Entanglement and universality in quantum computation

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Problem: What resources are universal for quantum computation?

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Problem: What resources are universal for quantum computation?

Church-Turing-Deutsch Conjecture:
All physical systems can be efficiently simulated on a quantum computer.
Universal set I

Control qubit \( |c\rangle \quad |c\rangle \)

Target qubit \( |t\rangle \quad |t \oplus c\rangle \)

\[ \mathcal{U} \]

interaction with the environment measurements

Universal set II

Quantum memory
2-qubit projective measurements

Can we build a programmable quantum computer?

\[ |\psi\rangle \rightarrow U|\psi\rangle \]

Fixed array

\[ U \rightarrow |U\rangle \rightarrow |P_{U,\psi}\rangle \]

No-programming theorem: (MAN, Chuang, PRL ’97)
Distinct unitary operators \( U_1, \ldots, U_n \) require orthogonal programs \( |U_1\rangle, \ldots, |U_n\rangle \).

A stochastic programmable quantum computer

\[ |\psi\rangle \rightarrow \text{Bell Measurement} \]

\[ |U\rangle \rightarrow \bigg\{ \bigg( I \otimes U \bigg) \frac{|00\rangle + |11\rangle}{\sqrt{2}} \bigg\} \rightarrow \bigg( U \sigma_j \bigg) |\psi\rangle \]

\[ |U\rangle \equiv (I \otimes U) \frac{|00\rangle + |11\rangle}{\sqrt{2}} \]

(Nielsen, Chuang, PRL 97)
Why it works

\[ |\psi\rangle \]

\[ \{ \frac{|00\rangle + |11\rangle}{\sqrt{2}} \} \]

\[ U \]

\[ U \sigma_j |\psi\rangle \]

\[ \sigma_j |\psi\rangle \]

(Nielsen, Chuang, PRL 97)

How to do single-qubit gates

\[ |\psi\rangle \]

\[ \{ \frac{|00\rangle + |11\rangle}{\sqrt{2}} \} \]

\[ U \sigma_k \]

\[ U \sigma_k U \sigma_k \sigma_j |\psi\rangle \]

\[ |U_k\rangle = (I \otimes U \sigma_k) \frac{|00\rangle + |11\rangle}{\sqrt{2}} \]

With probability \( \frac{1}{4}, j = k \), and the gate succeeds.

(Nielsen, quant-ph 2001)
Coping with failure

Action was $U\sigma_k\sigma_j, j \neq k$ - a known unitary error.

Now attempt the gate $U(U\sigma_k\sigma_j)^\dagger$.

Successful with probability $\frac{1}{4}$, otherwise repeat.

Failure probability $\varepsilon$ can be achieved with $O(\log\frac{1}{\varepsilon})$ repetitions.

(Nielsen, quant-ph 2001)

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How to do the controlled-not

\[ |\psi\rangle \]

\[ \xrightarrow{\text{Bell}^\otimes 2} \]

\[ \xrightarrow{\text{Measurement}} j, k = 0,1,2,3 \]

\[ |U_{hm}\rangle \]

\[ \xrightarrow{U(\sigma_i \otimes \sigma_m)(\sigma_j \otimes \sigma_k)} |\psi\rangle \]

\[ |U_{lm}\rangle = (I \otimes U\sigma_i \otimes \sigma_m)|\text{Bell}^\otimes 2 \]

With probability $\frac{1}{16}, j = l, k = m$, and the gate succeeds.

(Nielsen, quant-ph 2001)
Summary

An arbitrary quantum computation can be efficiently performed using just quantum memory and four-qubit projective measurements.


Problem: Is there a practical variant of this scheme?

Problem: What sets of measurement are sufficient to do universal quantum computation?

Universal set III

Suppose $H$ is any two-body entangling Hamiltonian, on $n$ qubits.

By alternating periods of evolution due to $H$ with single qubit gates, we may efficiently simulate an arbitrary quantum computation.

$$H = \sum_{j=1}^{n} \sum_{k=1}^{3} \alpha_k^j \sigma_k^j + \sum_{j,j'=-1}^{n} \sum_{k,k'=1}^{3} \beta_{jj'}^{kk'} \sigma_k^j \otimes \sigma_{k'}^{j'}$$

(Bristol, IBM, Innsbruck, Karlsruhe, LANL, MIT, UQ, mostly 2001)
Three observations

A. If we can simulate evolution due to $J$, and perform unitaries $U$ and $U^*$, then we can exactly simulate evolution due to $UJU^*$.

$$U e^{-iJ} U^* = e^{-iJ}$$

B. If we can simulate evolution due to $J_1$ and $J_2$ for a small time $\Delta$, then we can approximately simulate evolution due to $J_1 + J_2$ for time $\Delta$.

$$e^{-i(J_1 + J_2)\Delta} = e^{-iJ_1\Delta} e^{-iJ_2\Delta} + O(\Delta^2)$$

C. If we can simulate evolution due to $J$, then we can exactly simulate evolution due to $\alpha J$ for any positive $\alpha$.

A two-qubit example

Given: $H = Z \otimes I + 2X \otimes Z + Z \otimes Z$

$$(X \otimes I)H(X \otimes I)^\dagger = -Z \otimes I + 2X \otimes Z - Z \otimes Z$$

$$X \otimes Z = \frac{(X \otimes I)H(X \otimes I)^\dagger + H}{4}$$

$$\sigma_j \otimes \sigma_k = (U \otimes V)X \otimes Z (U \otimes V)^\dagger$$

$$-\sigma_j \otimes \sigma_k = (U \otimes I)\sigma_j \otimes \sigma_k (U \otimes I)^\dagger$$

Any desired $K$ can be written as a positive linear combination of $\pm \sigma_j \otimes I_x, \pm I \otimes \sigma_k$, and $\pm \sigma_j \otimes \sigma_k$ terms.
**General two-qubit case**

Given: $H = \sum_{j,k} h_{jk} \sigma_j \otimes \sigma_k$

Choose the largest entangling term: $\sigma_r \otimes \sigma_s$.

$$\sigma_r \otimes \sigma_s = \frac{1}{4h_{rs}} \sum_{j=0, r, K=0, s} (\sigma_j \otimes \sigma_k) H (\sigma_j \otimes \sigma_k)^\dagger$$

$$- \frac{h_{rs}}{h_{rs}} \sigma_r \otimes I - h_{rs} I \otimes \sigma_s - h_{rs} I \otimes I$$

$$\sigma_j \otimes \sigma_k = (U \otimes V) \sigma_r \otimes \sigma_s (U \otimes V)^\dagger$$

$$- \sigma_j \otimes \sigma_k = (U \otimes I) \sigma_r \otimes \sigma_s (U \otimes I)^\dagger$$

Any desired $K$ can be written as a positive linear combination of $\pm \sigma_j \otimes I$, $\pm I \otimes \sigma_k$, and $\pm \sigma_j \otimes \sigma_k$ terms.

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**Errors**

To get $10^{-3}$ accuracy in a cnot we need $\approx 10^7$ operations.

Using higher-order simulation techniques, can achieve $10^{-3}$ accuracy using $\approx 10^4$ operations.

Leveraging specific knowledge we can get, for example, $10^{-3}$ accuracy using $\approx 10^2$ operations.
Extension to n qubits

\( X_S = \) tensor product of \( X \) operators on system \( S \).

\[
H' = \frac{H + X_S H X_S + Y_S H Y_S + Z_S H Z_S}{4}
\]

Leaves the Hamiltonian on \( P \) invariant, and eliminates all couplings between \( P \) and \( S \), all single qubit terms in \( S \), and all asymmetric couplings in \( S \), that is, couplings like \( X \otimes Y \).

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Extension to n qubits

\[
H'' = \frac{H' + X_{S_0} H' X_{S_0} + Y_{S_0} H' Y_{S_0} + Z_{S_0} H' Z_{S_0}}{4}
\]

Leaves the Hamiltonian on \( P \) invariant, and eliminates all couplings between \( S_0 \) and \( S_1 \).

\[
H'' + X_{S_{00}S_{10}} H'' X_{S_{00}S_{10}} + Y_{S_{00}S_{10}} H'' Y_{S_{00}S_{10}} + Z_{S_{00}S_{10}} H'' Z_{S_{00}S_{10}}
\]

Leaves the Hamiltonian on \( P \) invariant, and eliminates all couplings between \( S_{00} \) and \( S_{01} \), and between \( S_{10} \) and \( S_{11} \).
Summary

Any (two-body) entangling Hamiltonian is sufficient to do universal quantum computation, provided we can also do local unitaries.

Problem: Can we turn this scheme into a practical method for doing quantum computation?