Gadgets and Gizmos for Adiabatic Quantum Computation

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Adiabatic Quantum Computing

Hamiltonian:	H(0)	$\xrightarrow{\text{smoothly}}$	H(T)
ground state:	$\vert 000...\rangle$	$\longrightarrow$	$\vert \text{answer}\rangle$

If $\left|\frac{dH}{dt}\right|$ is sufficiently small compared to the eigenvalue gap then the system will track the ground state.

Adiabatic Quantum Computing

- Originally proposed by Farhi *et al.* as a method for solving satisfiability problems (*e.g.* 3-SAT)

- Can simulate quantum circuits (Aharonov *et al.*)

- Can be simulated by quantum circuits using standard Trotterization

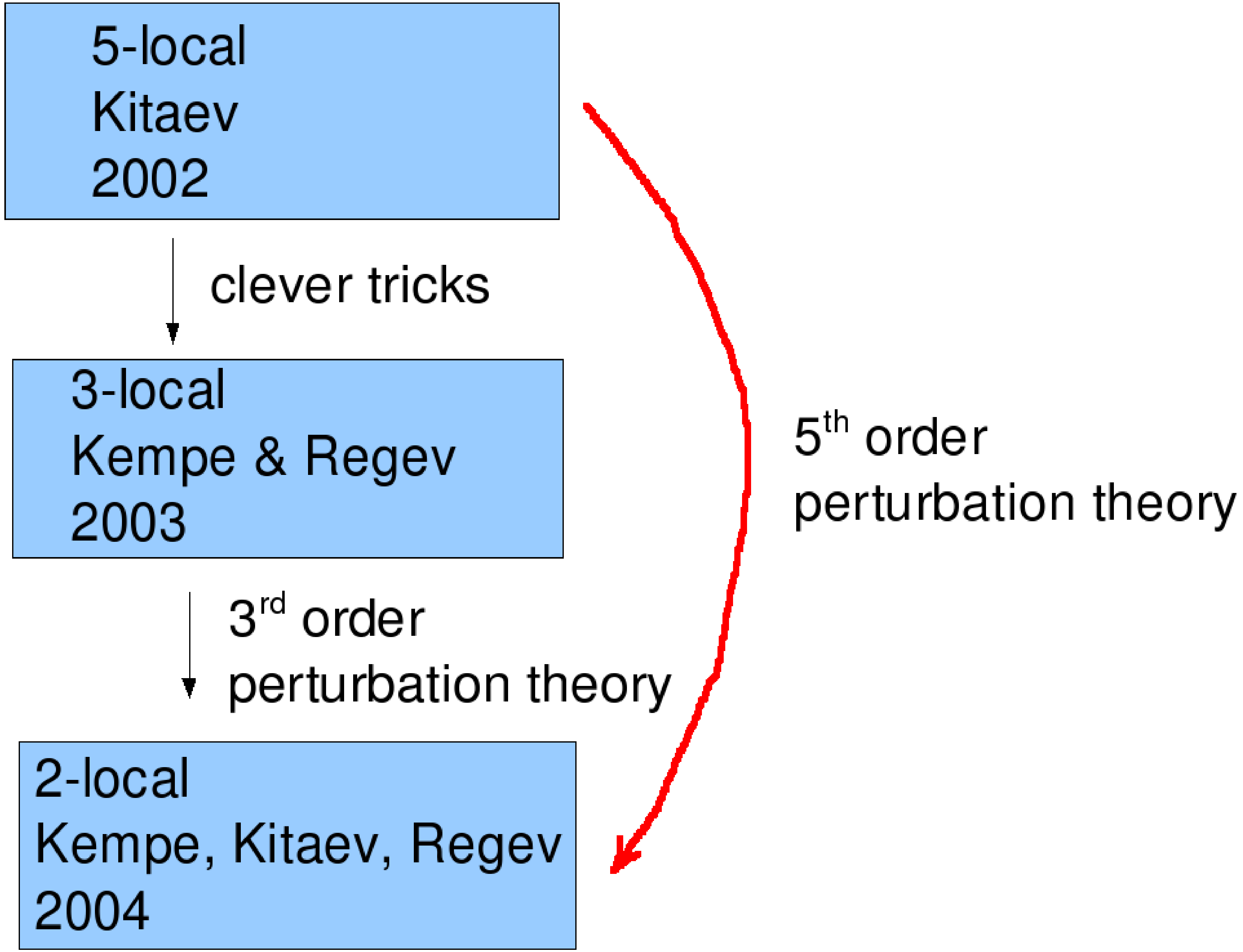
Local Interactions

- k-local Hamiltonian:

  $$H = \sum_i H_i$$

  where each H_i acts on at most k-qubits
  e.g. $Z \otimes Z \otimes I \otimes I \otimes I$
  - 1-local term: field
  - 2-local term: interaction (Ising, Heisenberg,...)

- spatial locality is a separate question


5-local
Kitaev
2002
clever tricks
3-local
Kempe & Regev
2003
3rd order
perturbation theory
2-local
Kempe, Kitaev, Regev
2004
5th order
perturbation theory

Part I: Perturbative Gadgets

main idea:

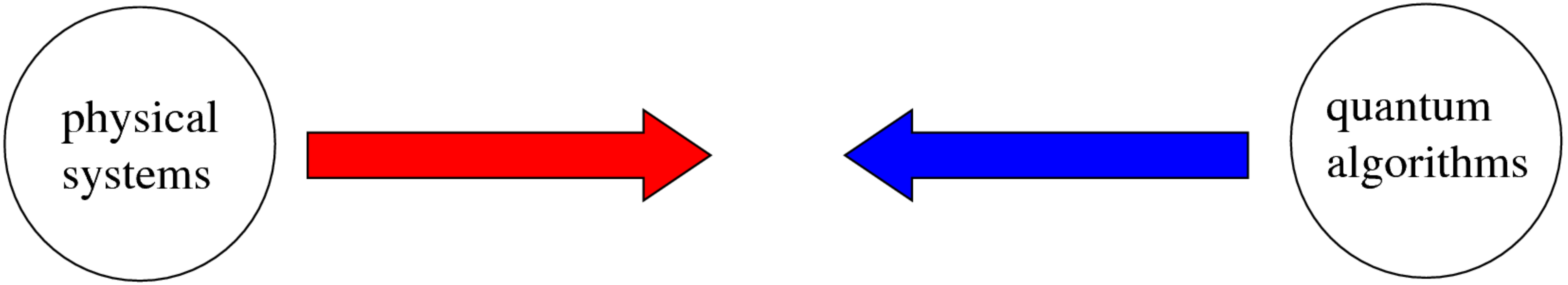
- construct 2-local Hamiltonian whose ground state approximately equals the ground state of a given k-local Hamiltonian

- Approximate equality of ground states is proven using k^{th}-order perturbation theory

S. Jordan and E. Farhi
Perturbative Gadgets at Arbitrary Orders
PRA 77, 062329 (2008) [ArXiv:0802.1874]

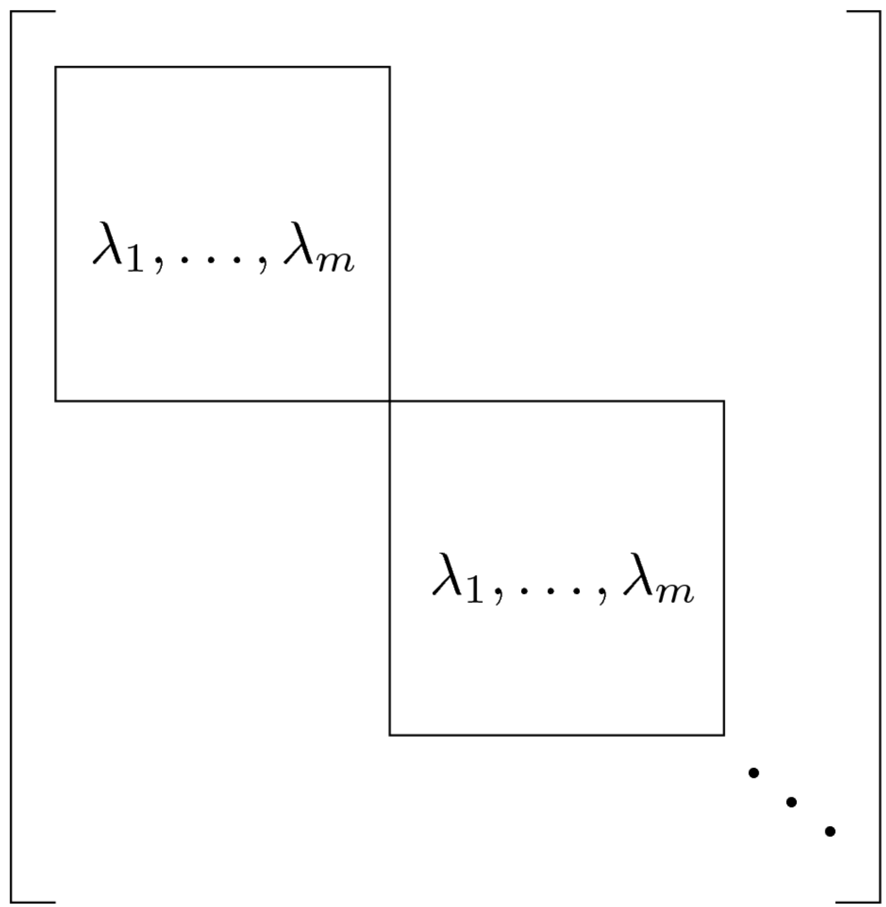
Other Applications of Gadgets

- spatial locality (Oliveira & Terhal)
- Hamiltonian of Z field and XZ interactions is sufficient for universal AQC (Biamonte & Love)
- QMA-completeness



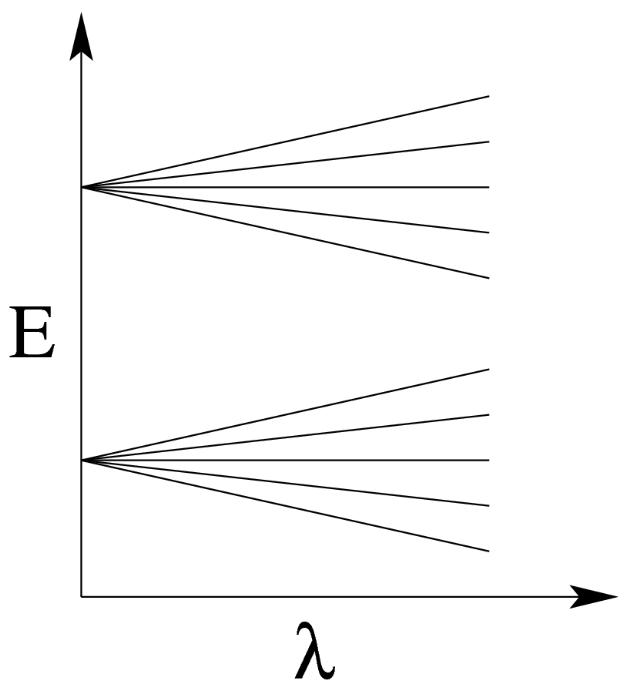
Subspace Computing

$$[H(t), R_i] = 0 \qquad i = 1, 2, \ldots, m$$



Effective Hamiltonian

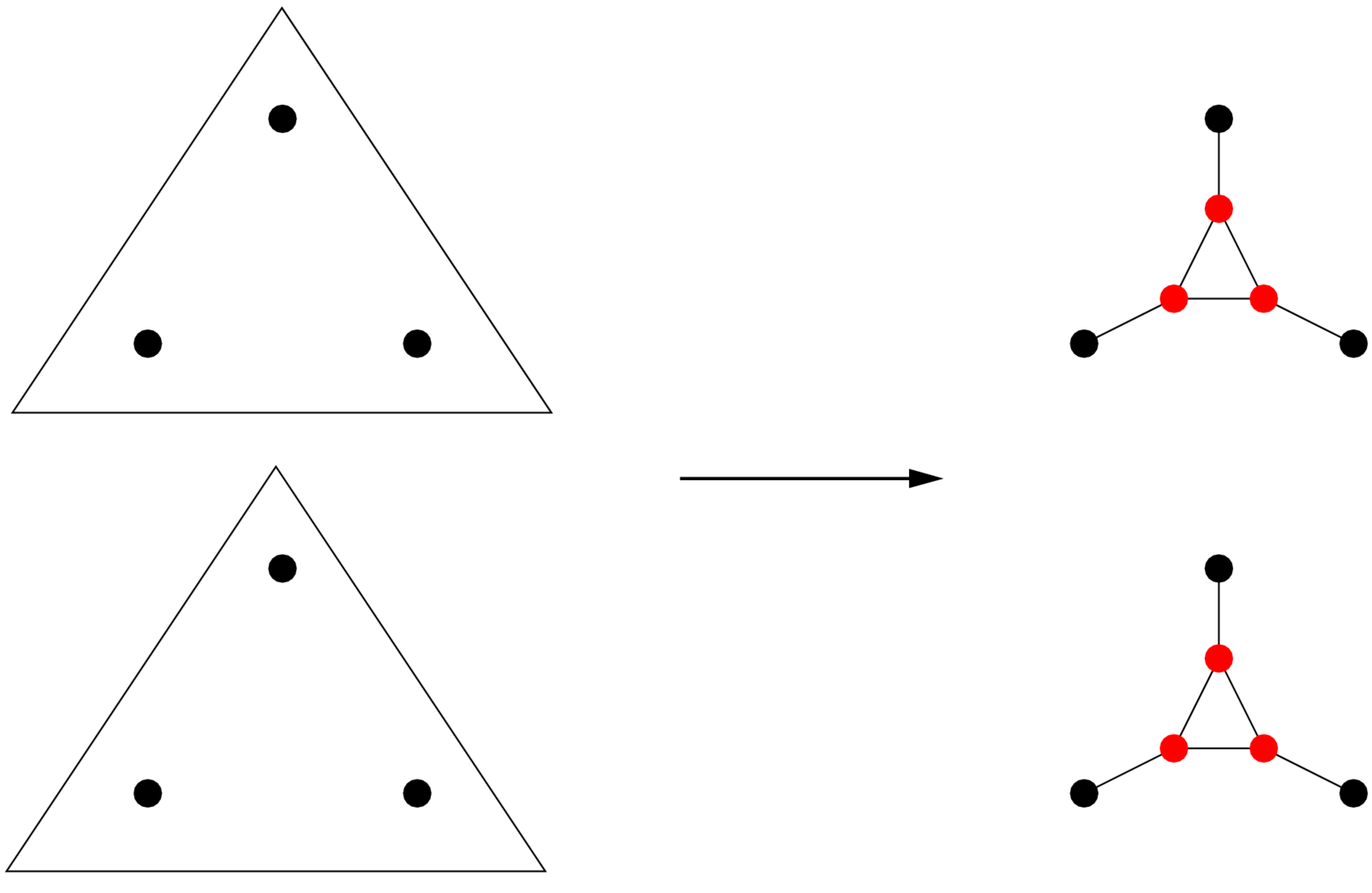
- H_0 has *d*-fold degenerate ground state
- we'll calculate the effective Hamiltonian on the *d* lowest energy states of the perturbed Hamiltonian



$$H = H_0 + \lambda V$$

$$H_{\mathrm{eff}} \equiv \sum_{i=1}^{d} E_i |\psi_i\rangle\langle\psi_i|$$

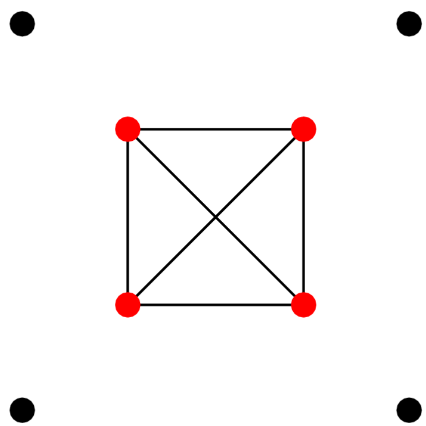
Gadgets



want: $H_{\text{eff}} = \sigma_1 \sigma_2 \sigma_3 \sigma_4$

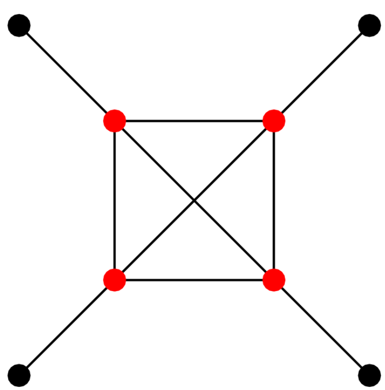
use:

unperturbed:



$$H^{\text{anc}} = \sum_{1 \leq i < j \leq k} \frac{1}{2}(I - Z_i Z_j)$$

perturbation:



$$V = \sum_{j=1}^{k} \sigma_j \otimes X_j$$

Subspace

- $X^{\otimes k}$ commutes with H^{gad} on any ancilla register
- Hamiltonian is block diagonal with blocks corresponding to eigenvalues of $X^{\otimes k}$
- If the system starts in the +1 block it will stay there
- we'll consider this block (a Hermitian matrix) to be the Hamiltonian of our system, and ignore the other blocks

Main Result

to simulate:

$$H^{\text{comp}} = \sum_{s=1}^{r} c_s H_s$$

$$H_s = \sigma_{s,1}\sigma_{s,2}\ldots\sigma_{s,k}$$

$$\sigma_{s,j} = \hat{n}_{s,j} \cdot \vec{\sigma}_{s,j}$$

use:

$$H^{\text{gad}} = \sum_{s=1}^{r} H_s^{\text{anc}} + \lambda \sum_{s=1}^{r} \sqrt[k]{c_s} V_s$$

$$H_s^{\text{anc}} = \sum_{1 \leq i < j \leq k} \frac{1}{2}(I - Z_{s,i} Z_{s,j})$$

$$V_s = \sum_{j=1}^{k} \sigma_{s,j} \otimes X_{s,j}$$

Time-independent Perturbation Theory

- $H = H_0 + \lambda V$
- eigenstates $|\psi_i(\lambda)\rangle$ and eigenvalues $E_i(\lambda)$ of H have unique Taylor expansion in λ

$$|\psi_i\rangle = \sum_{k=0}^{\infty} \lambda^k |\psi_i^{(k)}\rangle$$

$$E_i = \sum_{k=0}^{\infty} \lambda^k E_i^{(k)}$$

- convergence guaranteed if $\|\lambda V\| < \frac{\gamma}{2}$ (Kato)

Bloch's method

- expressions for $|\psi_i^{(k)}\rangle$ and $E_i^{(k)}$ are messy at large k
- Claude Bloch provides convenient way of organizing terms (1958)
- $H_{\text{eff}} = \mathcal{U}\mathcal{A}\mathcal{U}^\dagger$
- $\mathcal{A}$ is an operator on the unperturbed ground space (for gadgets ground space is simple)
- $\mathcal{A}$ and $\mathcal{U}$ have nice perturbative expansions to all orders

$$\mathcal{A} = \sum_{m=1}^{\infty} \mathcal{A}^{(m)}$$

$$\mathcal{A}^{(m)} = \sum_{(m-1)} P_0 V S^{l_1} V S^{l_2} \ldots V S^{l_{m-1}} V P_0$$

where:

$$S^l = \begin{cases} \sum_{j \neq 0} \frac{P_j}{(-E_j^{(0)})^l} & \text{if } l > 0 \\ -P_0 & \text{if } l = 0 \end{cases}$$

$\mathcal{U}$ has similar expansion

Effective Gadget Hamiltonian

$$\mathcal{A}^{(m)} = \sum_{(m-1)} P_0 V S^{l_1} V S^{l_2} \ldots V S^{l_{m-1}} V P_0$$

$$V = \sum_{j=1}^{k} \sigma_j \otimes X_j$$

$$\frac{1}{\sqrt{2}}(|0\rangle^{\otimes k} + |1\rangle^{\otimes k})$$

$$P_0 V P_0 = 0$$

$$P_0 V S^1 V P_0 \propto P_0$$

$$\vdots$$

$$P_0 (V S^1)^k P_0 \propto P_0 (\sigma_1 \otimes X_1)(\sigma_2 \otimes X_2) \ldots (\sigma_k \otimes X_k) P_0$$

$$= P_0 \sigma_1 \sigma_2 \ldots \sigma_k \otimes X^{\otimes k} P_0$$

$$= H^{\mathrm{comp}} \otimes P_+$$

Effective Gadget Hamiltonian

$$\mathcal{A} = f(\lambda)P_0 + \beta\lambda^k H^{\mathrm{comp}} \otimes P_+$$

$$H_{\mathrm{eff}} = \mathcal{U}\mathcal{A}\mathcal{U}^\dagger$$

A short calculation shows that up to an overall energy shift and terms of order λ^{k+1}

$$H_{\mathrm{eff}} = \beta\lambda^k H^{\mathrm{comp}} \otimes P_+$$

Gadget Summary

- if $|\psi\rangle$ is the ground state of a k-local Hamiltonian then $|\psi\rangle(|000\ldots\rangle + |111\ldots\rangle)^{\otimes r}$ can be obtained as the approximate ground state of a 2-local Hamiltonian
- if the gap of the k-local Hamiltonian is γ then the gap of the 2-local Hamiltonian is $\lambda^k \gamma$
- $\lambda < \frac{\gamma}{\|V\|}$ suffices
- Open Problem: How small does λ need to be?

Part II: Gizmos

- physical Hamiltonians often have restricted locality (e.g. 2-body interactions, spatially local)
  - use gadgets to achieve universality
- physical Hamiltonians often have restricted matrix elements (e.g. real, or negative off-diagonal)
  - replace matrix elements with higher dimensional representations

S. Jordan and P. Love
QMA-complete problems for stoquastic Hamiltonians and Markov Matrices
[ArXiv:0905.4755]

Definition: stoquastic Hamiltonian

a Hermitian matrix in which all off-diagonal entries are non-positive

- In physics not all bases are created equal!
- Basis of unentangled states is easy to prepare and measure
- We use σ_Z basis but results hold for any basis of tensor product states

- Some stoquastic Hamiltonians:
  - ferromagnetic Heisenberg model
  - transverse Ising model
  - adiabatic optimization algorithms
  - Josephson junction flux qubits

- Stoquastic adiabatic computation $\in \mathrm{BPP}_{\mathrm{path}}$
  - probably not universal!
- Stoquastic LOCAL HAMILTONIAN $\in$ AM
  - probably not QMA-complete!

[Bravyi, *et al.*]

However:

- Adiabatic computation in <u>highest</u> eigenstate of stoquastic Hamiltonian is universal
- Approximating <u>highest</u> energy of stoquastic Hamiltonian is QMA-complete

Intuition

- By Perron-Frobenius: highest eigenvector of stochastic matrix is probability distribution
  - lowest eigenvector of stoquastic Hamiltonian is $\propto$ a probability distribution
  - QMA $\longrightarrow$ AM, BQP $\longrightarrow$ $\mathrm{BPP}_{\mathrm{path}}$

- Other eigenstates have amplitudes of both signs

Unvirsal AQC with Stoquastic Hamiltonians

Begin by using result of Biamonte & Love:

Time-dependent Hamiltonians of the form:

$$H_{XZ} = \sum_i d_i X_i \sum_i h_i Z_i + \sum_{i,j} K_{ij} X_i X_j + \sum_{i,j} J_{ij} Z_i Z_j$$

achieve universal AQC

Main trick:

- We want to get rid of negative matrix elements
- So, we represent the group $\mathbb{Z}_2 = \{1, -1\}$

by $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

- H_{XZ} is of the form:

$$H_{XZ} = \sum_k \alpha_k S_k$$

where each coefficient α_k is negative and each S_k is one of $\pm X, \pm Z, \pm X_i X_j, \pm Z_i Z_j$

- All entries in S_k are +1, -1, or 0
- Make the replacements

$$1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; -1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; 0 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

call the result $\tilde{S}_k$

- by $1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; -1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; 0 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$

  $\tilde{H}_{XZ} = \sum_k \alpha_k \tilde{S}_k$ is stoquastic

- H_{XZ} acts on *n* qubits

  $\tilde{H}_{XZ}$ acts on *n+1* qubits

- The 2x2 blocks from the replacement act on the extra qubit

- If the ancilla is $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ then the blocks act like the original scalars
- if the ancilla is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ then the blocks replacing +1 and -1 both act like +1

$$1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ -1 \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ 0 \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- Thus: $\tilde{H}_{XZ} = H_{XZ} \otimes |-\rangle\langle-| + H'_{XZ} \otimes |+\rangle\langle+|$

- Adiabatic computing in the lowest energy state of the $|-\rangle$ subspace exactly mimics the original computation of H_{XZ}

- Can we shift this to the ground state?
  NO: The penalty X is not stoquastic.
  (-X is stoquastic so we can shift it to the top.)

Other applications of Gizmos

- Also proves QMA-hardness for estimating lowest energy in this subspace
- -X penalty shifts it to the highest state
- additional normalization makes the Hamiltonian a stochastic matrix
- finding lowest eigenvalue of stochastic matrix is QMA-complete

- we have QMA-completeness for a “classical” problem
- exponentially large stochastic matrices arise in Markov chains
- natural tensor product structure:

$$p_{t+1} = M p_t$$

$$q_{t+1} = N q_t$$

joint probability evolves with $M \otimes N$

Even more applications

- Frustration Free Hamiltonian: sum of local terms, ground state is simultaneous ground state of each one
- AQC in ground state of stoquastic frustration-free Hamiltonian is efficiently classically simulable [Bravyi & Terhal]
- But other eigenstates are still universal – use gizmos

joint work with Peter Love and David Gosset,
to be posted “any day now”

- Kitaev's original 5-local clock Hamiltonian for universal AQC is frustration-free
- Our stoquastizer gizmo preserves frustration-freeness
- BUT: Kitaev's Hamiltonian has complex matrix elements

- expand Kitaev's Hamiltonian as

$$H_5 = \sum_k \alpha_k S_k$$

  where each α_k is a negative real number and each S_k is a tensor product of five Pauli matrices

- The nonzero matrix elements of each S_k are all ± 1 or $\pm i$

- replace the group {1,i,-1,-i} by its regular representation

$$1 \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad i \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$-1 \rightarrow \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad -i \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- The resulting Hamiltonian acts on *n+2* qubits. If these are in the state $\frac{1}{2}\left(|0\rangle + i|1\rangle - |2\rangle - i|3\rangle\right)$ we get back the original Hamiltonian H_5

- Hence: universal AQC in the ground state of the $\frac{1}{2}\left(|0\rangle + i|1\rangle - |2\rangle - i|3\rangle\right)$ subspace of a 7-local frustration-free stoquastic Hamiltonian
- add a (somewhat complicated) penalty term $\Longrightarrow$ highest energy of 6-local frustration-free stoquastic Hamiltonian is QMA-complete
- stark contrast from ground state in P!

Summary

- simulate k-local Hamiltonian by 2-local Hamiltonian: use gadgets
- simulate Hamiltonian with arbitrary matrix elements using only real and positive: use gizmos
- open problem: gadgets with a weaker condition than $\lambda\|V\| < \frac{\gamma}{2}$