Learning much from little

Compressed sensing ideas for quantum state tomography and other instances of quantum systems identification

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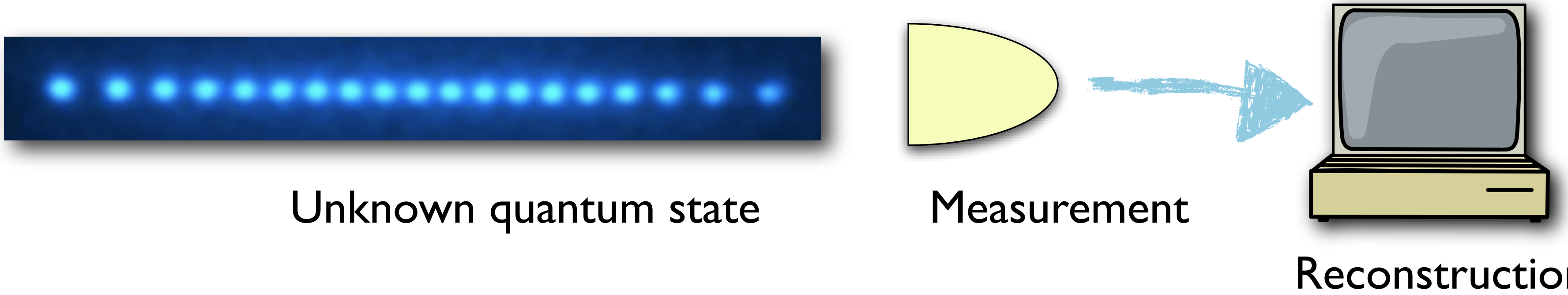
Mention joint work with **Toby Cubitt, Michael Wolf**, **Ignacio Cirac**

- Consider some **unknown** quantum state ρ of n spins, say, of ions in a trap
- We would like to measure that state



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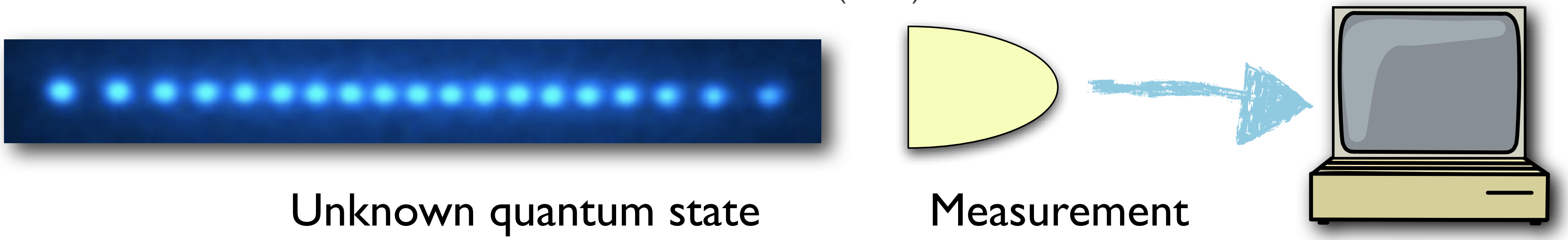
3.2Mb

Haeffner et al, *Nature* **438**, 64 (2005)



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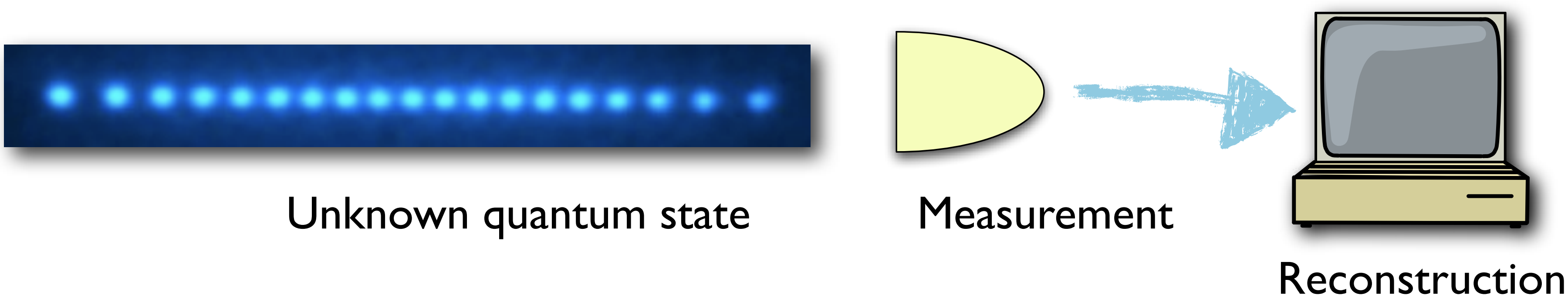
- We would like to measure that state
  - Assume that $\operatorname{rank} \rho = r$, with $r \ll d$, where $d = 2^n$
  - How many parameters do we need to know to specify ρ ?
    - Hmm, well, about rd



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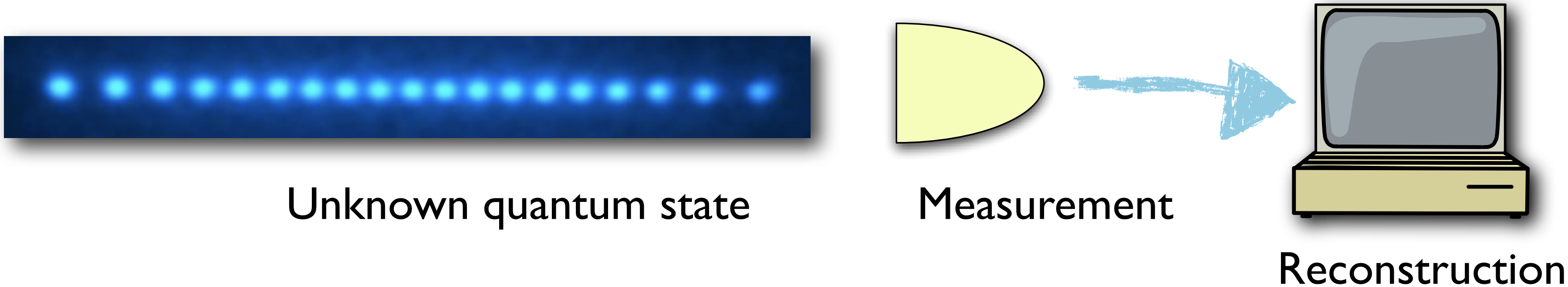
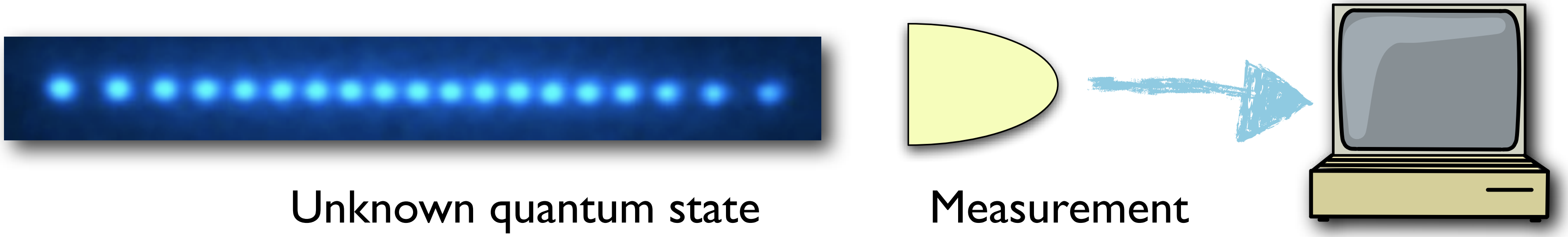
  - Assume that $\mathrm{rank}\,\rho = r$, with $r \ll d$, where $d = 2^n$
  - How many parameters do we need to know to specify ρ ?
    - Hmm, well, about rd
  - Now, how many numbers do we have to measure for full tomography?
    - Ok, surely about $d^2 \gg rd$
  - **What a terrible waste!**



- **Main question of first part of talk:**

  - Can one obtain complete information about an unknown quantum state using substantially fewer than d^2 measurement settings, if the state is (essentially) low rank?

- **Yes we can**



Unknown quantum state

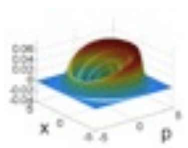
Measurement

Reconstruction

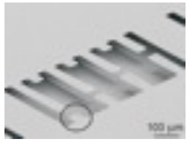
- **Guided tour through (the rest of) the talk:**

  - A classical analogue
  - The theorem
  - Some flavor of proof
  - Certified quantum state tomography

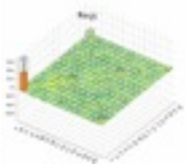
- **Long outlook: Other ideas related in spirit:**



  - Entanglement bounds in optical systems



  - Certifying spectral densities of environments of opto-mechanical systems



  - Detecting non-Markovian dynamics from a snapshot in time

- A classical analogue

- At given time few (r) out of many possible strings (d) sound
- Spectrum essentially described by $r \ll d$ numbers
- **Task:** Identify that spectrum using a few measurements

- **First idea:** Measure in frequency domain
  - Need d sensors!
- **Second idea:** Take few samples in time domain
  - *Shannon-Nyquist:* "If a function contains no frequencies higher than ω Hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2\omega)$ seconds apart"

- Compressed sensing

- **Classical compressed sensing:**
- Consider discrete time signal x , composed of at most r "frequencies"

$$x = \sum_{i=1}^{r} s_i \psi_i$$

so $x = \Psi s$, and perform measurements $y_i = \langle x, \phi_i \rangle$, $y = \Phi x$

Theorem (Candes, Tao, et al, 2004):

- Knowing *only* $O(r \log d)$ *different* such measurements, with randomly chosen measurement vectors ϕ_i , one can recover any discrete-time signal x composed of at most r frequencies
- Scheme is probabilistic, succeeds with overwhelming probability
- Recovery is *exact*
- *Computationally efficient*: Signal uniquely solves convex optimization problem

min $\|s'\|_{l_1}$

subject to $\Phi \Psi s' = y$

Candes, Tao, *IEEE Trans Inf Th* **51**, 4203 (2005)
Candes, Romberg, Tao, *IEEE Trans Inf Th* **52**, 489 (2006)

- Quantum compressed sensing

- Back to unknown rank- r **density matrices** ρ ...

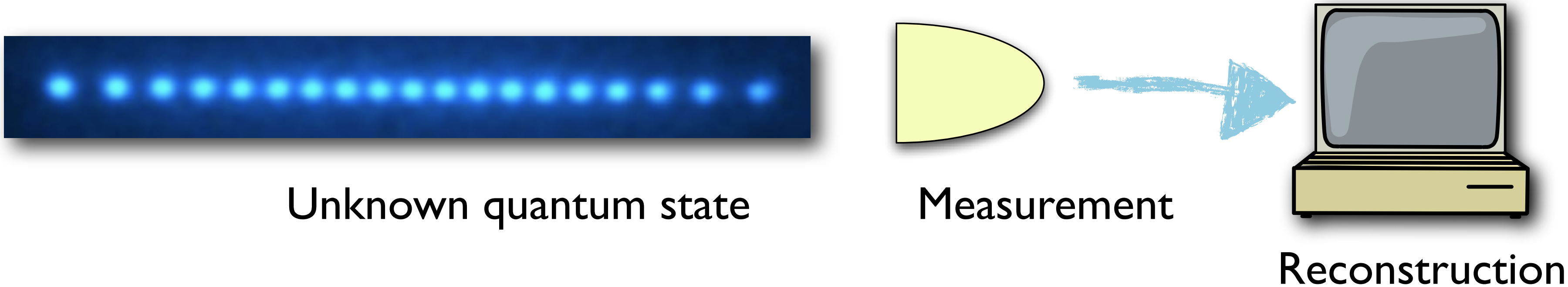
... which we would like to learn in an economic fashion

- Want to learn about a *sparse* object, without knowing *sparsity pattern, does* resemble compressed sensing

- Indeed, previous results extend to **matrix completion**: Reconstruct unknown matrix from only *few matrix elements*

Candes, Recht, arXiv:0805.4471
Candes, Tao, arXiv:0903.1476
Candes, Plan, arXiv:0903.3131

- Not quite applicable to quantum case



- **More natural in quantum case:**
  - Measure **Pauli matrix** expectation values $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$

    so collect data $\mathrm{tr}\rho(\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n})$

    - Physical dimension is $d = 2^n$, write

$$w = \bigotimes_{i=1}^{n} w_i, \;\; w_i \in \{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$$

$$w(A), \;\; A \in [1, d^2]$$

Unknown quantum state

Measurement

Reconstruction

- **Quantum compressed sensing:**

Theorem (Gross, Liu, Flammia, Becker, Eisert, 2009):

- Knowing $O(rd\log d)$ randomly chosen *Pauli expectation values* $\mathrm{tr}(w(A_i)\rho)$ one can recover any unknown density matrix ρ of rank r
- Scheme is probabilistic, succeeds with overwhelming probability
- Recovery is *exact*
- Achieved *computationally efficiently*: Quantum state uniquely solves convex optimization problem

min $\|\omega\|_1$

subject to $\mathrm{tr}(w(A_i)\omega) = \mathrm{tr}(w(A_i)\rho)\,,\ \ i = 1, \ldots, m$

$$\mathrm{tr}(\omega) = 1$$

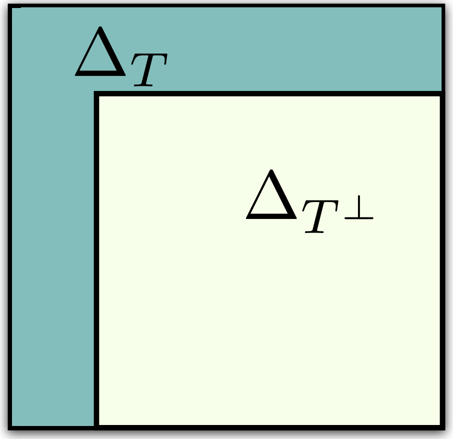
Gross, Liu, Flammia, Becker, Eisert, arXiv:0909.3304

- **Quantum compressed sensing:** Flavor of proof
- For $m = \kappa d r$ measurements, define measurement operator

$$\mathcal{R} : \rho \mapsto \frac{d}{m} \sum_{i=1}^{m} w(A_i) \mathrm{tr}(\rho w(A_i))$$

- For a state σ , consider deviation $\Delta = \sigma - \rho$ from "true state"

- Let T be column and row space of ρ , $\mathcal{P}_T$ projection onto T, decompose deviation as $\Delta = \Delta_T + \Delta_T^\perp$



- **Have uniqueness if for all deviations Δ either**
  - $\|\rho + \Delta\|_1 > \|\rho\|_1$ ("worse solution") or
  - $\mathcal{R}\Delta \neq 0$ ("infeasible")

- **Quantum compressed sensing:** Flavor of proof
- Now consider two cases: *Case (i)*: $\|\Delta_T\|_2 < d^2\|\Delta_{T^\perp}\|_2$

$$\Pr(\|\mathcal{P}_T\mathcal{R}\mathcal{P}_T - \mathbb{I}_T\| > t) < 4dre^{-t^2\kappa/4}$$

$$\|\mathcal{R}\Delta\|_2 > 0$$

"Infeasible"

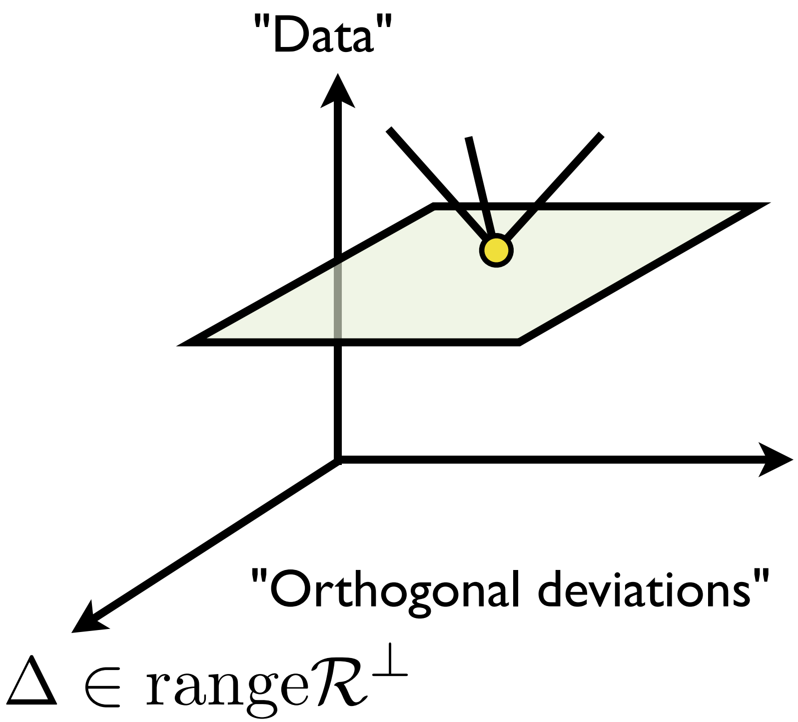
- **Matrix-valued Bernstein inequality** (Ahlswede, Winter, 2002):
- Let $S = \sum_{i=1}^{m} X_i$ with X_i i.i.d. matrix-valued random variables, $\mathbb{E}(X) = 0$,

set $\sigma^2 = \|\mathbb{E}(X^2)\|$, then, for $t < 2m\sigma^2/\|X\|$ one finds

$$\Pr(\|S\| > t) \leq 2de^{-t^2/(4m\sigma^2)}$$

Ahlswede, Winter, IEEE Trans Inf Th **48**, 569 (2002)

- **Quantum compressed sensing:** Flavor of proof
- Now consider two cases: *Case (ii)*: $\|\Delta_T\|_2 > d^2 \|\Delta_{T^\perp}\|_2$



Task: Find subgradient $Y \in \mathrm{range}\mathcal{R}$ such that

$$\|\rho + \Delta\|_1 > \|\rho\|_1 + \mathrm{tr}[Y\Delta] \geq \|\rho\|_1$$

for all $\Delta \in \mathrm{range}\mathcal{R}^\perp \neq 0$

$$\|\rho + \Delta\|_1 > \|\rho\|_1$$

"Not optimal"

- **Quantum compressed sensing:** Flavor of proof
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"Not optimal"

Sweat goes into construction of such Y, again
- using large deviation bounds, and
- an adaptive scheme of using data, "golfing"

$\|\mathcal{P}_T Y - \mathbb{I}_T\|_2 \leq 1(2d^2)$, $\|\mathcal{P}_T Y\|_2 < 1/2$

(End of proof)

- **Certified tomography:**

  - Nice, but how do we know that the state is low rank in the first place?

  - **One does not have to!** (Say, $r = 1$)
    - Make use of part of the data $O(rd \log d)$ to **estimate the purity** $\mathrm{tr}(\rho^2)$,
    - ... formulate a version of theorem allowing for **errors**
    - ... **use** the estimate for the purity in the bound

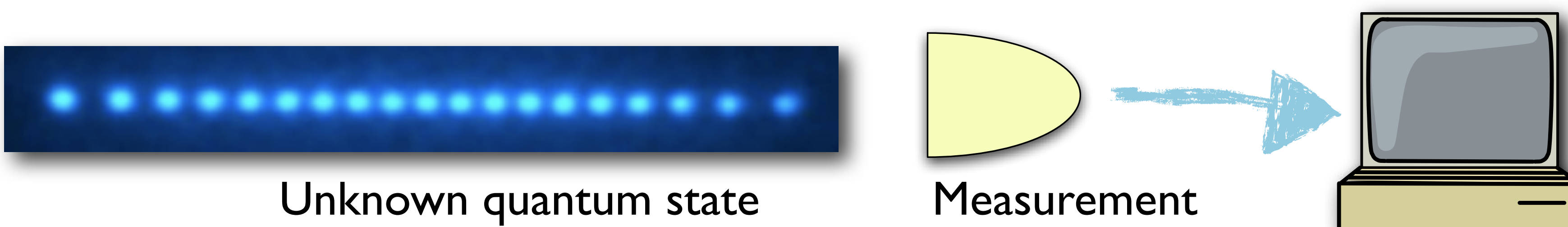
  - **Assumption-free quantum state tomography**

Gross, Liu, Flammia, Becker, Eisert, arXiv:0909.3304

- **Lesson of the main part of talk:**

> If a state is close to being low-rank, then perform the **same** measurements as for full quantum state tomography, but just **randomly so** and **much fewer** of them, and still faithfully (and efficiently) reconstruct the state

Gross, Liu, Flammia, Becker, Eisert, arXiv:0909.3304

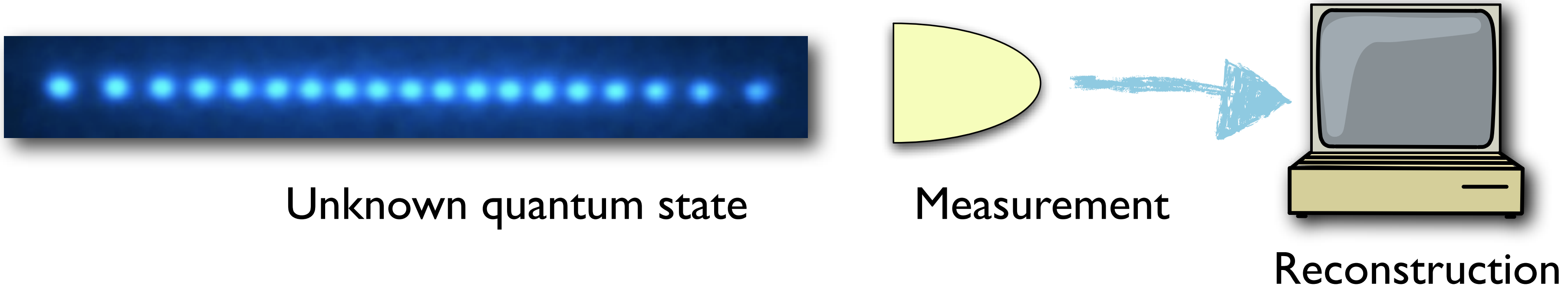


- (Methods general enough to get simpler - and in effort scaling improved - proof of matrix completion)

Gross, arXiv:0910.1879

- Long outlook: Related ideas

- **Trying to further "learn much from little"**
  - **Directly measure** interesting quantities in experiments, without detour via quantum process or state tomography
  - Do it with **error bars**
  - Measure the **"unexpected"**



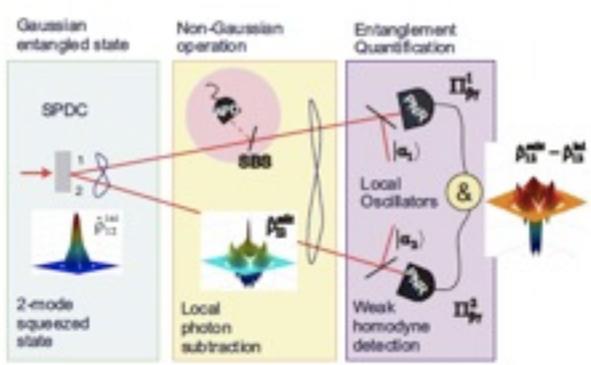
1. Directly estimating entanglement

- Estimate the **quantitative entanglement content** of states

- ...from much less than tomographic knowledge

> - Find good and feasible lower bounds to solution of
>
> $$\begin{array}{ll} \text{min} & E(\rho) \\ \text{subject to} & \mathrm{tr}(\rho W_i) = c_i \end{array}$$
>
> for entanglement measure E and some expectation values of W_i

- Applied to continuous-variable **entanglement distillation** schemes, where tomographic knowledge is too expensive/noisy

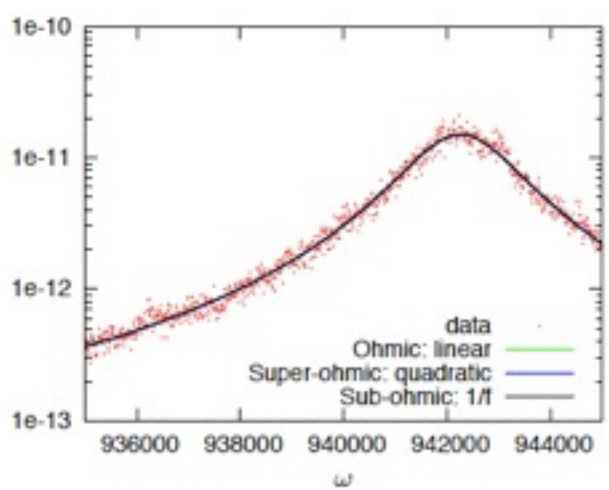


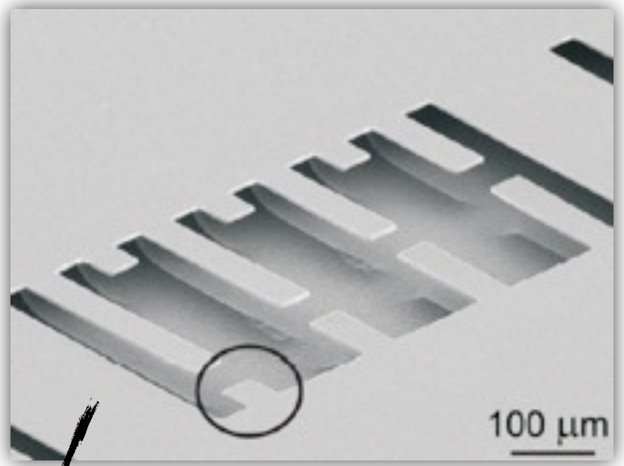
Lundeen, Feito, Coldenstrodt-Ronge, Pregnell, Silberhorn, Ralph, Eisert, Plenio, Walmsley, *Nature Physics* **5**, 27 (2009)
Puentes, Datta, Feito, Eisert, Plenio, Walmsley, arXiv:0911.2482

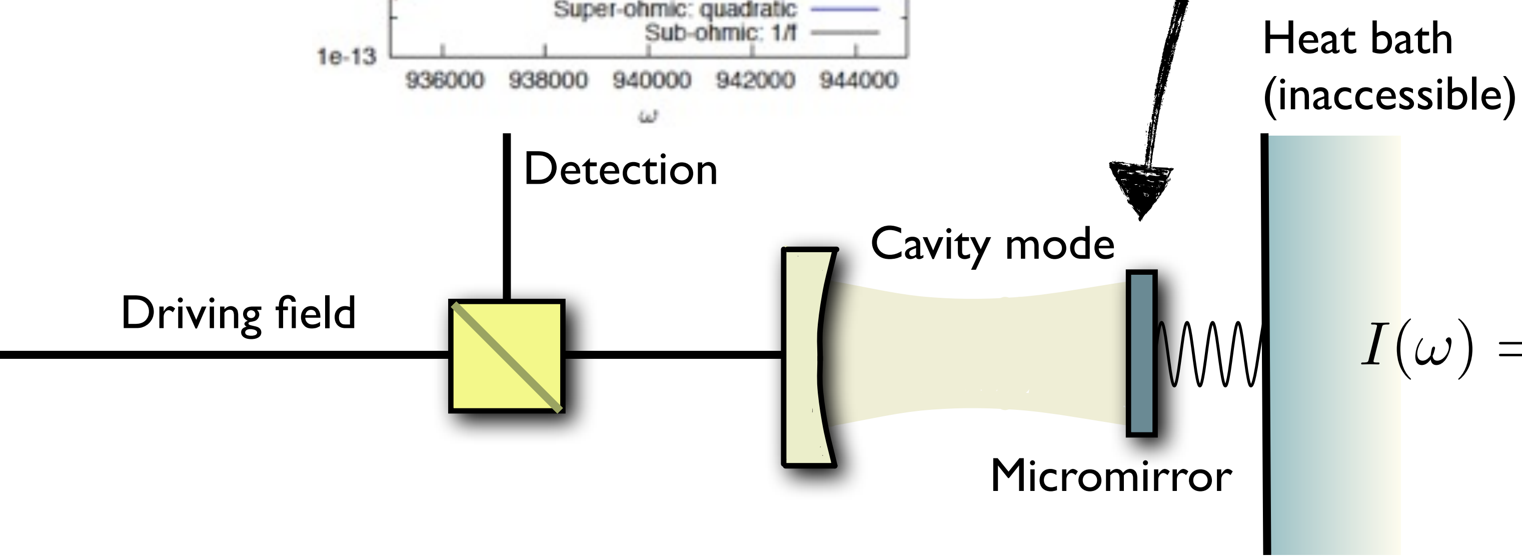
Eisert, Brandao, Audenaert, *New J Phys* **8**, 46 (2007)
Guehne, Reimpell, Werner, *Phys Rev Lett* **98**, 110502 (2007)

2. Assessing decoherence of optomechanical systems:

- Learn about otherwise inaccessible **spectral density** of the heat bath of mechanical mode from spectral properties of light leaving the optical cavity
- *Certify non-Ohmic baths*





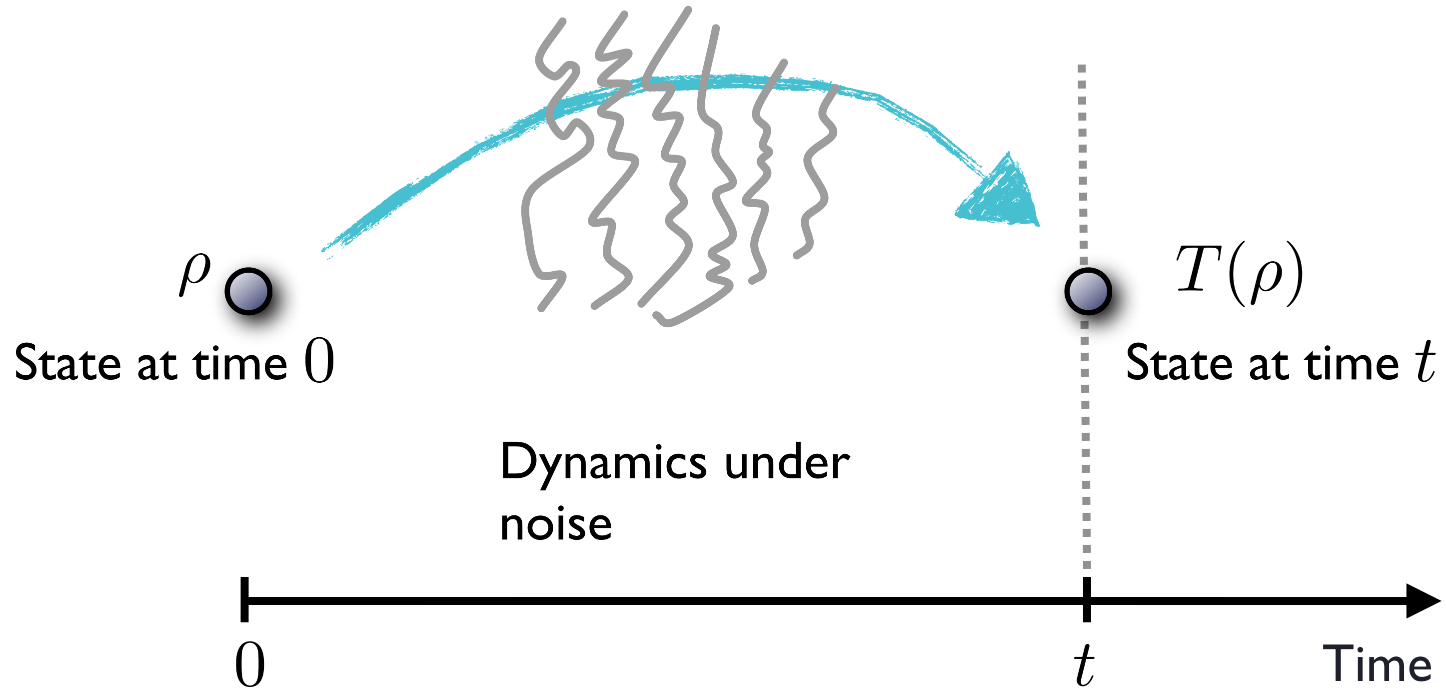


Spectral density

$$I(\omega) = \sum_n \frac{c_n^2}{2 m_n \omega_n} \delta(\omega - \omega_n)$$

Trubarov, Kieling, Groeblacher, Aspelmeyer, Eisert, in preparation (2009)

3. Detecting non-Markovian dynamics from a snapshot in time?



Cubitt, Eisert, Wolf, arXiv:0908.2128
Wolf, Eisert, Cubitt, Cirac, *Phys Rev Lett* **101**, 150402 (2008)

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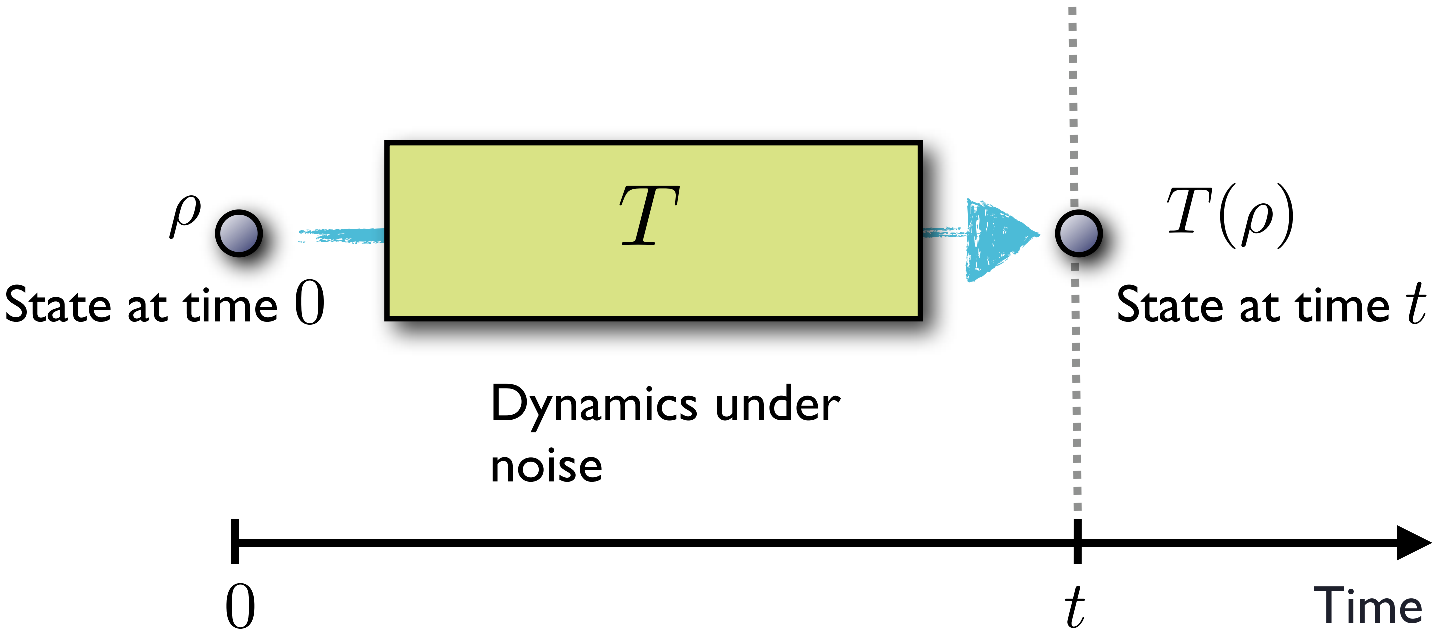
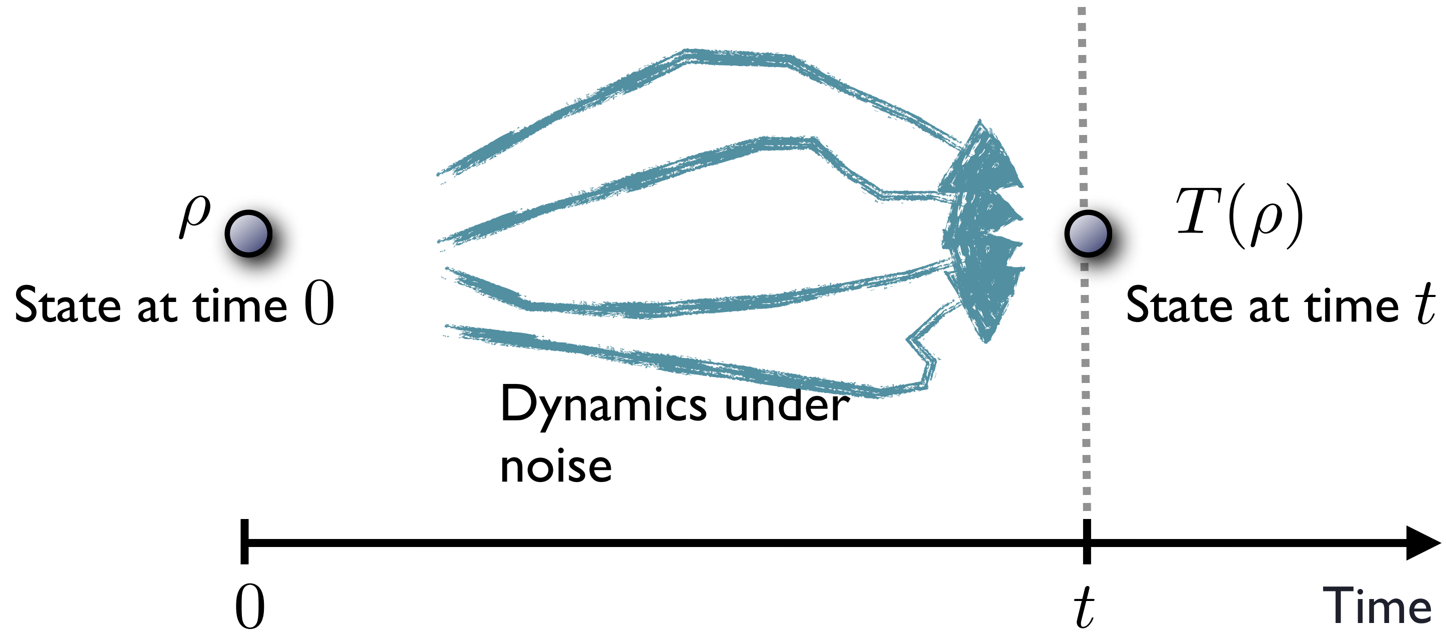


- **Dynamical map:** Completely positive map T specifying dynamics after given time
- **Typical setting in process tomography:** Do process tomography at many time slices

3. Detecting non-Markovian dynamics from a snapshot in time?



- **But could we have known whether dynamics was Markovian from just a single snapshot in time?**

- **Quantum channels:**

  - Channel $T: M_d \rightarrow M_d$ has **matrix form** $\hat{T}$

$$\hat{T}_{j,k} = \mathrm{tr}[O_j T(O_k)]$$

and **Choi matrix** $\hat{T}^\Gamma$, $\langle j,k|\hat{T}^\Gamma|a,b\rangle = \langle j,a|\hat{T}|k,b\rangle$

  - Channel is **Markovian**, if $T = e^L$ for some generator L (setting time $t=1$)

  - **"Lindblad form"** of generator:

$$L(\rho) = i[\rho, H] + \sum_{\alpha,\beta} G_{\alpha,\beta} \left(F_\alpha \rho F_\beta^\dagger - \frac{1}{2}\{F_\beta^\dagger F_\alpha, \rho\}_+ \right)$$

- **How do we now “test for Markovianity”?**
- Jordan normal form

$$\hat{T} = \sum_r \lambda_r P_r + \sum_c (\lambda_c P_c + \bar{\lambda}_c \mathbb{F} \bar{P}_c \mathbb{F})$$

Real part

Complex part

- Logarithm:

$$\log \hat{T} = L_0 + 2\pi i \sum_c m_c (\lambda_c P_c + \bar{\lambda}_c \mathbb{F} \bar{P}_c \mathbb{F})$$

- Needless to say, has infinitely many branches
- Is one of the branches a valid Lindblad generator?

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- **Lemma:** A Hermitian linear map $L : M_d \longrightarrow M_d$ is a valid Lindblad generator iff it satisfies normalization $L^*(\mathbb{1}) = 0$ and

$$(\mathbb{I} - \omega) L^\Gamma (\mathbb{I} - \omega) \geq 0$$

ω maximally entangled state

- Putting things together:

> - **Theorem:** A channel T **is Markovian** ("could have come from Markovian dynamics") iff there is an integer solution to
>
> $$A_0 + \sum_c m_c A_c \geq 0$$

- Known matrices:

$$A_0 = (\mathbb{I} - \omega) L_0^\Gamma (\mathbb{I} - \omega)$$
$$A_c = 2\pi i (\mathbb{1} - \omega)(P_c - \mathbb{F}\bar{P}_c\mathbb{F})^\Gamma (\mathbb{1} - \omega)$$

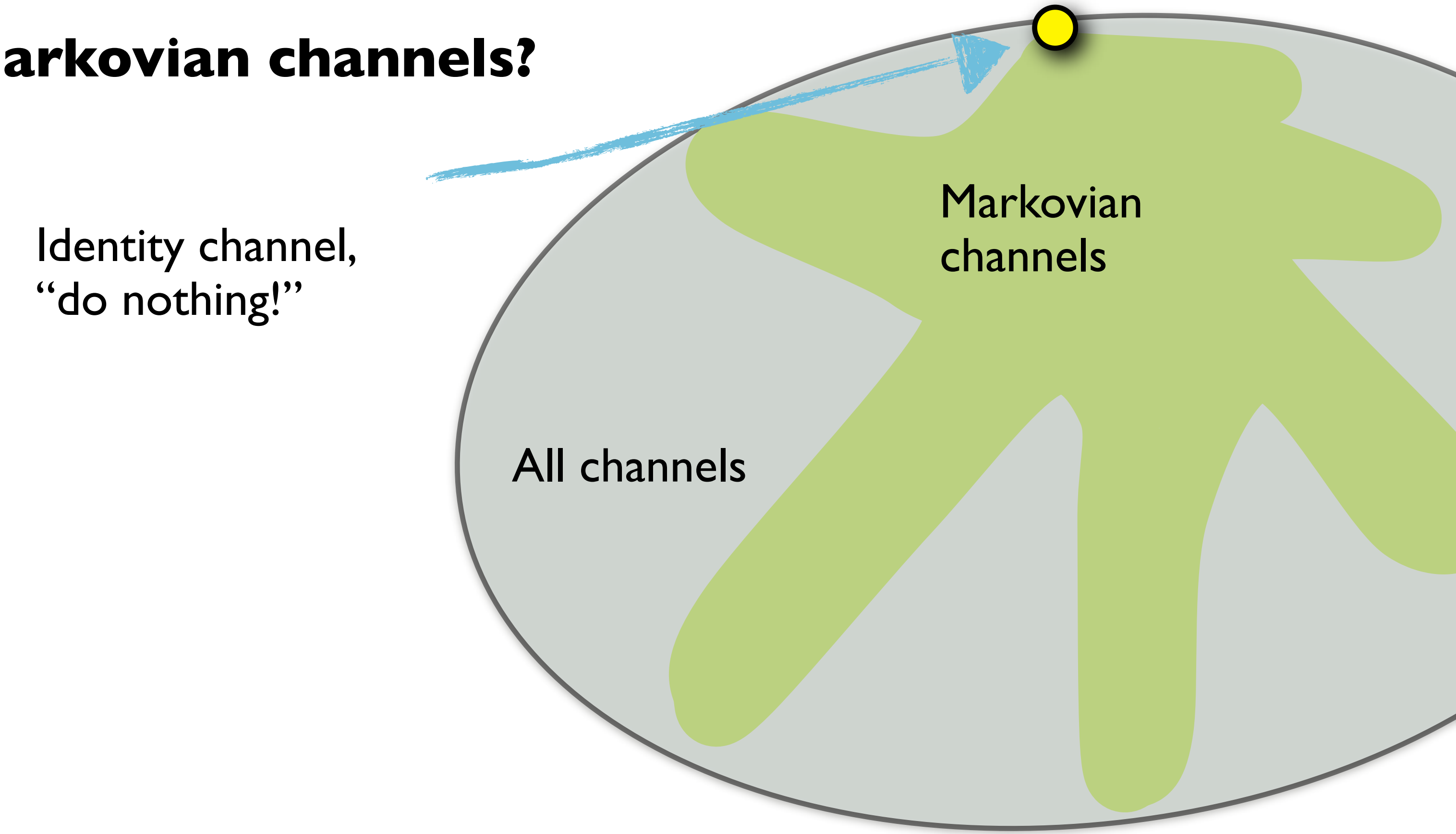
→ **Test for Markovianity!**

(Efficient in input length, practical; interestingly NP hard in physical dimension, just as the classical embedding problem)

→ Can be made quantitative **measure of Markovianity**

Cubitt, Eisert, Wolf, arXiv:0908.2128
Wolf, Eisert, Cubitt, Cirac, *Phys Rev Lett* **101**, 150402 (2008)

- **Where are the Markovian channels?**

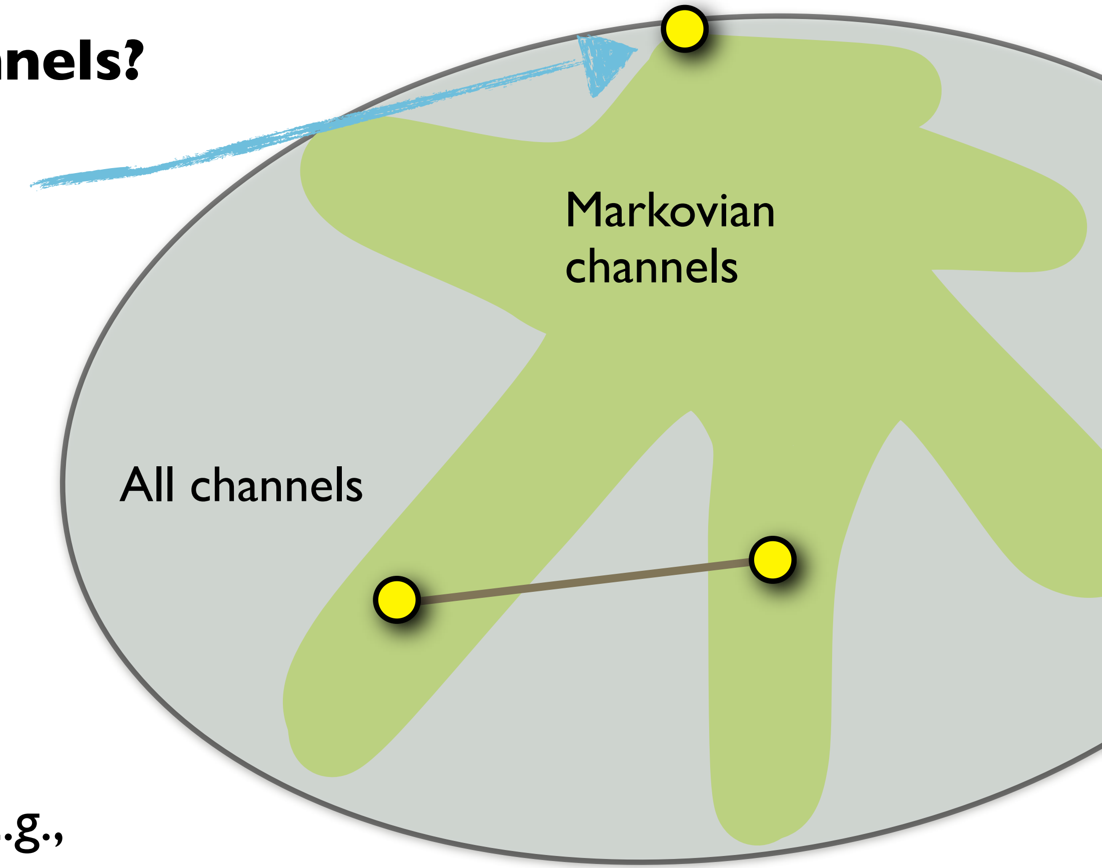


- For qubit channels: **Only 2% Markovian***

* Drawn according to Haar measure for unitaries on system+ environment

- **Where are the Markovian channels?**

Identity channel,
"do nothing!"



- **Strange enough: Non-convex!** E.g.,

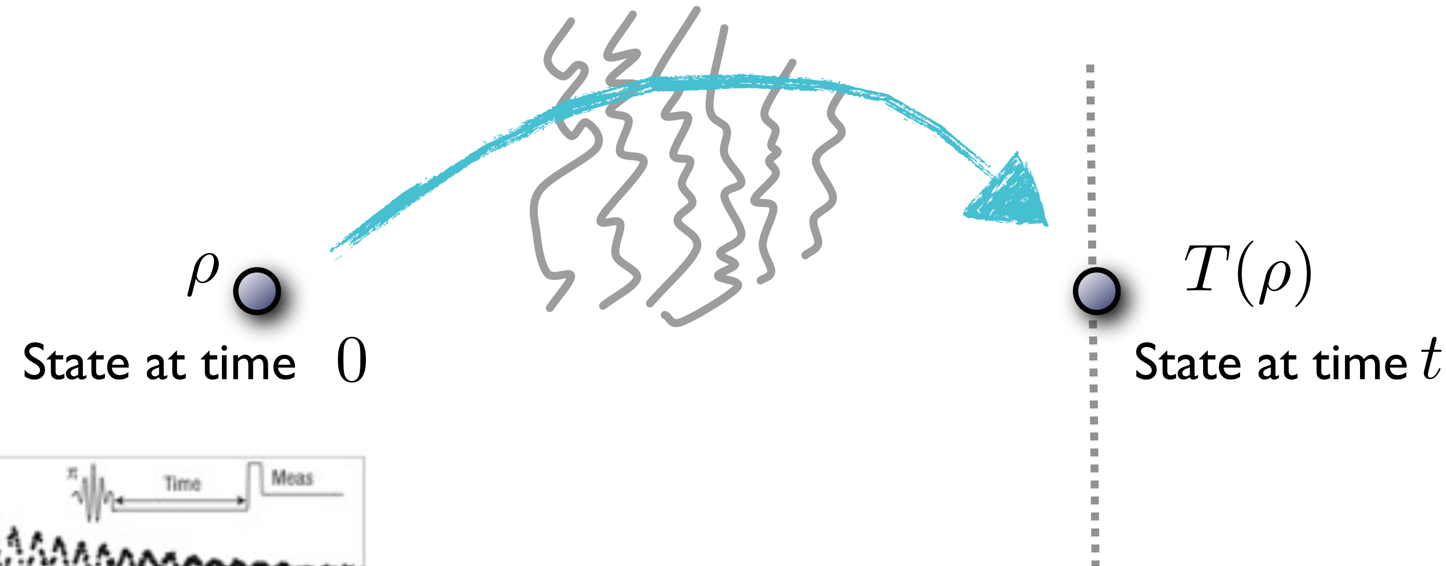
$$T(\rho) = (\lambda)T_1(\rho) + (1-\lambda)T_2(\rho)$$

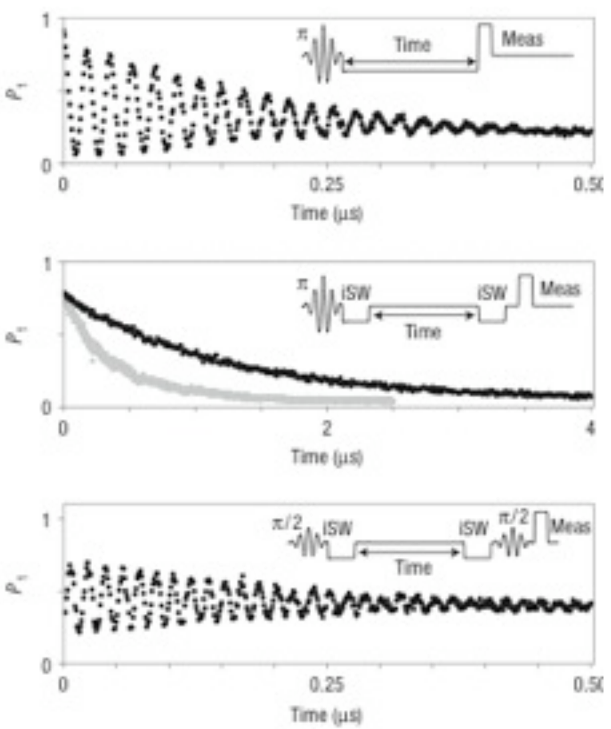
$\pi/4$ Rabi oscillation channel

Dephasing channel

- Non-Markovian effects can arise from environments in mixture of states each of which would lead to Markovian dynamics

- Test for **Markovianity** at a single time

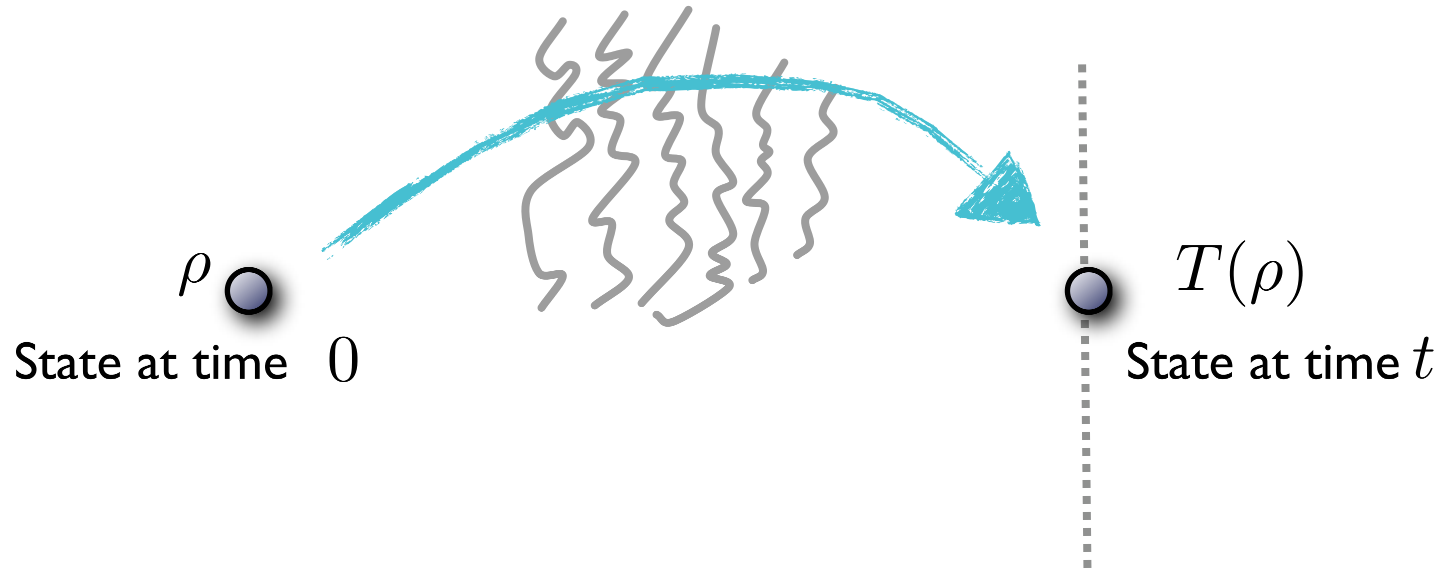




Neeley, Ansmann, Bialczak, Hofheinz, Katz, Lucero, O'Connell, Wang, Cleland, Martinis, *Nature Physics* **4**, 523 (2008)

- Interestingly, for some times, single snapshots of *phase qubit evolution* certify strongly non-Markovian dynamics

- Test for **Markovianity** at a single time



- Can one even get an estimate for the **bath-correlation time**, without making a model of environment, without even thinking about it?

- Many snapshots?

- Summary

- **"Learn much from little"**

1. Compressed sensing approach to quantum state tomography:

"Get reliable estimates from few measurement settings, within the paradigm of compressed sensing"

2. Related ideas, like detecting forgetfulness of channels from a snapshot in time:

"Measure once, and get a meaningful statement about a continuous process"

Thanks for your attention