Control 101
The challenge of finding the right pulse
Control theory
Leakage elimination
Fock state preparation
Optimal control of open systems
Summary

Optimal control of imperfect qubits

F.K. Wilhelm[1] F. Motzoi [1] J.M. Gambetta [1] B. Khani [1] P. Rebentrost[1 2] I. Serban[1 3] Thomas Schulte-Herbrüggen [4]

[1]University of Waterloo, Canada

[2]Harvard University, USA

[3]Leiden University, Netherlands

[4]Munich University of Technology, Germany

KITP @ UCSB 2009

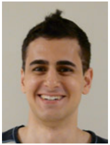
Research group

CIFAR: Jay Gambetta

PD: Seth Merkel

PhD: Felix Motzoi

MSc: Pierre-Luc Dallaire-Demers



MSc: Botan Khani

CIFAR:Bill Coish

PD: Mo-hammad Ansari

PhD: Peter Groszkowski

PhD: Farzad Qassemi Maloomeh

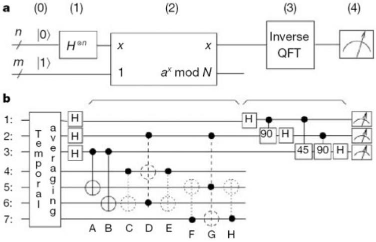
MSc: Cheng Shen

Contents

1. Control theory and the GRAPE algorithm
   - The challenge of finding the right pulse
   - Control theory
2. Leakage elimination
3. Fock state preparation
4. Optimal control of open systems

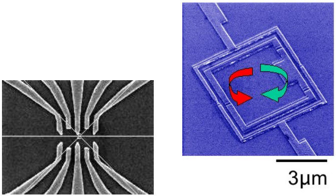
Quantum computing and quantum gates

- N-qubit quantum computer universal $\Leftrightarrow$ any $U \in U\left(2^N\right)$.
- Building blocks: Single qubit rotations + entangling two-qubit gate
- Need error rate below some threshold p



From device to gate

Qubit candidate device



↓ known properties

↓ fabrication parameters

Hamiltonian $H(t)$

$$H(t) = H_0 + H_{\mathrm{control}}(t) + H_{\mathrm{dec}} + H_{\mathrm{junk}}$$

↓ Schrödinger equation

↓ optimized controls

Quantum gate U_{Gate}

Approximate quantum map

$$F \simeq U_{\mathrm{gate}} \otimes \bar{U}_{\mathrm{gate}}$$

Basic problem setting

- Our physical system gives us a Hamiltonian

$$H(t) = H_{\mathrm{d}} + \sum_j u_j(t) H_j$$

with *drift* H_{d}, controls u_j and *control Hamiltonians* H_j.

- Our goal: Build a *propagator*

$$U_{\mathrm{gate}} = U(t,0) = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right)$$

using physical $u_j(t)$.

Rotating wave and area theorem.

Spin in static *z* plus rotating *xy* field

$$H(t) = -\gamma \vec{B}(t) \cdot \vec{\sigma} = \frac{1}{2} \begin{pmatrix} E & \lambda(t)e^{i\omega t} \\ \lambda(t)e^{-i\omega t} & -E \end{pmatrix}$$

in co-rotating frame

$$H'(t) = \frac{1}{2} \begin{pmatrix} E-\omega & \lambda(t) \\ \lambda(t) & -(E-\omega) \end{pmatrix}$$

On resonance: $E - \omega = 0$ $[H'(t), H'(t')] = 0$, thus

$$\mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right) = \exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right) =$$

$$= \cos\phi(t) - i\sigma_x \sin\phi(t) \qquad \phi(t) = \frac{1}{\hbar}\int_0^t dt' \lambda(t')$$

Area theorem

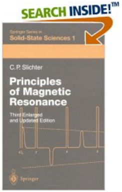
Beyond the area theorem

The area theorem does in general not hold for $[H'(t), H'(t')] \neq 0$

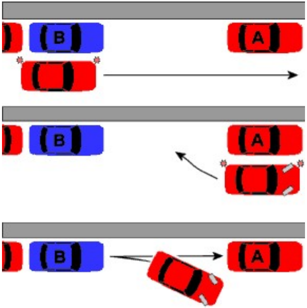
- out of resonance
- for non-rotating wave Hamiltonians *and* strong driving (non-RWA) i.e. high pulses $\rightarrow$ fast gates
- for multi-qubit systems

Complex control sequences

There are ingenious NMR solutions based on 50 years of quantum control
... do we have to do it again?

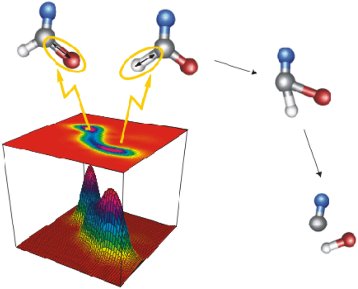


Analogous situation:
Steering / parallel parking



Control theory

- Established discipline in applied math / engineering
- Applied to quantum systems for state transfers e.g. in quantum chemistry (Rabitz ...)
- Developed for NMR by N. Khaneja (Harvard), S.J. Glaser, T. Schulte-Herbrüggen . . . (TUM)



Basic idea

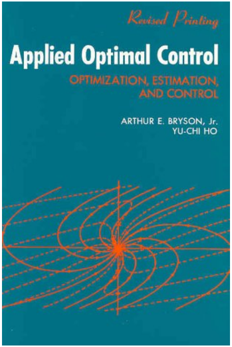
Take any *dynamical system* with variables x_i and controls u_j with EOM

$$\dot{x} = f(x, u, t)$$

Optimize a *performance index* at final time t_f, $\phi(x(t_f), u(t_f))$ using

$$\begin{aligned} J &= \phi(x(t_f), u(t_f)) + \\ &\int_{t_i}^{t_f} dt \lambda^T(t)(\dot{x} - f(x, u, t)) \end{aligned}$$

with initial conditions $x(t_i)$.



From Rockets to Propagators

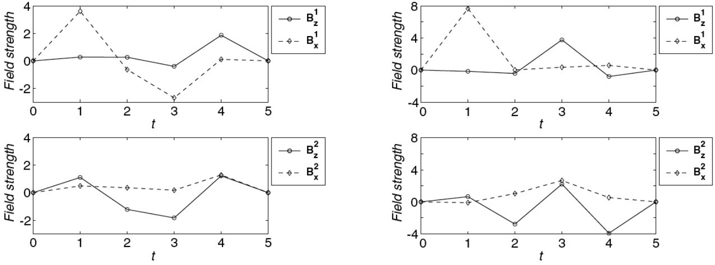
- Control problem for a quantum gate:

$$
\begin{aligned}
x &\mapsto U(t) \quad U(t_i) = \hat{1} \\
f &\mapsto -i(H_d + \sum_i u_i(t) H_i) U \\
\phi &= \left\| U_{\mathrm{gate}} - U(t_f) \right\|^2 = 2N - 2\mathrm{Re}\mathrm{Tr}(U_{\mathrm{gate}}^\dagger U(t_f))
\end{aligned}
$$

- So we need to *maximize* $\mathrm{Tr}(U_{\mathrm{gate}}^\dagger U(t_f))$.
- *Problem*: Fixes global phase, too
- *Solution*: Maximize $\Phi = |\mathrm{Tr}(U_{\mathrm{gate}}^\dagger U(t_f))|^2$ instead.

Numerical solution

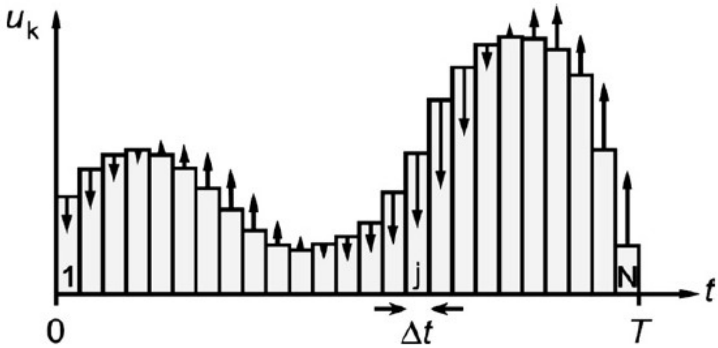
Numerical solution: Minimize J directly.
Problem: Computationally hard optimization, numerical gradients $\frac{\partial \phi}{\partial u_i}$ time-consuming ($\approx$ hours on supercomputer).



A.O. Niskanen, J.J. Vartiainen and M.M. Salomaa, PRL **90**, 197901 (2003).

Challenge

In the discretized grid, how does Φ change when the control is changed in one point?



Gradient Ascent Pulse Engineering (GRAPE) I

Rewrite performance index

$$\begin{aligned}\Phi &= |\mathrm{Tr}(U_{\mathrm{gate}}^{\dagger}U(t_f))|^2 = \left|\mathrm{Tr}(U^{\dagger}(t_j,t_N)U_{\mathrm{gate}})^{\dagger}U(t_j,t_1)\right|^2 \\ &= \left|\mathrm{Tr}\left(U_{j+1}^{\dagger}\dots U_N^{\dagger}U_{\mathrm{gate}}\right)^{\dagger}U_j\dots U_1\right|^2\end{aligned}$$

Trotterized time-step propagators

$$U_i = \exp\left(-i\Delta t\left(H_d + \sum u_k(t_i)H_k\right)\right) \quad (1)$$

Using

$$\frac{d}{dx}e^{A+Bx}\bigg|_{x=0} = e^A\int_0^1 d\tau e^{-A\tau}Be^{A\tau} \quad (2)$$

Gradient Ascent Pulse Engineering (GRAPE) II

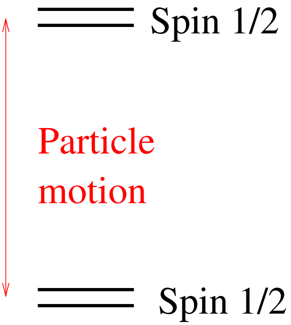
we can derive $\frac{\partial \Phi}{\partial u_k}$ analytically

$$\frac{\partial \Phi}{\partial u_k(t_j)} = \delta t \mathrm{Re}\left[\left(\mathrm{Tr} U_{j+1}^\dagger \ldots U_N^\dagger U_{\mathrm{gate}} H_k U_j \ldots U_1\right) \left(\mathrm{Tr} U_1^\dagger \ldots U_j^\dagger U_{\mathrm{gate}} U_N \ldots U_{j+1}\right)\right]$$

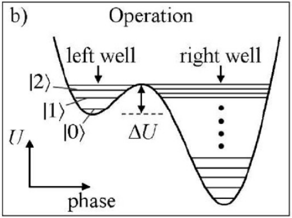
N. Khaneja, T. Reiss, C. Kehlet, T. Schulte-Herbrüggen, S.J. Glaser, Journal of magnetic resonance **172**, 296 (2005).

Leakage

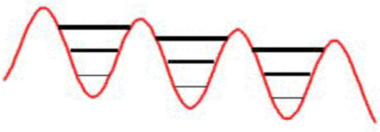
Nuclear/electron spin:



Phase qubit:



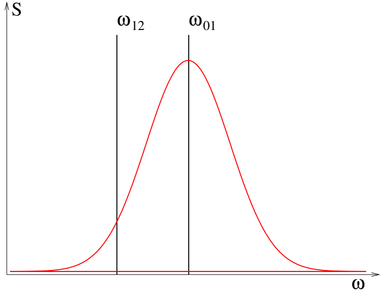
Optical lattice:



- Harmonic oscillator is not a qubit (only classical states accessible)
- Decoherence / complexity-optimized qubits often have weak nonlinearity: Almost HOs

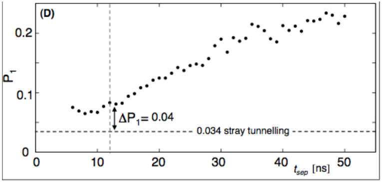
Spectral limitation

- Strategies aim at *never* occupying leakage state
- Rabi pulse at ω_{01}, duration T, bandwidth $\simeq$ Rabi frequency $\simeq \pi/T$
- Resonance frequency ω_{01}, leakage frequency ω_{12}
- Need to constrain $|\omega_{01} - \omega_{12}| \ll \pi/T$: Speed limit



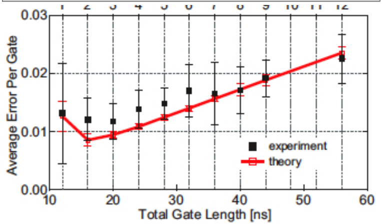
Experimental problem

Qubits with good control and long T_1, T_2:



Phase qubit: Leakage error at short pulses.

E. Lucero *et al.*, PRL 2008



Transmon: Leakage limits randomized benchmarking quality.

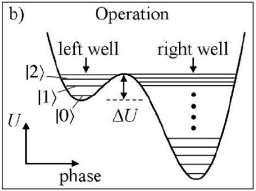
J. Chow *et al.*, PRL 2009

Weak nonlinearities

Phase qubit, transmon, vibrational qubits

$$\delta\omega = \omega_{01} - \omega_{12} \simeq 0.1\omega_{01}$$

Drive resonantly on ω_{01}



Fast gate → large bandwidth → leakage to the higher level

Hamiltonian

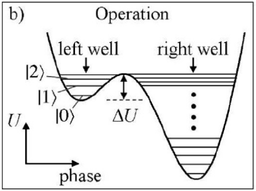
$$H = \begin{pmatrix} 0 & \lambda(t)\cos\omega_{01}t & 0 \\ \lambda(t)\cos\omega_{01}t & \omega_{01} & \sqrt{2}\lambda(t)\cos\omega_{01}t \\ 0 & \sqrt{2}\lambda(t)\cos\omega_{01}t & \omega_{01}+\omega_{21} \end{pmatrix}$$

Weak nonlinearities

Phase qubit, transmon, vibrational qubits

$$\delta\omega = \omega_{01} - \omega_{12} \simeq 0.1\omega_{01}$$

Drive resonantly on ω_{01}



Fast gate $\rightarrow$ large λ $\rightarrow$ leakage to the higher level

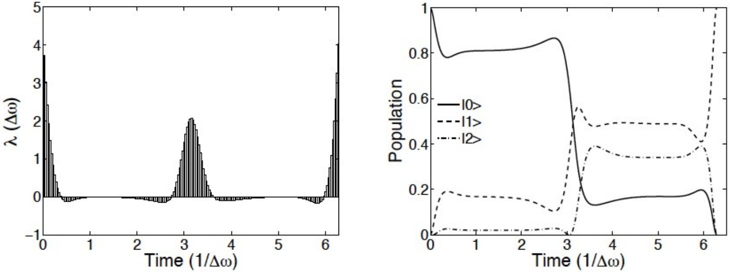
RWA Hamiltonian

$$H' = \begin{pmatrix} 0 & \lambda(t) & 0 \\ \lambda(t) & 0 & \sqrt{2}\lambda(t) \\ 0 & \sqrt{2}\lambda(t) & -\delta\omega \end{pmatrix}$$

Solution

Minimize the Rabi time.
Optimal time: $t_g \delta f = 1 + \epsilon$

$$U_{\text{gate}} = e^{i\phi_1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & e^{i\phi_2} \end{pmatrix}$$



Working transition: $R(\pi/4)R(\pi/2)R(\pi/4) = R(\pi)$
Leakage transition: $R(\pi/4)R(-\pi/2)R(\pi/4) = \hat{1}$.

P. Rebentrost and FKW, PRB 2009

Two quadrature solution

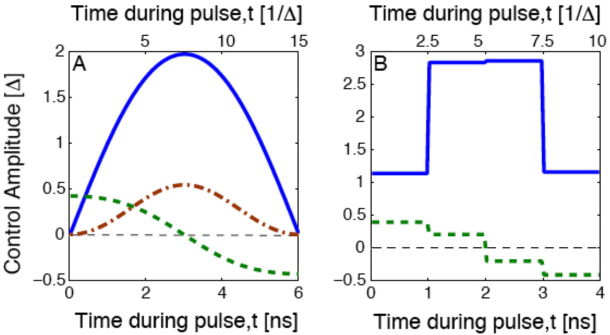
- Turn all knobs at the same time, use I-Q-mixer
- In- and out of phase components
  $\lambda_1(t)\cos\omega t + \lambda_2(t)\sin\omega t$
- Rotating frame, $z = \lambda_1 + i\lambda_2$

$$H' = \begin{pmatrix} 0 & z(t) & 0 \\ z^*(t) & 0 & \sqrt{2}z(t) \\ 0 & \sqrt{2}z^*(t) & -\delta\omega \end{pmatrix}$$

- Control both real *and* imaginary parts of z

Of course, it will be better, but how much?

Numerical sulution



- $\lambda_2 \propto \dot{\lambda}_1$!!
- Requires detuning or phase ramping
- Phase ramping: rotate λ_1, λ_2 into $\tilde{\lambda}_1$, $\tilde{\lambda}_2$

DRAG — why the derivative?

Toggling frame $H'(t) = V(t)H(t)V^\dagger(t) + i\dot{V}V^\dagger$

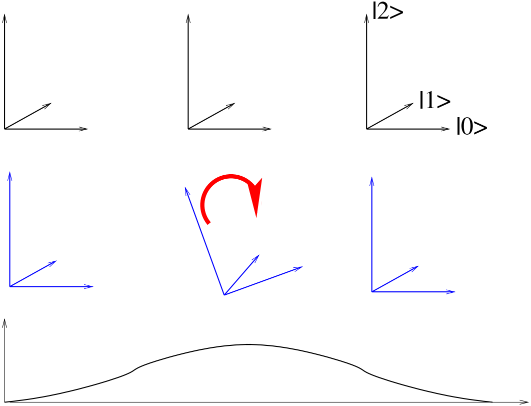
$$V(t) = \exp\left(-i\lambda_1 Y_3/\delta\omega\right) \quad Y_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix}$$

- $\lambda_i(0) = \lambda_i(t_g) = 0$: Gates are qubit gates
- $H_{\mathrm{eff}} = H_{\mathrm{diag}} + \lambda_1\hat{\sigma}_x + \left[\lambda_2 + \frac{\dot{\lambda}_1}{\delta\omega}\right] Y_3 + \frac{\lambda_1^2}{\sqrt{2}\delta\omega}(|0\rangle\langle 2| + \mathrm{h.c.})$
- Eliminate leakage by $\lambda_2 = -\dot{\lambda}_1/\delta\omega$
- remove higher order terms by higher order corrections

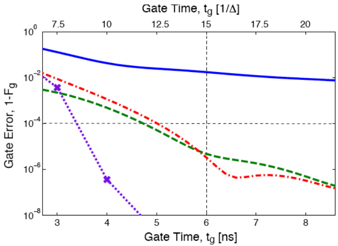
Derivative Removal by Adiabatic Gate

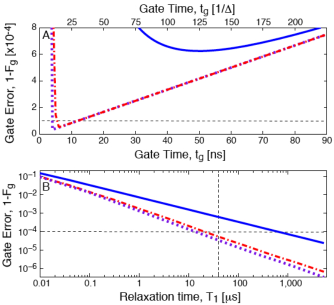
Physical picture

- traditional thinking: Limit bandwidth
- DRAG: Preserve adiabaticity + move on closed loop



Performance of DRAG + GRAPE



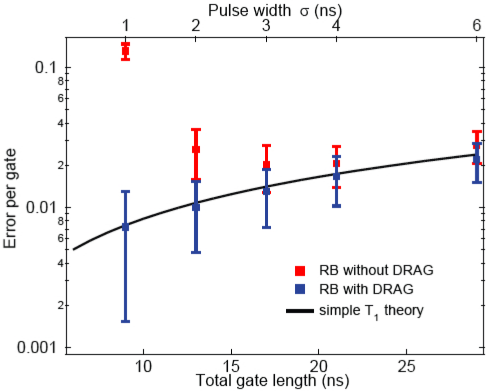


Gaussian, GRAPE,
Gauss-DRAG,
softbox-DRAG

Top: $T_1 = 40\mu s$
Bottom: Error vs. T_1

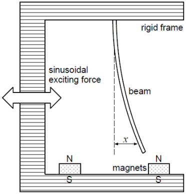
F. Motzoi, J.M. Gambetta, P. Rebentrost, FKW, PRL 2009

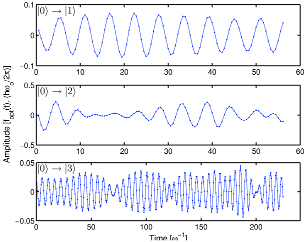
It works!

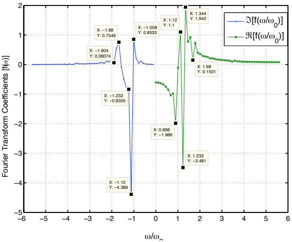


$T_1 \simeq 1.2\mu s$, J.M. Chow *et al.*, arXiv:0908.1955

Weakly nonlinear oscillators







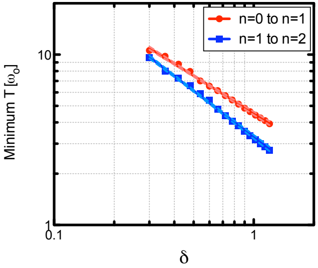
- Nanomechanicals
- Large Josephson junctions
- Ion traps
- Light in nonlinear media

Hamiltonian (Duffing)

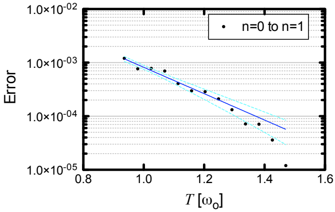
$$H = \hbar\omega_0 \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\delta}{12} (a + a^\dagger)^4.$$

Fock-State preparation

Minimal time for preparation



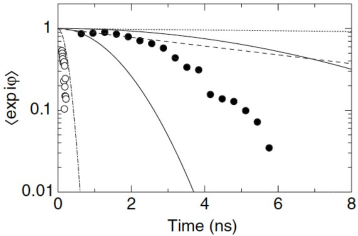
Error at fixed time



- Power law $t_{\min} \propto \delta^{-\alpha}$ with $\alpha_{01} = 0.73 \pm 0.029$ and $\alpha_{12} = 0.90 \pm 0.031$
- Qualitative difference to simple Landau-Zener limt $t_g \propto 1/\delta$

B. Khani, J.M. Gambetta, F. Motzoi, FKW, Physics Scripta, in press; arXiv:0909.4788

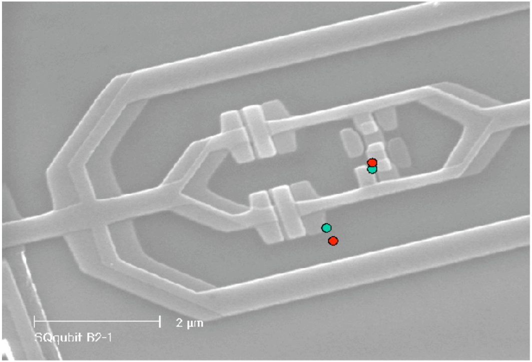
Finding controls in hostile environments



- Phase error rate $1/T_2$ increased by echo.
- Based on knowing that $H_{\rm decohernce}(t)$ is slow

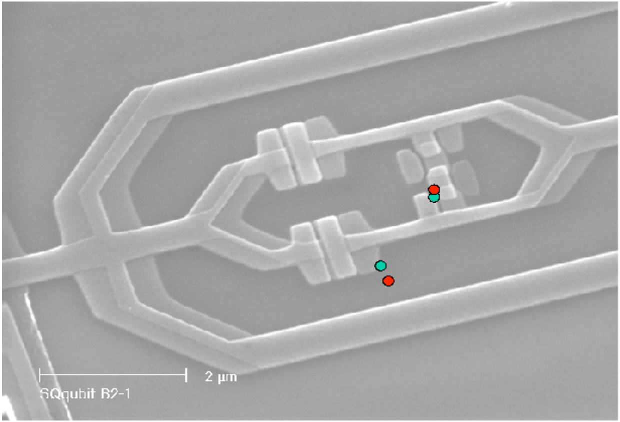
- Error rate depends on how $H_{\rm control}(t)$ is chosen
- Usually found by manual construction or NMR tricks
- Control theory: Find controls systematically

Slow fluctuators

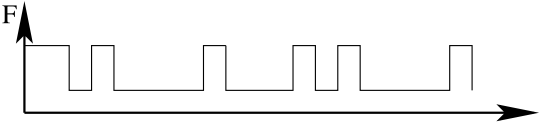


$$\hat{H}_S = E_1(t)\hat{\sigma}_z + \Delta\hat{\sigma}_x + E_2\hat{\tau}_z + \Lambda\hat{\sigma}_z X(t)$$
$$\langle X(t)X(0)\rangle_\omega \propto 1/\omega$$

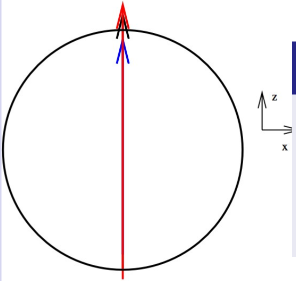
Simplified materials noise model



Classical limit = telegraph noise:



Optimum working point



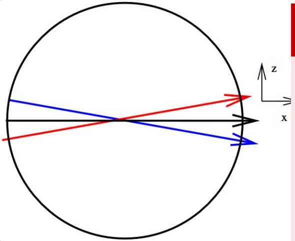
Change of precession frequency

No transition

$1/T_1 \to \infty$

$1/T_2 = S(0)$

Low-frequency noise power (high)



No change of precession

$\partial|B|/\partial B_z = O(B_z/B)$

Environment coupling needs transition

$1/T_1 = S(B)$

$1/T_2 = 1/(2T_1)$

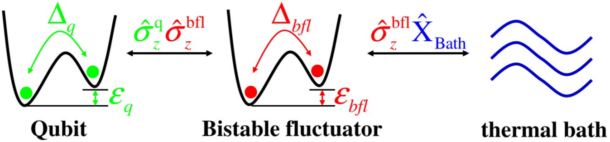
High-frequency noise (low)

Initial boost of T_2 from 10 ns to 500 ns.

D. Vion *et al.*, Nature 2002.

Open system control problem

Decoherence time scales $T_{1/2}$: Fastest = best ?



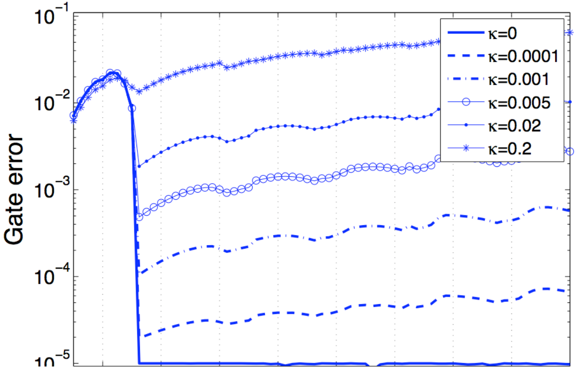
- Long correlation time switching: Nonmarkovian qubit dynamics
- Use master equation for qubit+fluctuator system $\rho_{\mathrm{q+fl}}$
- Trace out fluctuator *after* solving $\rho_q = \mathrm{Tr}_{\mathrm{fl}} \rho_{q+fl}$

E. Paladino *et al.*, PRL 2001

openGRAPE

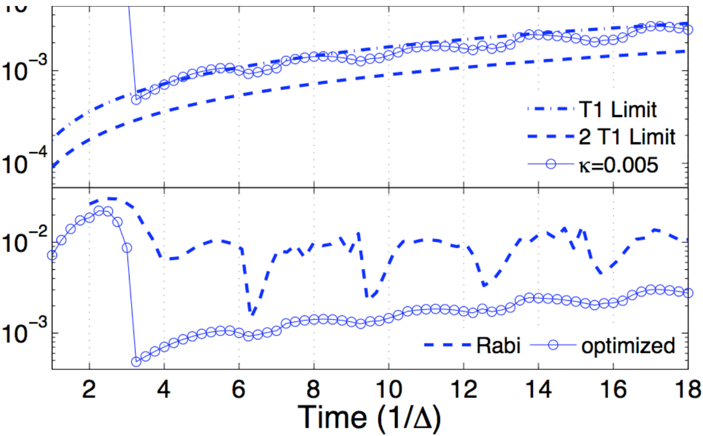
Set up optimal control problem for quantum map

$$F\left(\rho_q(0)\right) = \rho_q(t_g) \qquad F_{\text{target}} = U \otimes \bar{U}.$$



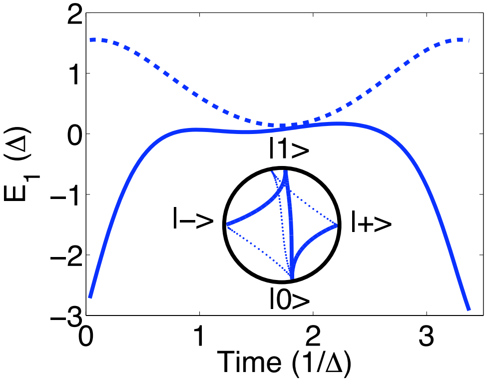
- lower time limit π/Δ
- increasing error at long times, oscillations
- no error at no bath coupling

Pulse performance



- T_1-limited
- Rabi pulse performs well at magic times $n\pi/\Delta$
- Cancelling counter-rotating terms at short times

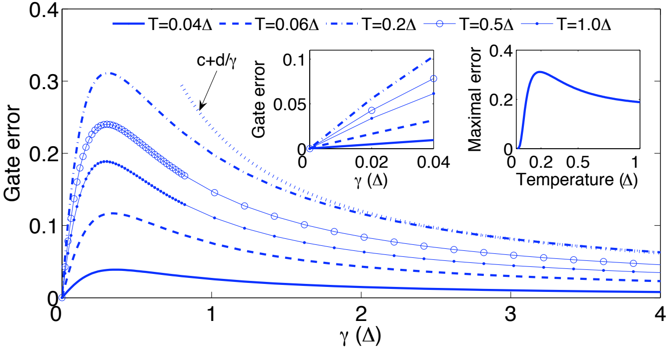
Pulse shapes



- Use Δ (X-field) to take a spin between states
- Anharmonic short Rabi burst: cancels counter-rotating term

P. Rebentrost, I. Serban, T. Schulte-Herbrüggen, FKW, PRL 2009

Gauging irreducible decoherence



- low κ static error — perfect correction
- large κ — motional narrowing
- $\kappa \simeq \Delta$ — cannot be corrected

Summary

- Pulse shaping as new resource for improving qubits
- Removal of leakage errors by DRAG
- Optimized optimal working point

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