[S_y^a, S_z^a] = iS_x^a \quad \Delta S_y^a \Delta S_z^a \geq \frac{1}{2} \left| \langle S_x^a \rangle \right|
\]

we treat $S_x^a$ classically and rescale

\[
[X^a, P^a] = i \quad \Delta X^a \Delta P^a \geq \frac{1}{2}
\]

canonical commutation relations

- coherent spin state = vacuum state
- there are many cv quantum states around it:

\[
|\psi^a\rangle = \int dX^a |X^a\rangle \psi(X^a)
\]
2. Light (polarization) as cv quantum system

- one dimensional light propagation problem: polarization

\[ E^{(±)}(z, t) = \sqrt{\frac{\hbar \omega_0}{4 \pi \epsilon_0 A}} \sum_{i=\sigma^+,\sigma^-} a_i(z, t) e^{i(k_0 z - \omega_0 t)} \]

\[ [a_i(z, t), a_j^\dagger(z', t)] = \delta_{ij} \delta(z - z') \]

- input
  strong coherent input with linear polarization
  \[ \langle a_i(0, t) \rangle = \alpha_t \]
  photon number
  \[ 2N_p = 2c \int_0^T |\alpha_t|^2 \, dt \gg 1 \]

- Stokes parameter

\[ S_{x}^{p} = \frac{c}{2} \int_0^T (a_1^\dagger a_2 + a_2^\dagger a_1) \, d\tau \]

\[ S_{y}^{p} = \frac{c}{2i} \int_0^T (a_1^\dagger a_2 - a_2^\dagger a_1) \, d\tau \]

\[ S_{z}^{p} = \frac{c}{2} \int_0^T (a_1^\dagger a_1 - a_2^\dagger a_2) \, d\tau \]
light cont.

• input and output

\[ e_x \uparrow \quad \text{atoms} \quad \rightarrow z \]

Rem.: free field \( a_i(z, t) = a_i(t - z/c) \)

commutation relations for a free field:

\[ [S_y, S_z] = iS_x \]

• for linear light polarization the expectation value of the Stokes vector is

\[ \langle \vec{S}^p \rangle = (\langle S_x^p \rangle, \langle S_y^p \rangle, \langle S_z^p \rangle) = (\frac{N_p}{2}, 0, 0) \]

we treat \( S_x^p \rightarrow \langle S_x^p \rangle = N_p \) classically

\[ [X^p, P^p] = i \]

\[ |\psi^p\rangle = \int dX^p |X^p\rangle \psi(X^p) \]
3. Dynamics: quantum noise propagation problem

- Maxwell-Bloch equations

\[
\frac{\partial}{\partial z} a_i(z, \tau) = i \frac{|g|^2 \rho A \sigma_{ii}}{\Delta c} a_i(z, \tau) - \frac{|g|^2 \rho A \gamma \sigma_{ii}}{2\Delta^2 c} a_i(z, \tau) + \text{noise}
\]

\[
\frac{\partial}{\partial \tau} \sigma_{12} = \frac{i |g|^2 (a_2^\dagger a_2 - a_1^\dagger a_1)}{\Delta} \sigma_{12} - \frac{|g|^2 \gamma'}{2\Delta^2} (a_2^\dagger a_2 + a_1^\dagger a_1) \sigma_{12} + \text{noise}
\]

approx. by a spatial mean field

H \sim S_{z}^a S_{z}^p \sim P_{p} P_{a} \quad \text{Kuzmich Polzik}

effective Hamiltonian: dispersive interaction
Dynamics: effective Hamiltonian

- We model the atom-light interaction as effective time evolution

\[
X_p \rightarrow X_p' = U_{\text{eff}}^{\dagger} X_p U_{\text{eff}} \\
P_p \rightarrow P_p' = U_{\text{eff}}^{\dagger} P_p U_{\text{eff}}
\]

- Whole system + process

\[
U_{\text{eff}} = \exp(\kappa P_p P_a)
\]

- We can describe dynamics in Heisenberg or Schrödinger picture
### 3.1 Dynamics in the Heisenberg picture

- **effective dynamics**

\[
\begin{align*}
X_p &\rightarrow X_p' = U_{\text{eff}}^* X_p U_{\text{eff}} \\
P_p &\rightarrow P_p' = U_{\text{eff}}^* P_p U_{\text{eff}}
\end{align*}
\]

\[
U_{\text{eff}} = \exp(i\kappa P_p P_a)
\]

- **Heisenberg equation:**

  **light:**

  \[
  \begin{align*}
  X_p' &= X_p - \kappa P_a \\
P_p' &= P_p
  \end{align*}
  \]

  **atoms:**

  \[
  \begin{align*}
  X_a' &= X_a - \kappa P_p \\
P_a' &= P_a
  \end{align*}
  \]
Teleportation Step 1: atomic EPR correlations

- first round

**Diagram:**

- Coherent light in
- Atoms 1 connected to atoms 2
- Measurement of $X_{p1}$ projects the two atomic ensembles into an (approximate) eigenstate of $P_{a1} + P_{a2}$

**Equation:**

$$X'_{p1} = X_{p1} - \kappa (P_{a1} + P_{a2})$$

**Expression:**

$$\Delta(P_{a1} + P_{a2})^2 = \frac{1}{1+2\kappa^2} = e^{-2r}$$

Measurement of $X_{p1}$ projects the two atomic ensembles into an (approximate) eigenstate of $P_{a1} + P_{a2}$.
cont.

- second round: we first rotate the atomic spins

\[ X_{a1} \rightarrow -P_{a1} \quad X_{a2} \rightarrow +P_{a2} \]

\[ P_{a1} \rightarrow +X_{a1} \quad P_{a2} \rightarrow -X_{a2} \]

and then we measure

\[ X'_{p2} = X_{p2} - \kappa (X_{a1} - X_{a2}) \]

• result: we have generated a cv EPR state

\[
\begin{align*}
\Delta(P_{a1} + P_{a2})^2 &= \frac{1}{1+2\kappa^2} = e^{-2r} \\
\Delta(X_{a1} - X_{a2})^2 &= \frac{1}{1+2\kappa^2} = e^{-2r}
\end{align*}
\]

for \( \kappa \sim 5 \) we have \( r \sim 2 \)
Teleportation Step 2: Bell measurement

\[
X'_{p1} = X_{p1} - \kappa (X_{a1} - X_{a3})
\]
\[
X'_{p2} = X_{p2} - \kappa (P_{a1} + P_{a3})
\]

\[
F = \frac{1}{1 + \frac{1}{1 + 2\kappa^2} + \frac{1}{2\kappa^2}}
\]

for \( \kappa \sim 5 \) we have \( F \sim 96\% \)

- combining 1 and 2: teleportation
Noise & Imperfections

- noise, and modelling of the noise as "beam splitters"

\[ F' \leq \frac{1}{1 + \sqrt{\eta_t}} \]

Even with a notable transmission loss rate \( \eta_t \sim 0.2 \), quantum teleportation with a remarkable high fidelity \( F \sim 0.7 \) is still achievable.
3.2 Wave function

- initial state

\[ |\psi\rangle = |\psi_p\rangle |\psi_a_1\rangle |\psi_a_2\rangle \]

\[ \sim \int_{-\infty}^{\infty} dp |p\rangle e^{-\frac{1}{2}p^2} \int_{-\infty}^{\infty} dp_{a_1} |p_{a_1}\rangle e^{-\frac{1}{2}p_{a_1}^2} \int_{-\infty}^{\infty} dp_{a_2} |p_{a_2}\rangle e^{-\frac{1}{2}p_{a_2}^2} \]
interaction
  – time evolution

\[ U = \exp(i\kappa \hat{p}\hat{p}_{a2}) \exp(i\kappa \hat{p}\hat{p}_{a1}) \]

– wave function

\[ |\psi\rangle \sim \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp_{a1} \int_{-\infty}^{\infty} dp_{a2} |p\rangle |p_{a1}\rangle |p_{a2}\rangle e^{i\kappa p(p_{a1}+p_{a2})} e^{-\frac{1}{2}(p^2+p_{a1}^2+p_{a2}^2)} \]

light + atom entangled by interaction

initial vacuum noise
• measurement
  – the atomic wave function after the measurement is

\[ |\psi\rangle_a = \langle \bar{x} | \psi \rangle \sim \]

\[ \sim \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp_{a1} \int_{-\infty}^{\infty} dp_{a2} |p_{a1}\rangle |p_{a2}\rangle e^{ip[\kappa(p_{a1}+p_{a2})+\bar{x}]} e^{-\frac{1}{2}(p^2+p_{a1}^2+p_{a2}^2)} \]
- **measurement**
  - the atomic wave function after the measurement is

\[
|\psi\rangle_a = \langle \bar{x}|\psi\rangle \sim \\
\sim \int_{-\infty}^{\infty} dp_{a1} \int_{-\infty}^{\infty} dp_{a2} |p_{a1}\rangle |p_{a2}\rangle e^{-\frac{1}{2}(\kappa(p_{a1}+p_{a2})+\bar{x})^2} e^{-\frac{1}{2}(p_{a1}^2+p_{a2}^2)}
\]

initial: uncorrelated

\[\Delta p = \frac{1}{\sqrt{2}}\]
measurement
- the atomic wave function after the measurement is

\[ |\psi\rangle_a = \langle \bar{x} | \psi \rangle \sim \]

\[ \sim \int_{-\infty}^{\infty} dp_{a1} \int_{-\infty}^{\infty} dp_{a2} |p_{a1}\rangle |p_{a2}\rangle e^{-\frac{1}{2}(\kappa(p_{a1}+p_{a2})+\bar{x})^2} e^{-\frac{1}{2}(p_{a1}^2+p_{a2}^2)} \]

initial: uncorrelated

\[ \Delta p = \frac{1}{\sqrt{2}} \]

EPR

\[ \Delta(p_{a1} + p_{a2})^2 = \frac{1}{1 + 2\kappa^2} \equiv e^{-2r} \]