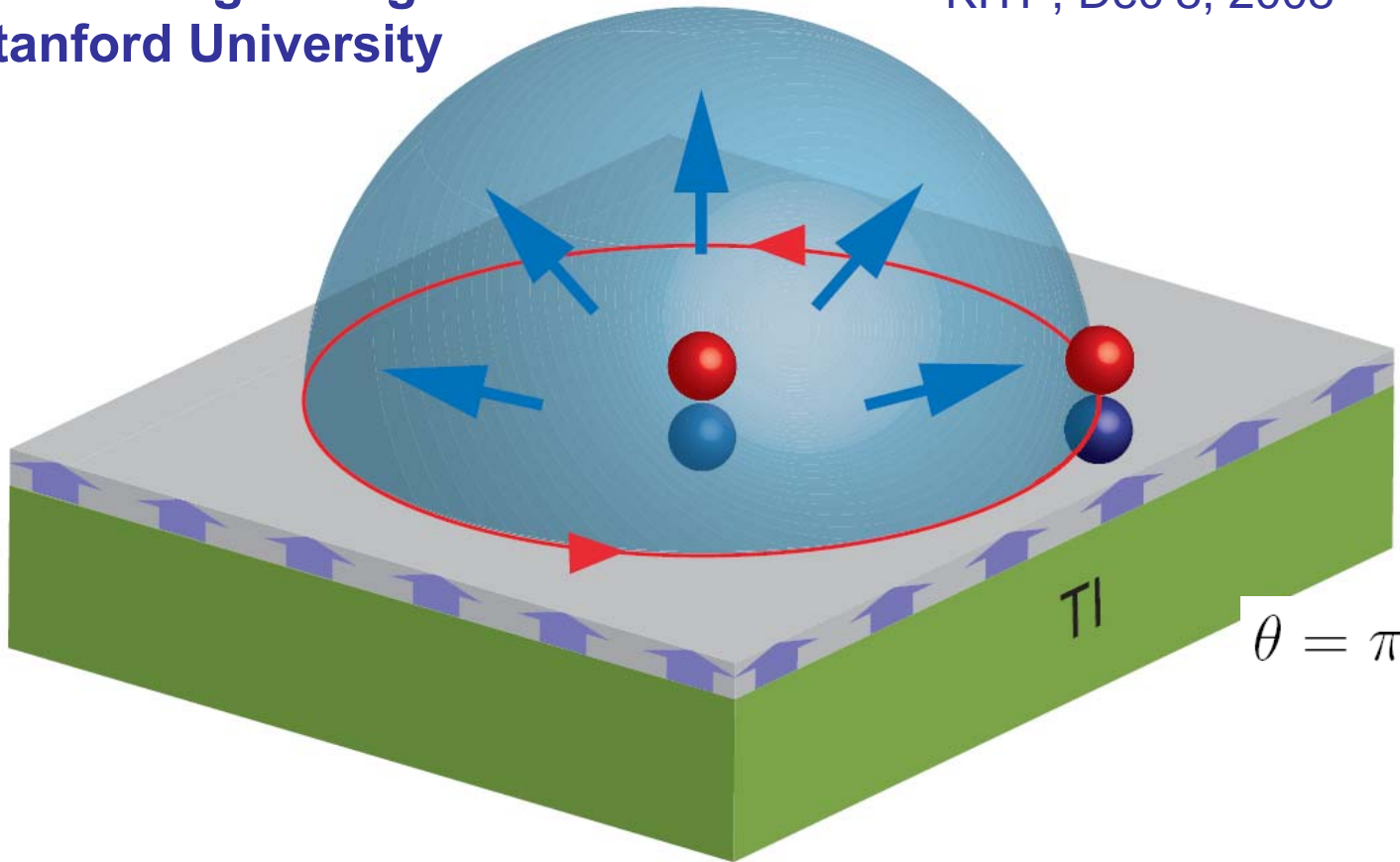
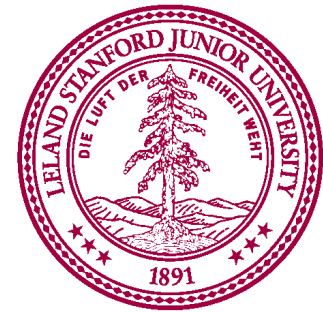


Topological Field Theory of Time Reversal Invariant Insulators

Shoucheng Zhang
Stanford University

KITP, Dec 8, 2008



$$S_{\theta} = \frac{\theta}{2\pi} \frac{\alpha}{16\pi} \int d^3x dt \epsilon_{\mu\nu\rho\tau} F^{\mu\nu} F^{\rho\tau} = \frac{\theta}{2\pi} \frac{\alpha}{4\pi} \int d^3x dt \partial^{\mu} (\epsilon_{\mu\nu\rho\sigma} A^{\nu} \partial^{\rho} A^{\sigma})$$

The search for new states of matter

The search for new elements led to a golden age of chemistry.

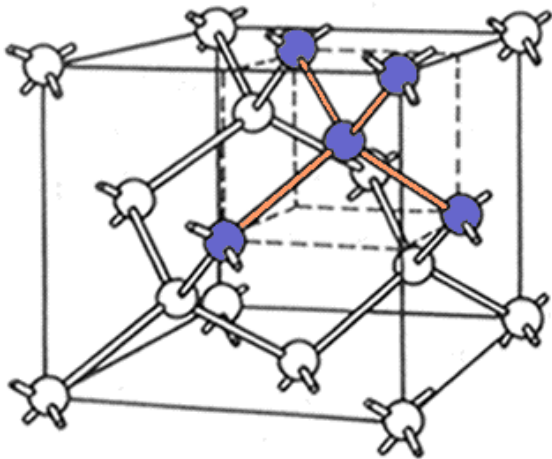
The search for new particles led to the golden age of particle physics.

In condensed matter physics, we ask what are the fundamental states of matter?

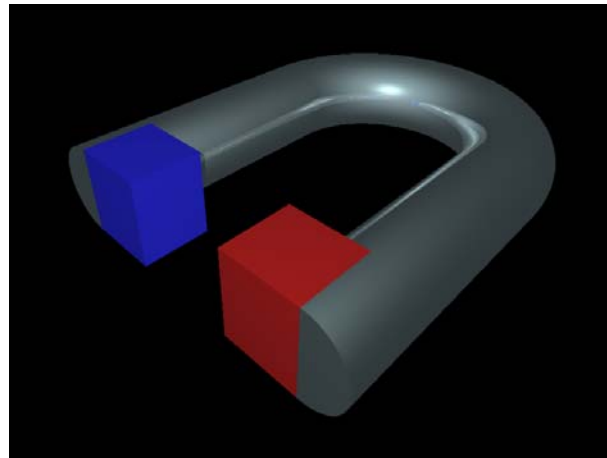
In the classical world we have solid, liquid and gas. The same H_2O molecules can condense into ice, water or vapor.

In the quantum world we have metals, insulators, superconductors, magnets etc.

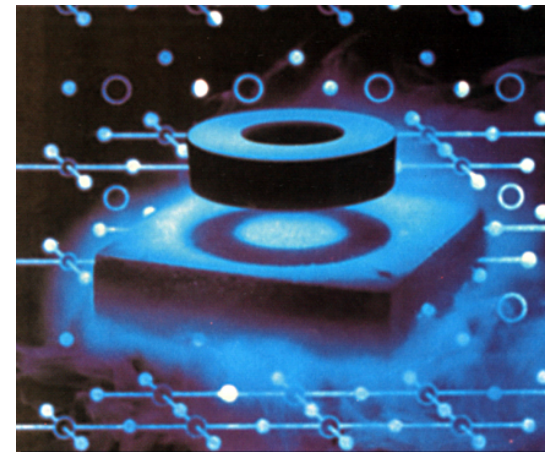
Most of these states are differentiated by the broken symmetry.



Crystal: Broken translational symmetry



Magnet: Broken rotational symmetry



Superconductor: Broken gauge symmetry

The quantum Hall state, a topologically non-trivial state of matter

$$\sigma_{xy} = n \frac{e^2}{h}$$

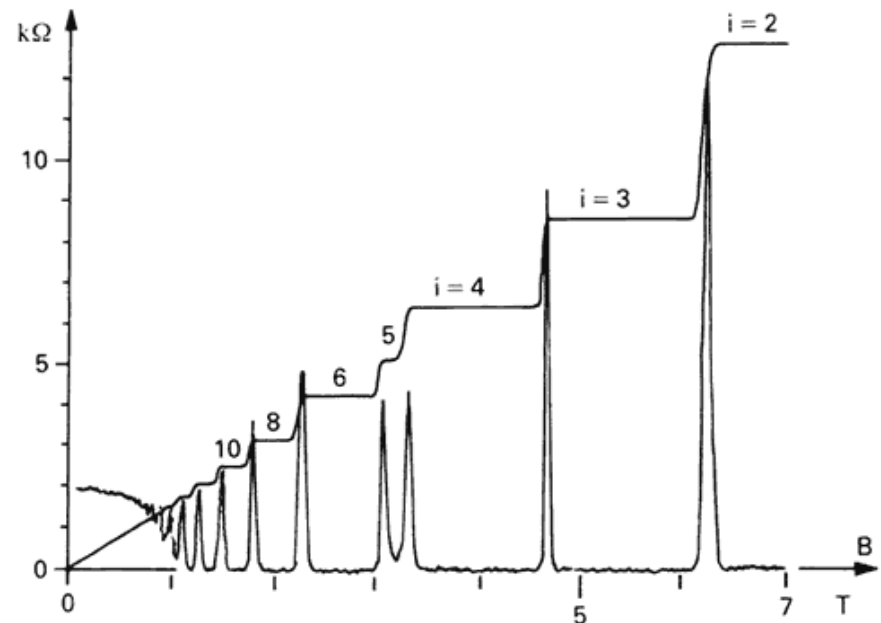
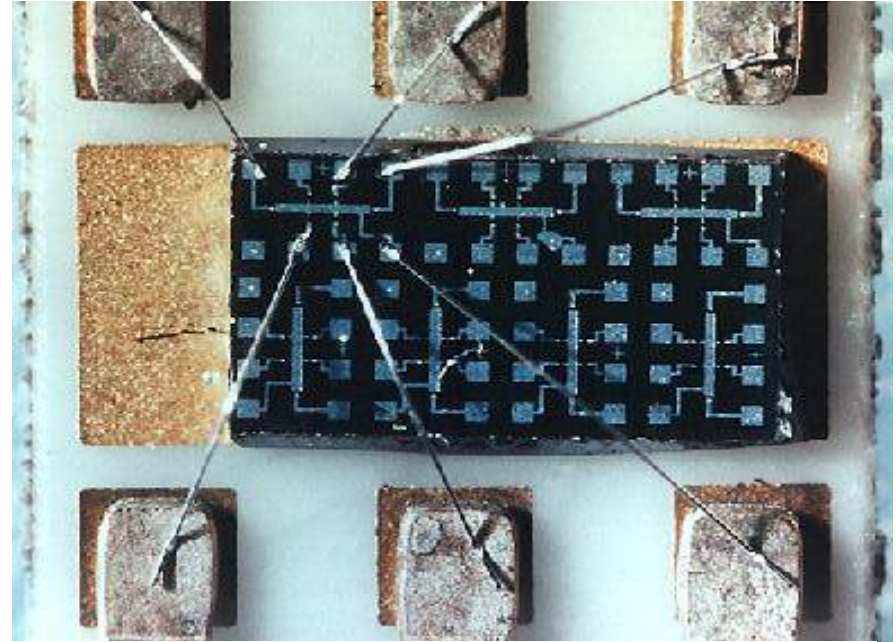
- TKNN integer = the first Chern number.

$$n = \int \frac{d^2k}{(2\pi)^2} \varepsilon^{\mu\nu} F_{\mu\nu}(k)$$

- Topological states of matter are defined and described by topological field theory:

$$S_{eff} = \frac{\sigma_{xy}}{2} \int d^2x dt \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

- Physically measurable topological properties are all contained in the topological field theory, e.g. QHE, fractional charge, fractional statistics etc...



Electromagnetic response of an insulator

- Electromagnetic response of an insulator is described by an effective action:

$$S_{eff} = \frac{1}{8\pi} \int d^3x dt (\epsilon \vec{E}^2 - \frac{1}{\mu} \vec{B}^2)$$

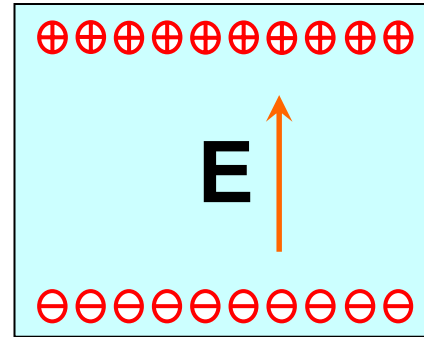
- However, another quadratic term is also allowed:

$$S_{\theta} = \frac{\theta}{2\pi} \frac{\alpha}{2\pi} \int d^3x dt \vec{E} \cdot \vec{B}$$

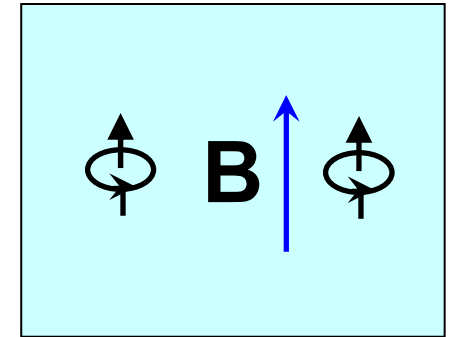
- Physically, this term describes the magneto-electric effect. Under time reversal:

$$\vec{E} \Rightarrow \vec{E} ; \vec{B} \Rightarrow -\vec{B}$$

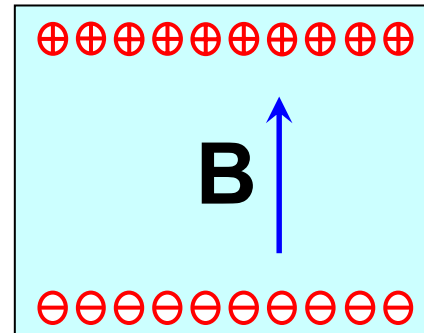
$$\theta \Rightarrow -\theta$$



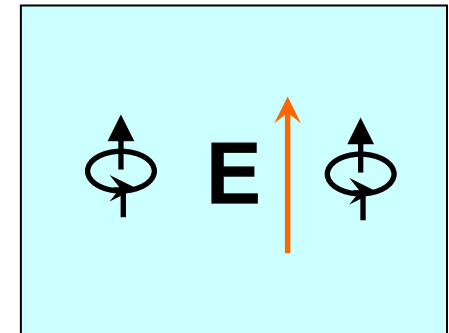
$$4\pi\mathbf{P}=(\epsilon-1)\mathbf{E}$$



$$4\pi\mathbf{M}=(1-1/\mu)\mathbf{B}$$



$$4\pi\mathbf{P}=\alpha \theta/2\pi \mathbf{B}$$



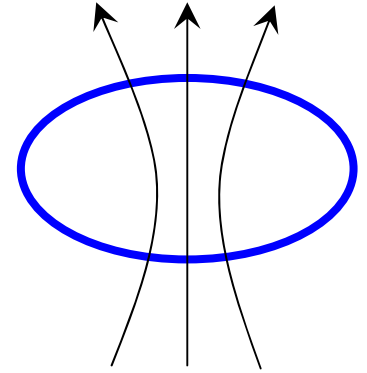
$$4\pi\mathbf{M}=\alpha \theta/2\pi \mathbf{E}$$

θ periodicity and time reversal

- Consider an analog system of a period ring. The flux enters the partition function as:
- Therefore, the physics is completely invariant under the shift of $\Phi \Rightarrow \Phi + 2\pi n$

$$\Phi = \int dx_{\mu} A^{\mu}$$

$$e^{i\Phi/\Phi_0}$$



- Under time reversal, $\phi \Rightarrow -\phi$, therefore, time reversal is recovered for two special values of ϕ , $\phi=0$ and $\phi=\pi$.

- The ME term is a total derivative, independent of the bulk values of the fields:

$$S_{\theta} = \frac{\theta}{2\pi} \frac{\alpha}{16\pi} \int d^3x dt \epsilon_{\mu\nu\rho\tau} F^{\mu\nu} F^{\rho\tau} = \frac{\theta}{2\pi} \frac{\alpha}{4\pi} \int d^3x dt \partial^{\mu} (\epsilon_{\mu\nu\rho\sigma} A^{\nu} \partial^{\rho} A^{\sigma})$$

- Integrated over a spatially and temporally periodic system,

$$\int c dt d^3x \vec{E} \cdot \vec{B} = \int dx dy B_z \int c dt dz \partial_t A_z = n \Phi_0^2$$

- Its contribution to the partition function is given by $e^{i\theta n}$. Therefore, the partition function is invariant under the shift:

$$\theta \Rightarrow \theta + 2\pi n$$

Time reversal symmetry is recovered at

$$\theta = 0, \quad \theta = \pi$$

Classification of all TRI insulators

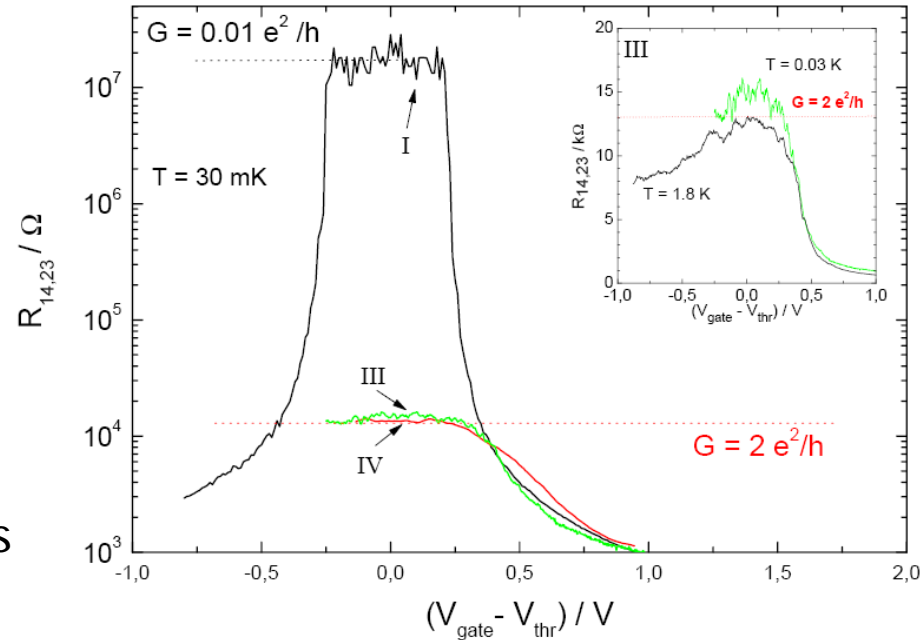
• Consider the most general, *periodic* model of a TRI insulator, with strong interactions or disorders. Couple fermions to external EM gauge fields and integrate out the fermions. All TRI insulators fall into two distinct classes:

$\theta = 0$ Topologically trivial insulators

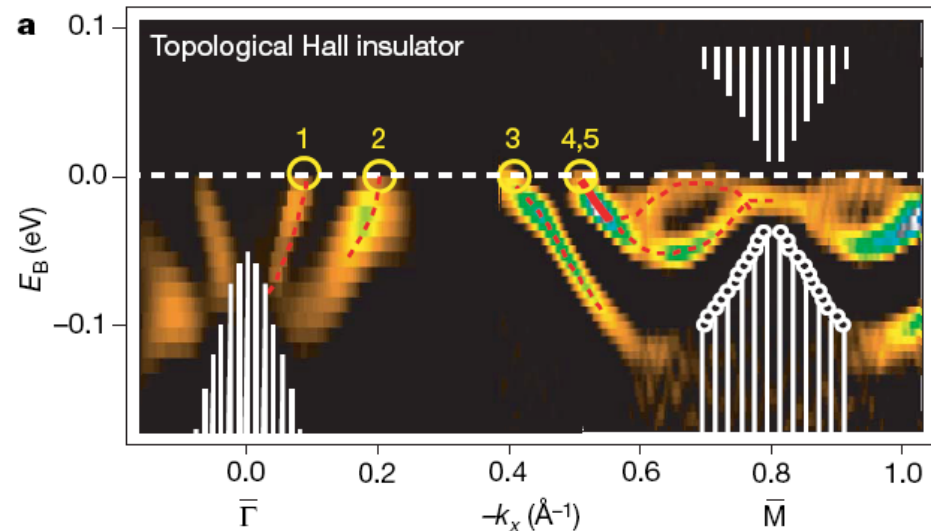
$\theta = \pi$ Topologically non-trivial insulators

- Kane and Mele
- Bernevig and Zhang
- Bernevig, Hughes and Zhang
- Koenig et al
- Wu, Bernevig and Zhang
- Xu and Moore
- Fu and Kane
- Moore and Balents
- Roy
- Qi, Hughes and Zhang
- Hsieh et al

Molenkamp group



Hasan group



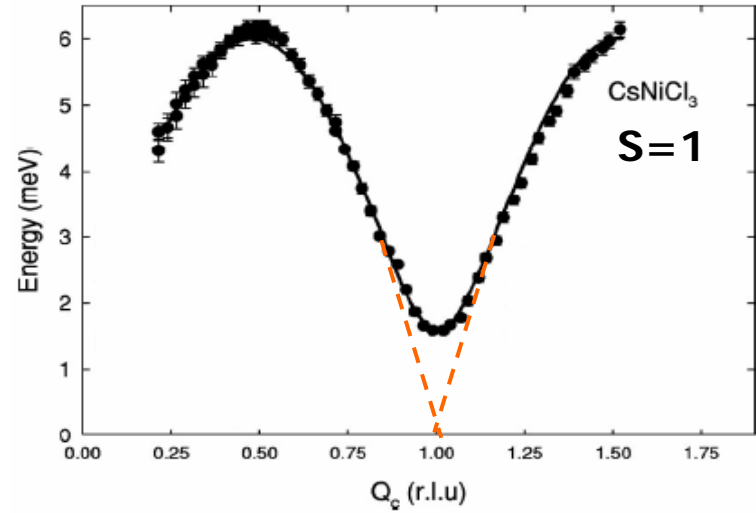
θ terms in condensed matter and particle physics

- Quantum spin chains:

$$S[\theta] = \theta \int dt dx \mathbf{n} \cdot (\partial_x \mathbf{n} \times \partial_t \mathbf{n}), \quad \theta = \frac{S}{2}$$

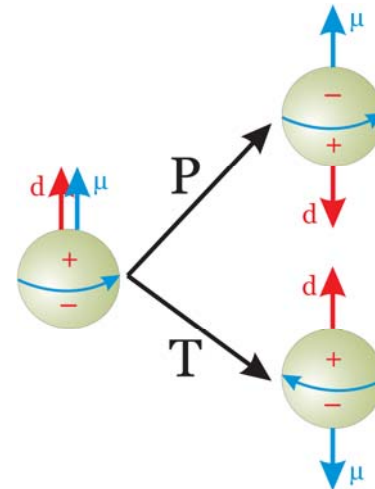
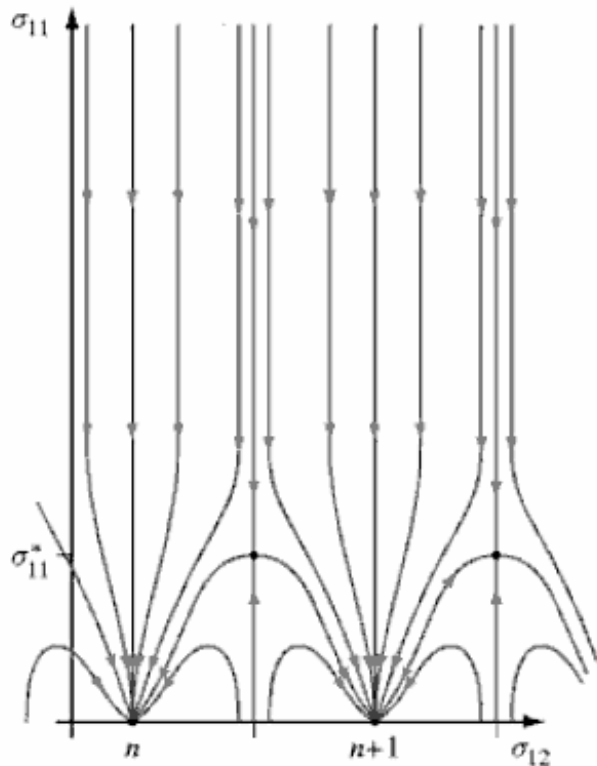
- Quantum Hall transitions:

$$S[\theta] = \theta \int d^2x \epsilon^{\mu\nu} \text{tr} (Q D_\mu Q D_\nu Q), \quad \theta = -\frac{\sigma_{xy}}{8}$$



- θ vacuum of QCD

$$S[\theta] = \theta \int d^4x \epsilon^{\mu\nu\rho\tau} \text{tr} (F_{\mu\nu}^a F_{\rho\tau}^a)$$



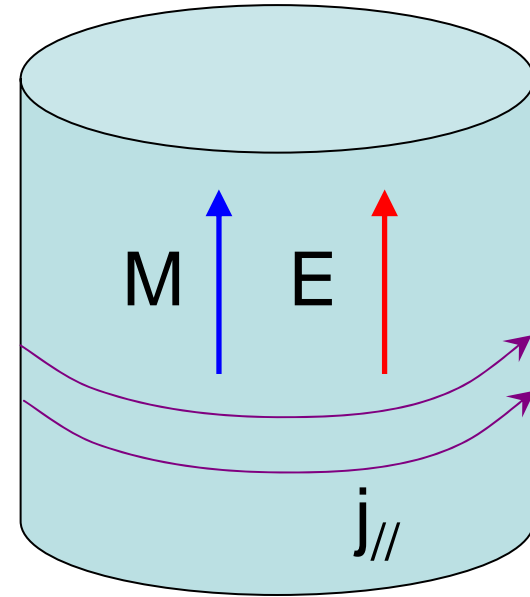
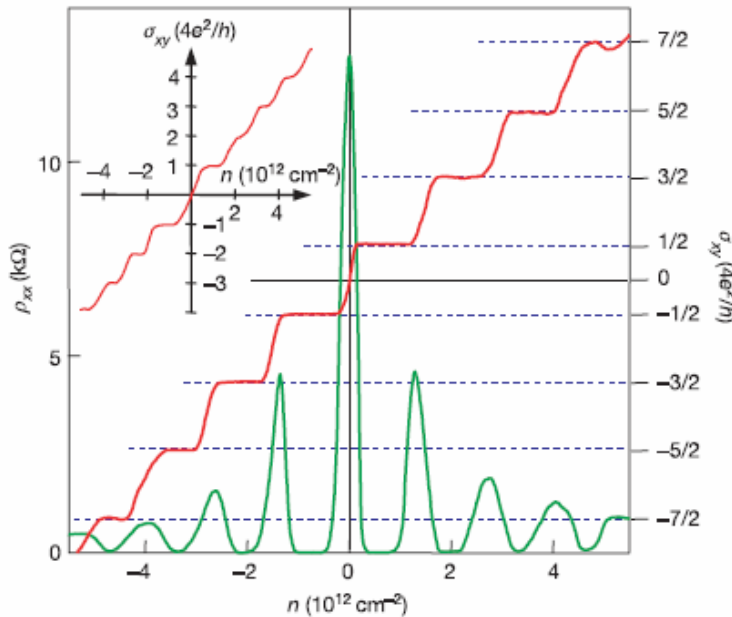
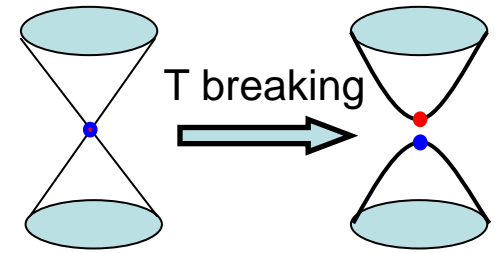
θ term with open boundaries

- $\theta = \pi$ implies QHE on the boundary with

$$\sigma_{xy} = \frac{1}{2} \frac{e^2}{h}$$

$$S_\theta = \frac{\theta}{2\pi} \frac{\alpha}{16\pi} \int d^3x dt \epsilon_{\mu\nu\rho\tau} F^{\mu\nu} F^{\rho\tau} = \frac{\theta}{2\pi} \frac{\alpha}{4\pi} \int d^3x dt \partial^\mu (\epsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma)$$

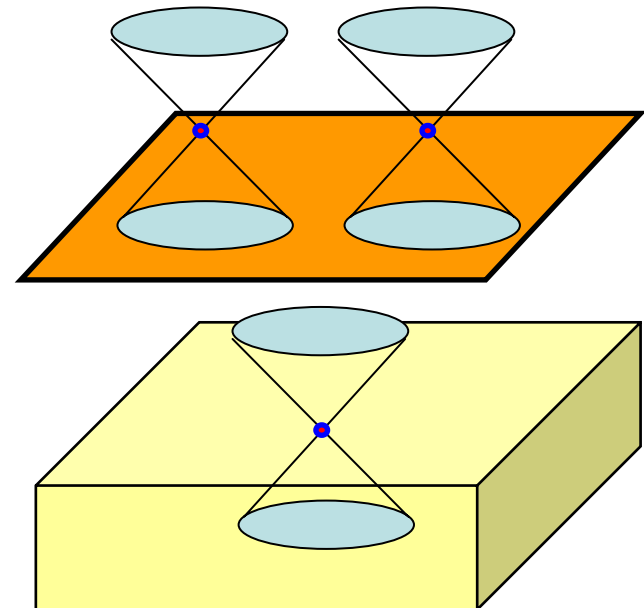
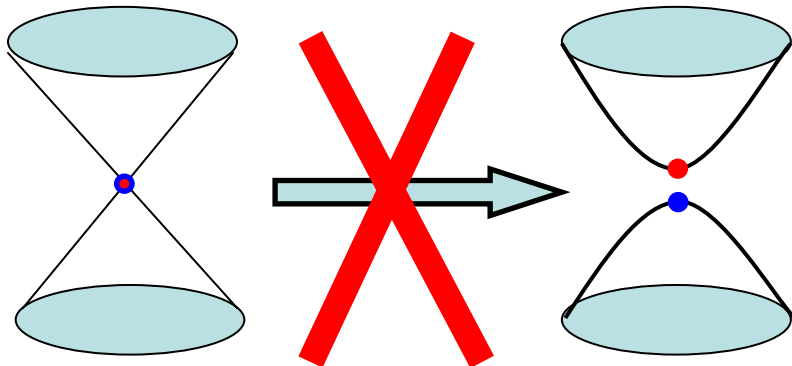
- For a sample with boundary, it is only insulating when a small T-breaking field is applied to the boundary. The surface theory is a CS term, describing the half QH.
- Each Dirac cone contributes $\sigma_{xy} = 1/2 e^2/h$ to the QH. Therefore, $\theta = \pi$ implies an odd number of Dirac cones on the surface!



- Surface of a TI = $1/4$ graphene

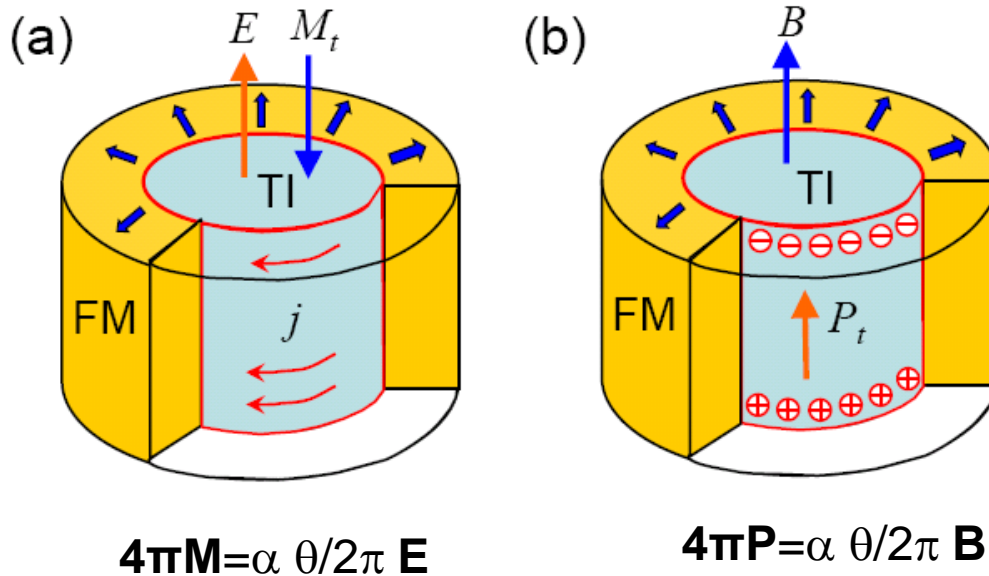
Topological stability of the surface states

- For a sample with boundary, physics is not periodic in θ . However, T-invariant perturbations, like disorder, can induce plateau transitions with $\Delta\sigma_{xy}=1 e^2/h$, or $\Delta\theta=2\pi$. For TI with $\theta=\pi$, the surface QH can never disappear, no matter how strong the disorder! $\sigma_{xy}=1/2 e^2/h \Rightarrow \sigma_{xy}=-1/2 e^2/h$.
- States related by interger plateau transition defines an equivalence class. There are only two classes!
- No-go theorem: it is not possible to construct a 2D model with an odd number of Dirac cones, in a system with $T^2=-1$ TR symmetry. Surface states of a TI with $\theta=\pi$ is a holographic liquid! [Wu, Bernevig & Zhang](#), [Nielsen & Ninomiya](#)
- TI surface states can not rust away by surface chemistry.



The Topological Magneto-Electric (TME) effect

- Equations of axion electrodynamics predict the robust TME effect.



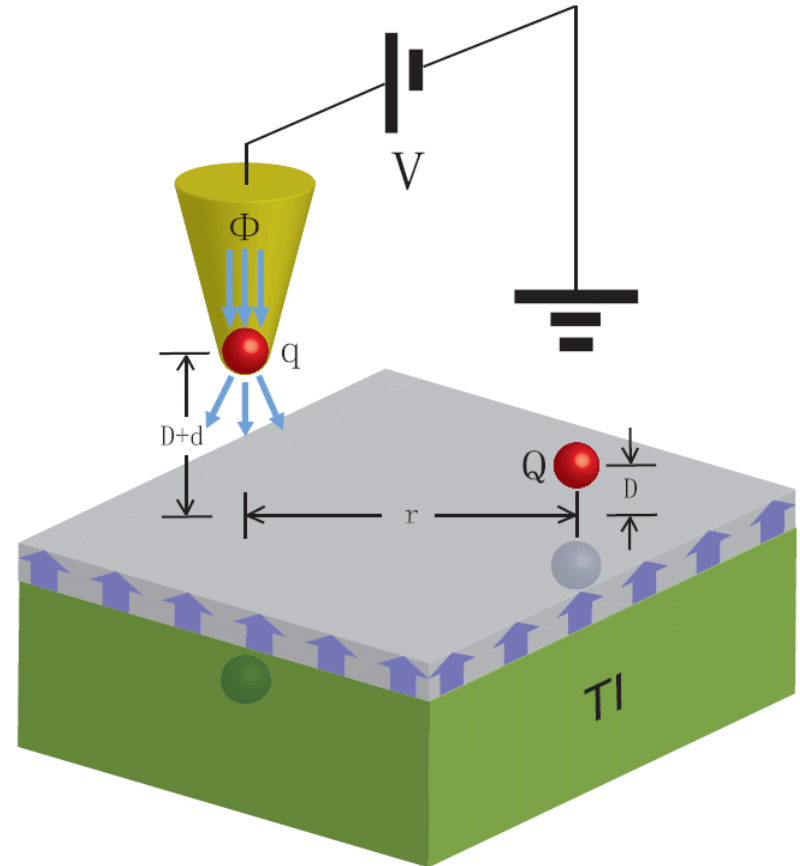
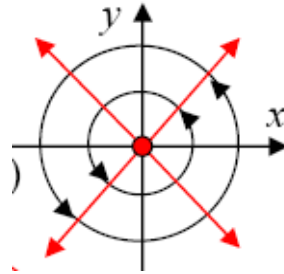
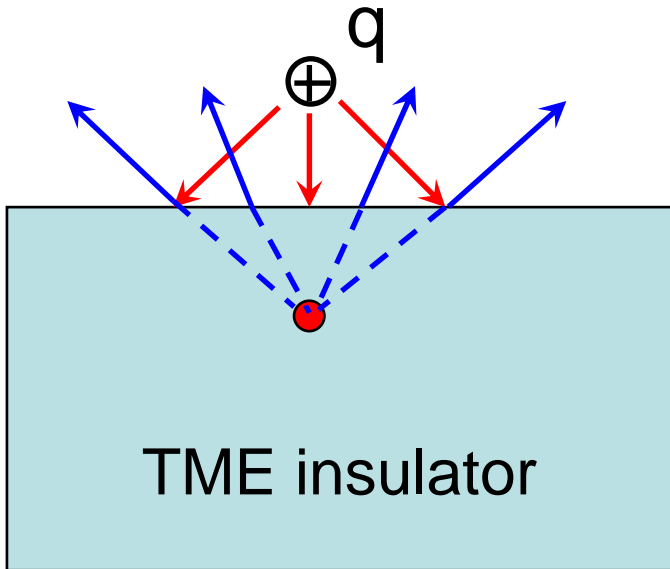
$$\begin{aligned}\nabla \cdot \mathbf{D} &= 4\pi\rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \\ \mathbf{D} &= \mathbf{E} + 4\pi\mathbf{P} - 2P_3\alpha\mathbf{B} \\ \mathbf{H} &= \mathbf{B} - 4\pi\mathbf{M} + 2P_3\alpha\mathbf{E}\end{aligned}$$

Wilczek, axion electrodynamics

- $P_3 = \theta/2\pi$ is the electro-magnetic polarization, microscopically given by the CS term over the momentum space. Change of $P_3 = 2^{\text{nd}}$ Chern number!

$$\begin{aligned}P_3(\theta_0) &= \int d^3k \mathcal{K}^\theta \\ &= \frac{1}{16\pi^2} \int d^3k \epsilon^{\theta ijk} \text{Tr} \left[\left(f_{ij} - \frac{1}{3} [a_i, a_j] \right) \cdot a_k \right]\end{aligned}$$

Seeing the magnetic monopole thru the mirror of a TME insulator



$$g = \frac{\alpha P_3}{1 + \alpha^2 P_3^2} q$$

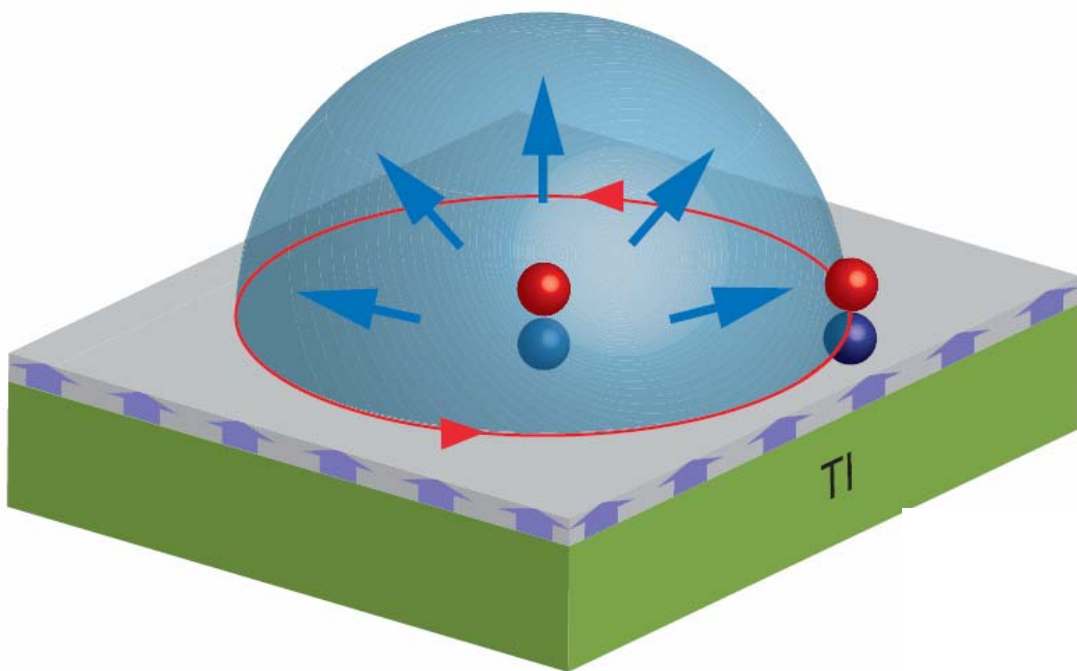
higher order
feed back

(for $\mu=\mu'$, $\varepsilon=\varepsilon'$)

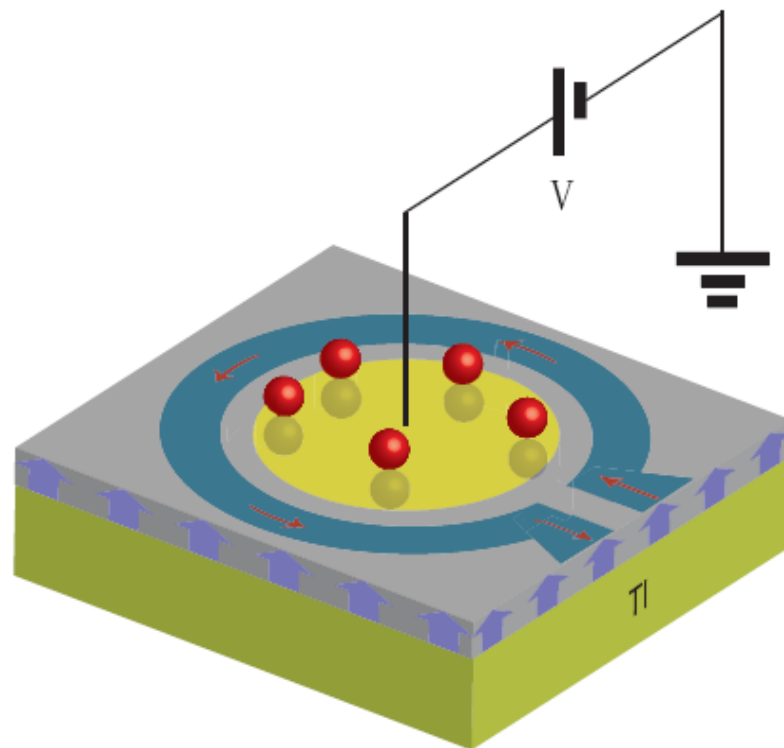
similar to Witten's dyon effect

Magnitude of B:
 $10^6 V/m \rightarrow 0.25G$

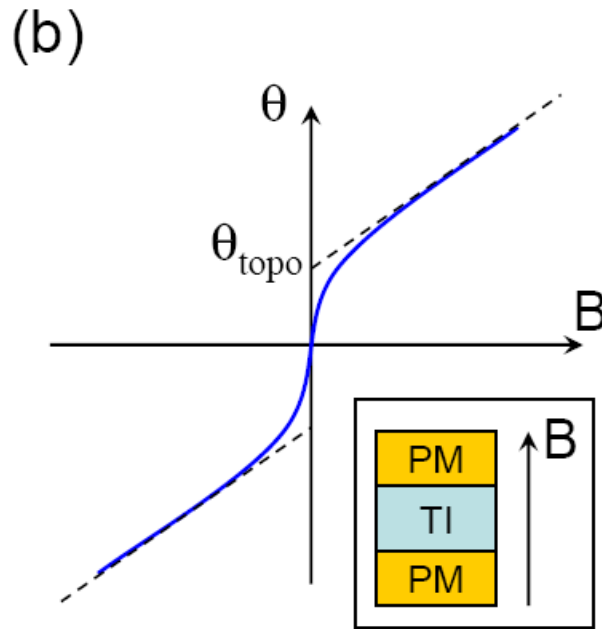
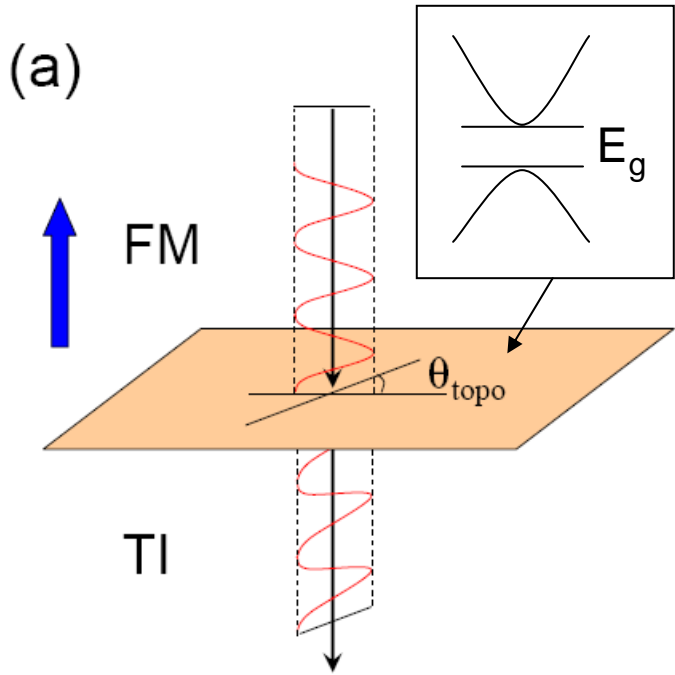
An electron-monopole dyon becomes an anyon!



$$\theta = 2\alpha^2 P_3$$



Low frequency Faraday/Kerr rotation



Adiabatic Requirement:
 $\hbar\omega \ll E_g$
 (surface gap)

$$\theta(B) = uB + \text{sgn}(B) \arctan \left(\frac{(2n - 1)\alpha}{\sqrt{\epsilon/\mu} + \sqrt{\epsilon'/\mu'}} \right)$$

normal contribution

Topological contribution

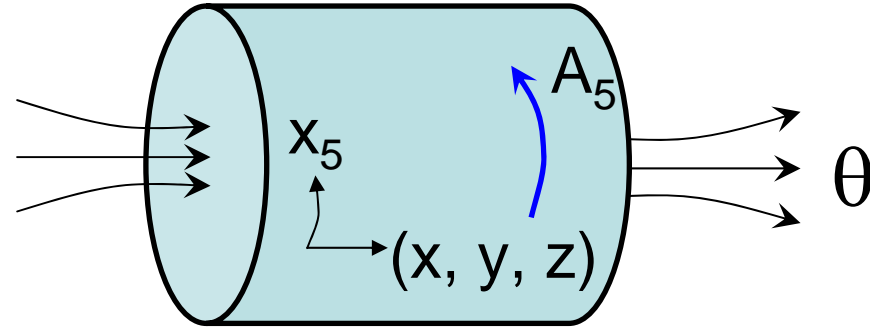
$$\theta_{\text{topo}} \sim 3.6 \xi \cdot 10^{-3} \text{ rad}$$

Dimensional reduction

- From 4D QHE to the 3D topological insulator

Zhang & Hu

$$\begin{aligned}
 S_{4DQH} &= \int d^4 x dt \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} \\
 &\Rightarrow \int d^3 x dt \left(\int dx_5 A_5(x, t) \right) \varepsilon^{\nu\rho\sigma\tau} F_{\nu\rho} F_{\sigma\tau} \\
 &\Rightarrow S_{3D} = \int d^3 x dt \theta(x, t) \varepsilon^{\nu\rho\sigma\tau} F_{\nu\rho} F_{\sigma\tau}
 \end{aligned}$$



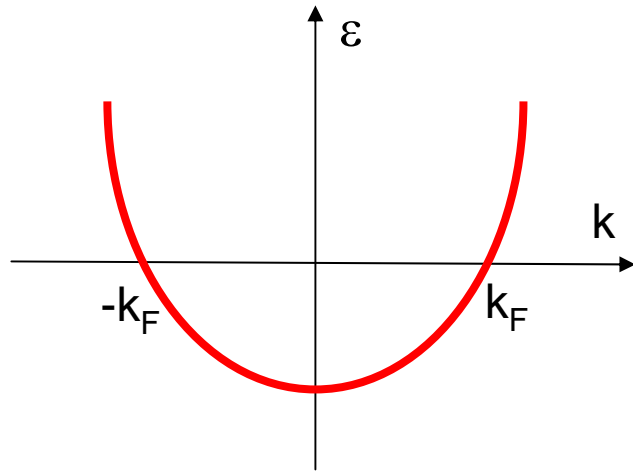
- From 3D axion action to the 2D QSH

$$\begin{aligned}
 S_{3D} &= \int d^3 x dt \varepsilon^{\nu\rho\sigma\tau} A_\nu \partial_\rho \theta \partial_\sigma A_\tau \\
 &\Rightarrow \int d^2 x dt \varepsilon^{\rho\sigma\tau} \left(\int dz A_z(x, t) \right) \partial_\rho \theta \partial_\sigma A_\tau \\
 &\Rightarrow S_{2D} = \int d^2 x dt \varepsilon^{\rho\sigma\tau} \partial_\sigma \varphi \partial_\rho \theta A_\tau
 \end{aligned}$$

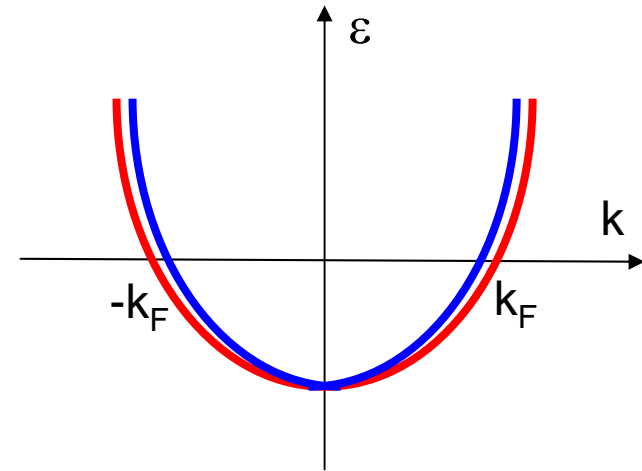
$$\begin{aligned}
 J_{2D}^\mu &= \frac{e}{2\pi^2} \varepsilon^{\mu\rho\sigma} \partial_\sigma \varphi \partial_\rho \theta \\
 &\Rightarrow J_{1D}^\mu = \frac{e}{2\pi} \varepsilon^{\mu\sigma} \partial_\sigma \varphi
 \end{aligned}$$

Goldstone & Wilzcek

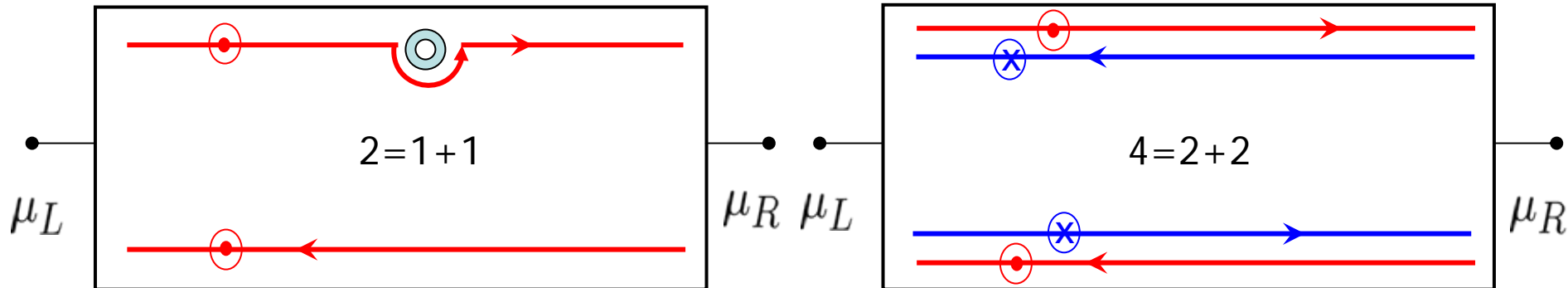
Chiral (QHE) and helical (QSHE) liquids in D=1



The QHE state spatially separates the two chiral states of a spinless 1D liquid



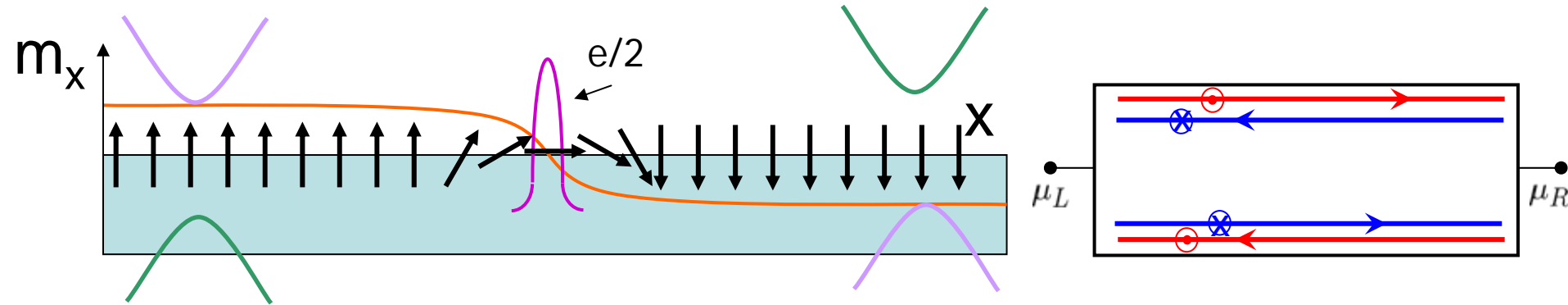
The QSHE state spatially separates the four chiral states of a spinful 1D liquid



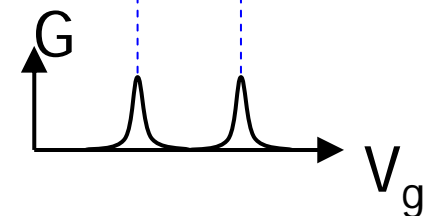
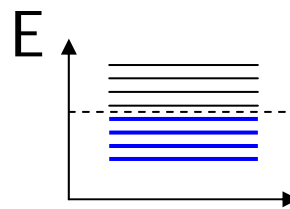
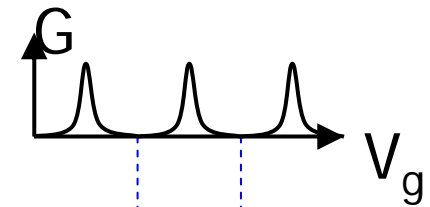
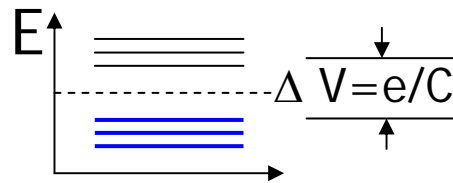
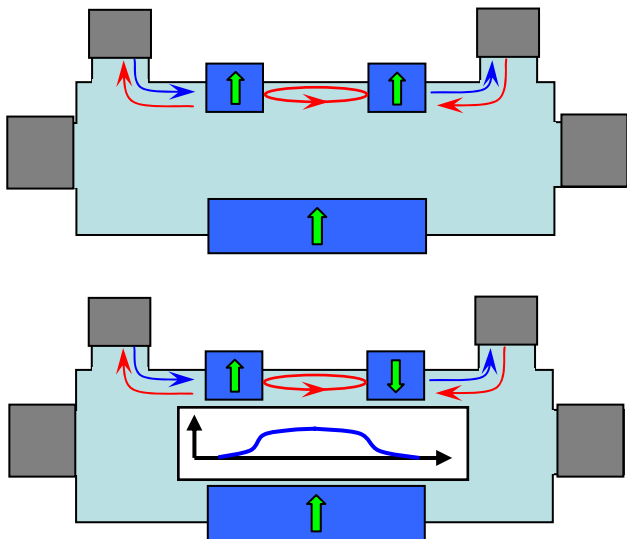
No go theorems: chiral and helical states can never be constructed microscopically from a purely 1D model [Wu, Bernevig, Zhang](#); [Nielsen, Ninomiya](#)

Fractional charge in the QSH state, E&M duality!

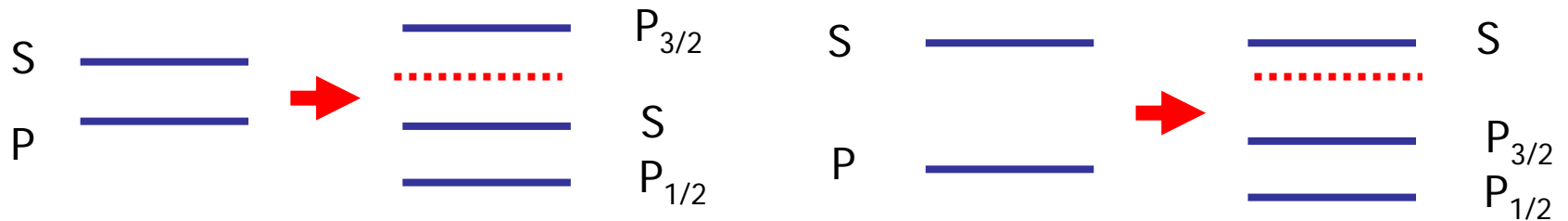
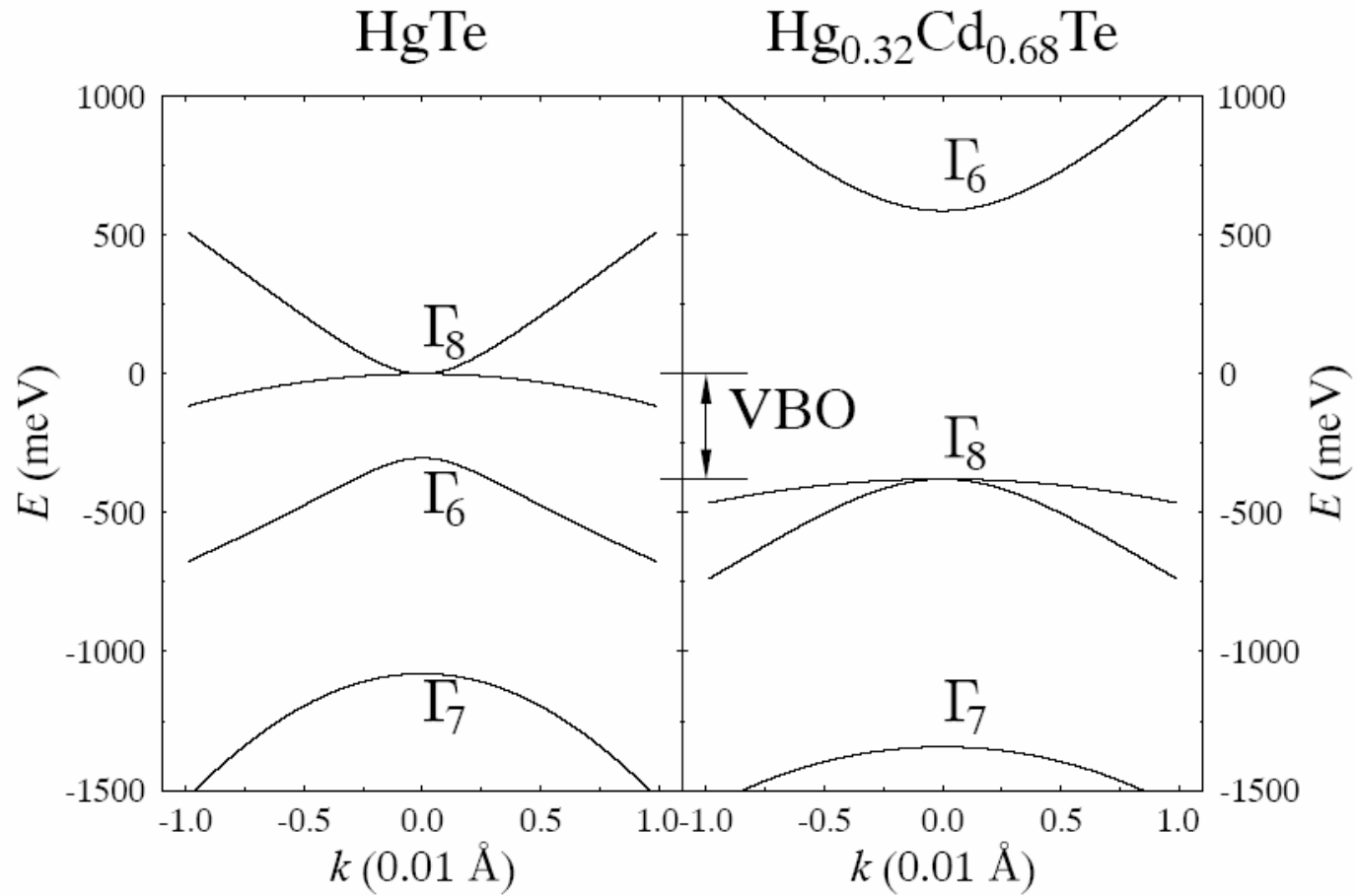
- Since the mass is proportional to the magnetization, a magnetization domain wall leads to a mass domain wall on the edge.



- The fractional charge $e/2$ can be measured by a Coulomb blockade experiment, one at the time! [Qi, Hughes & Zhang](#)

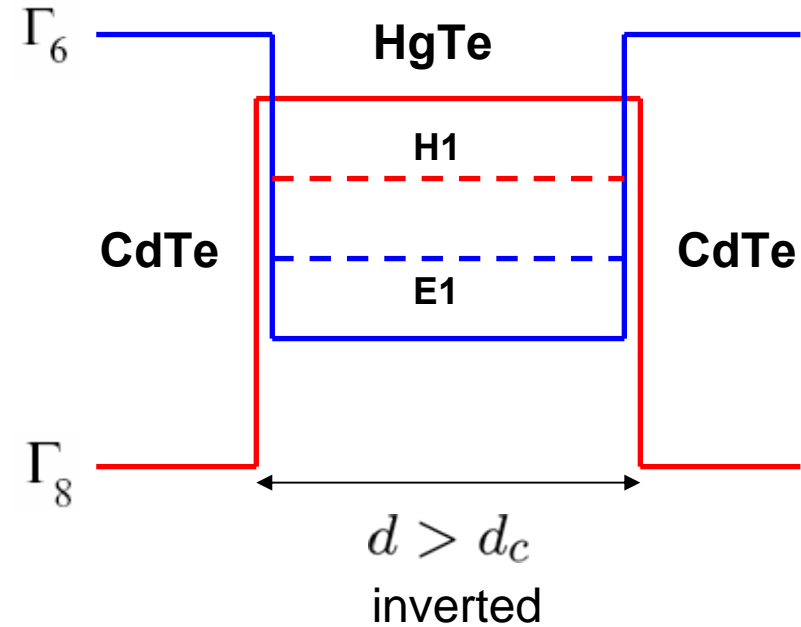
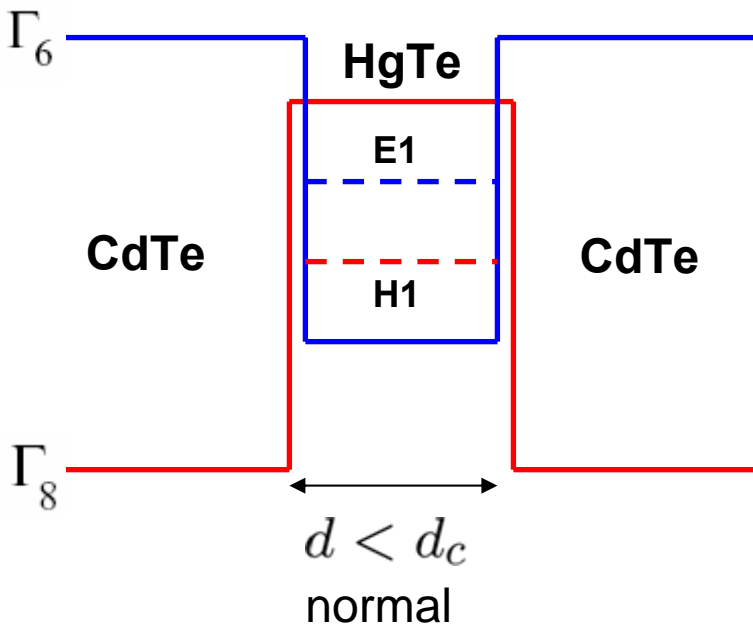
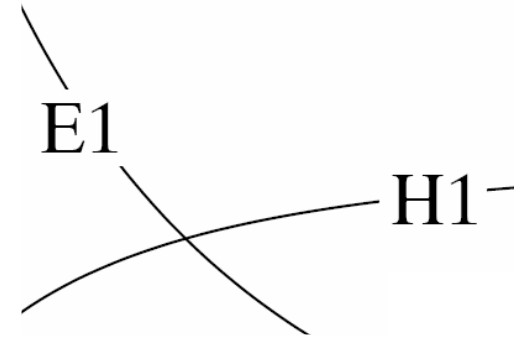
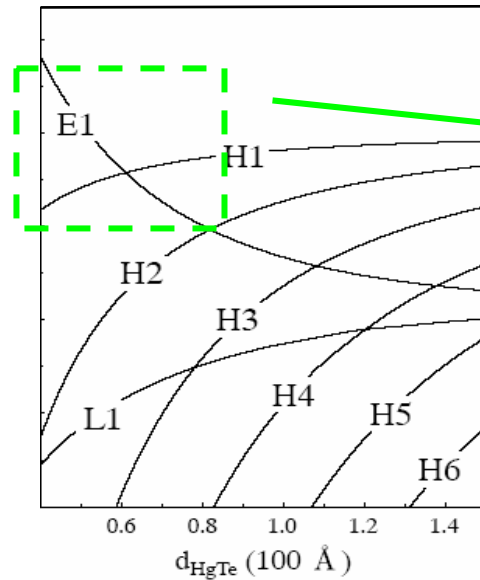


Band Structure of HgTe



Quantum Well Sub-bands

Let us focus on E1, H1 bands close to crossing point



Effective tight-binding model

Square lattice with 4-orbitals per site:

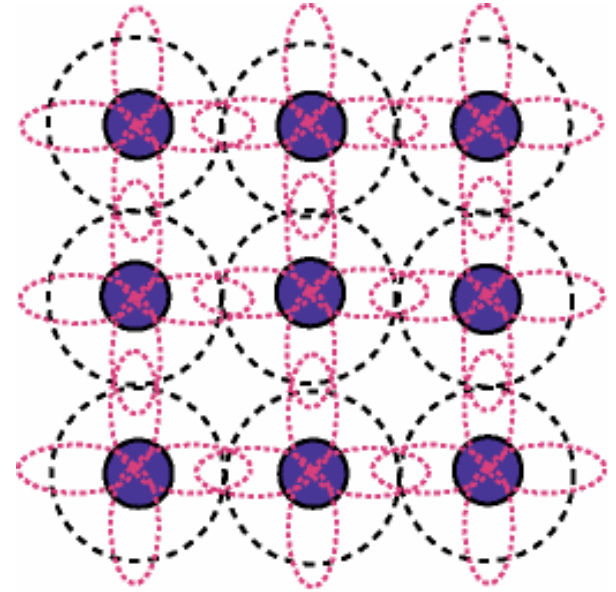
$$|s, \uparrow\rangle, |s, \downarrow\rangle, |(p_x + ip_y, \uparrow)\rangle, |-(p_x - ip_y), \downarrow\rangle$$

Nearest neighbor hopping integrals. Mixing matrix elements between the s and the p states must be odd in k.

$$H_{eff}(k_x, k_y) = \begin{pmatrix} h(k) & 0 \\ 0 & h^*(-k) \end{pmatrix}$$

$$h(k) = \begin{pmatrix} m(k) & A(\sin k_x - i \sin k_y) \\ A(\sin k_x + i \sin k_y) & -m(k) \end{pmatrix} \equiv d_a(k) \tau^a$$

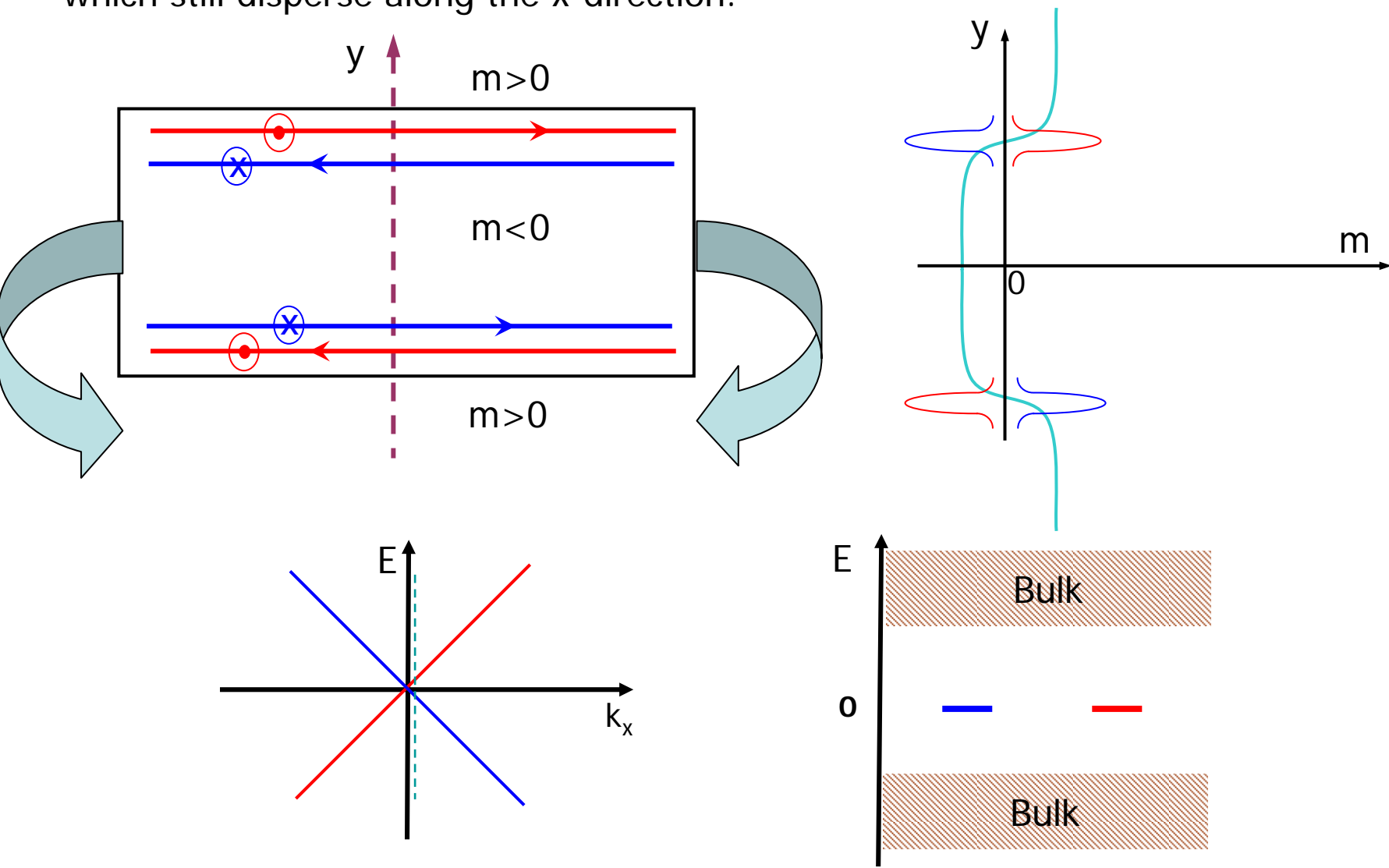
$$\Rightarrow \begin{pmatrix} m & A(k_x - ik_y) \\ A(k_x + ik_y) & -m \end{pmatrix}$$



Relativistic Dirac equation in 2+1 dimensions, with a mass term tunable by the sample thickness d ! $m < 0$ for $d > d_c$.

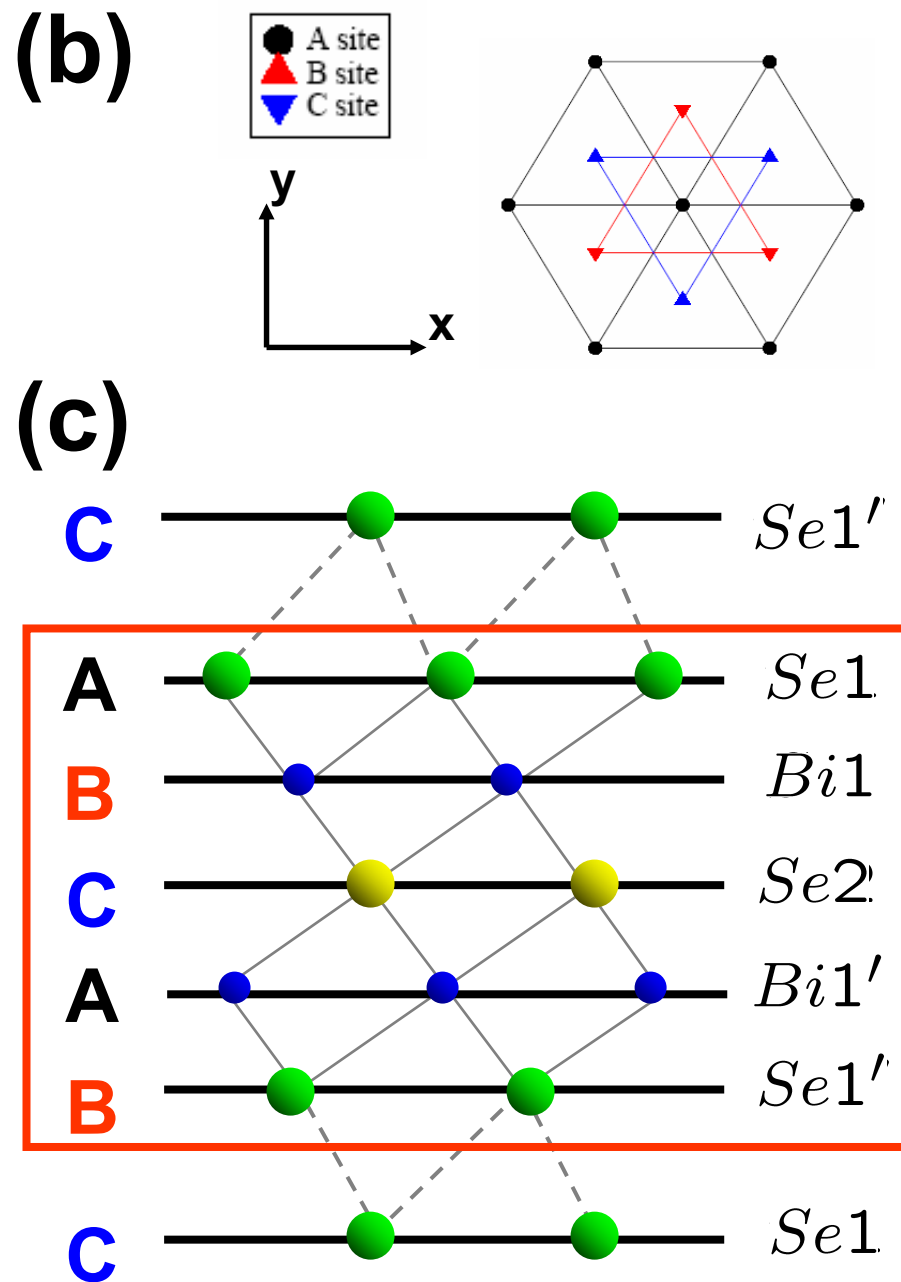
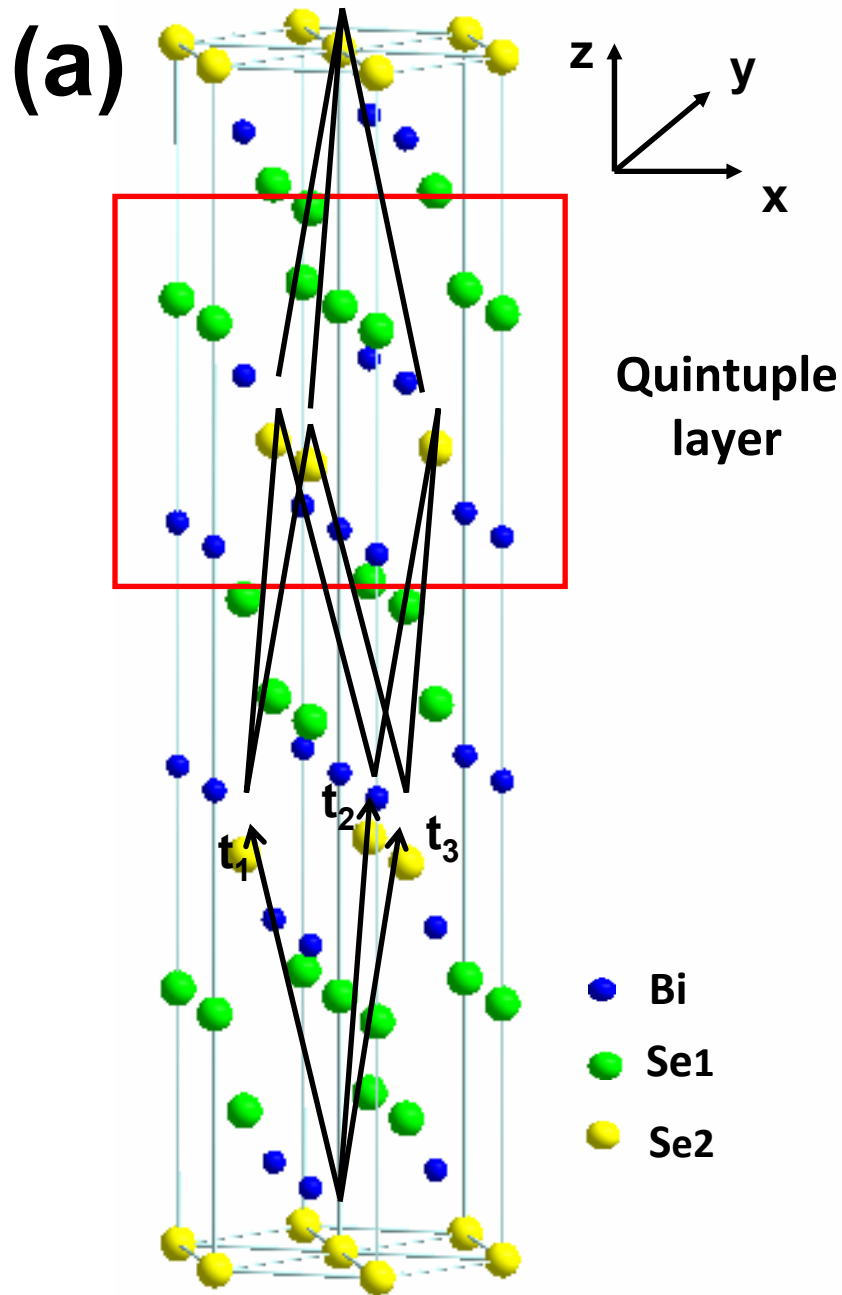
Mass domain wall

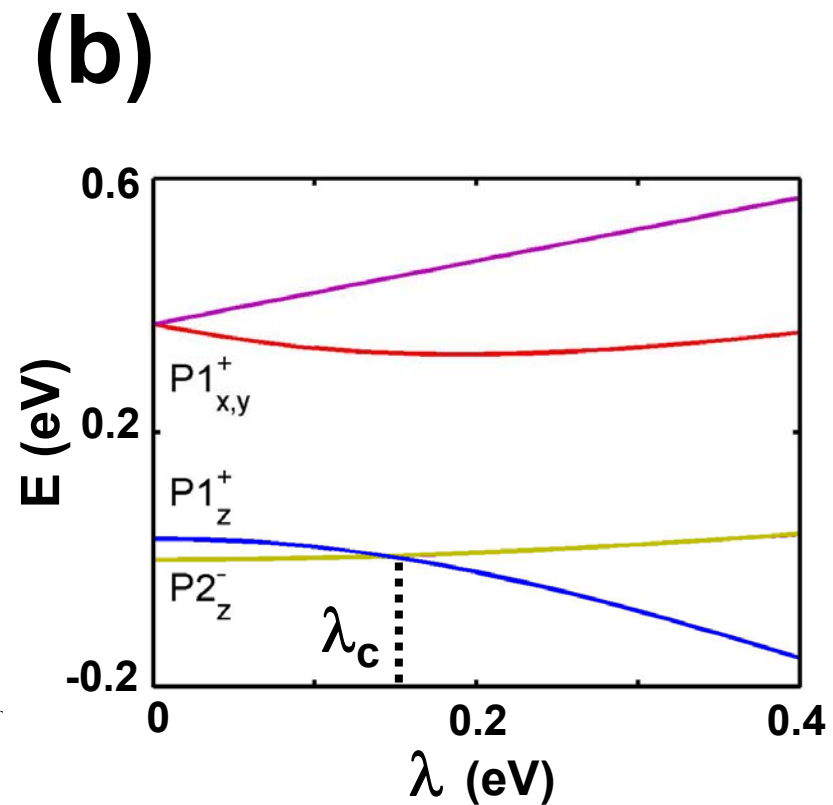
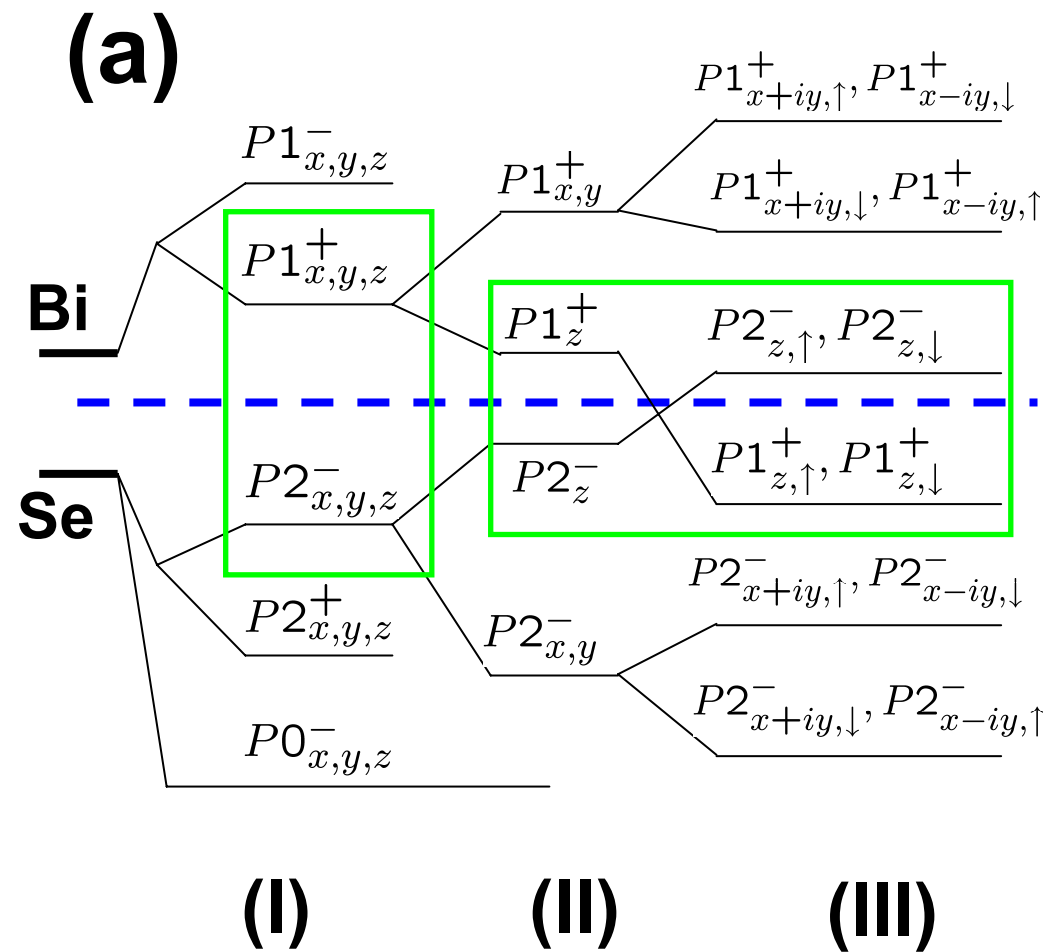
Cutting the Hall bar along the y -direction we see a domain-wall structure in the band structure mass term. This leads to states localized on the domain wall which still disperse along the x -direction.

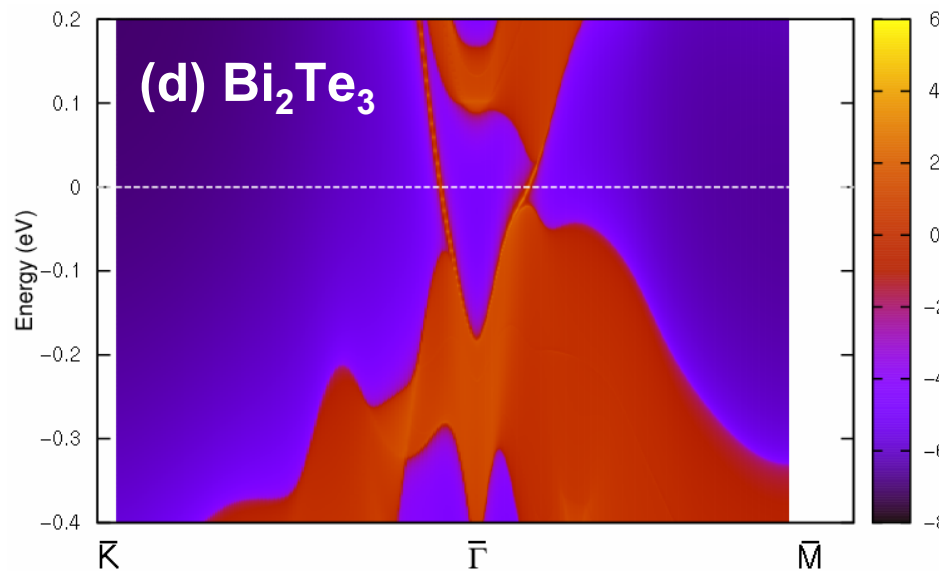
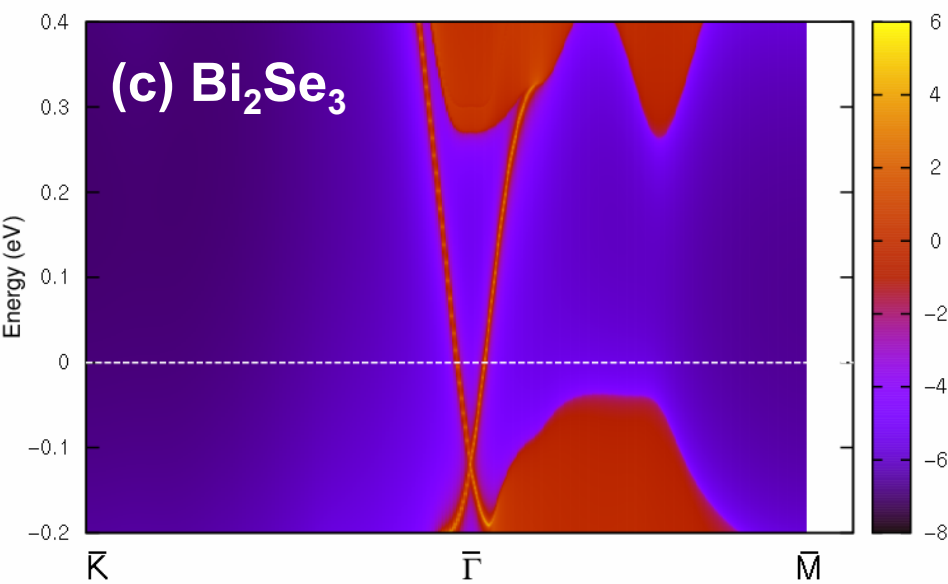
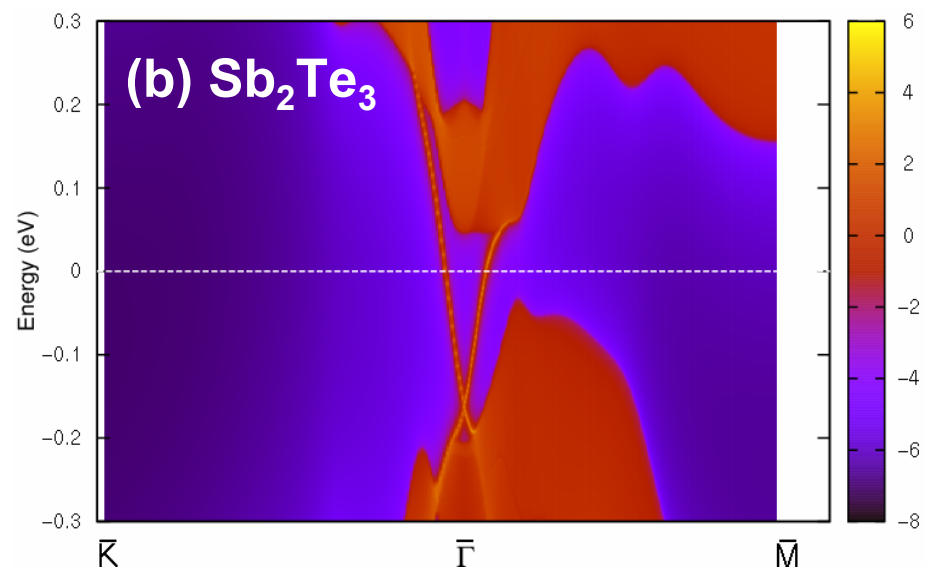
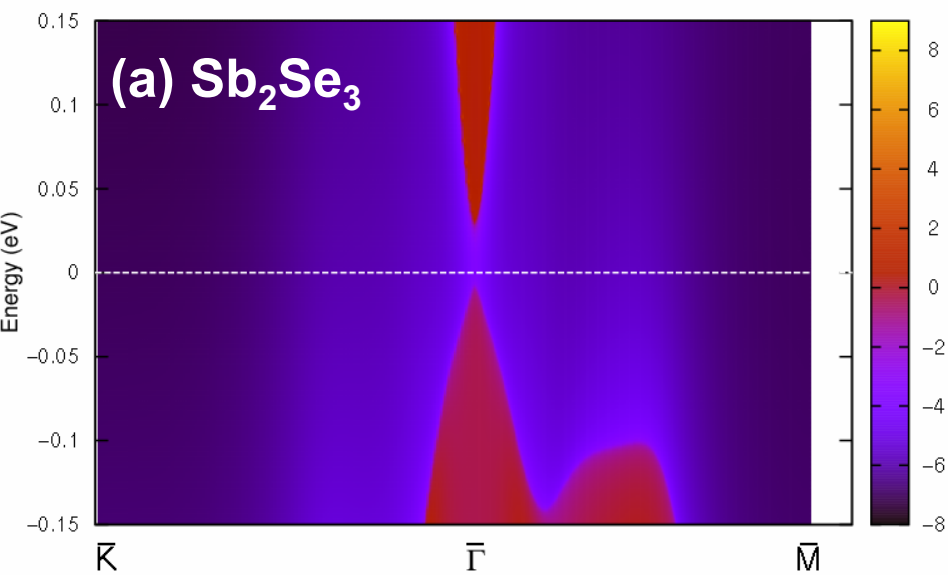


Topological insulators at room temperature

IOP&Stanford





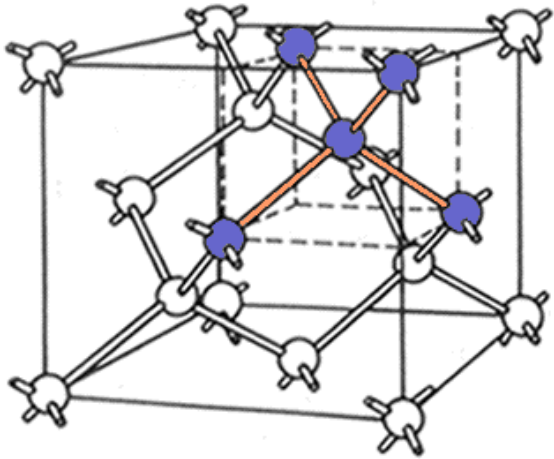


Effective model

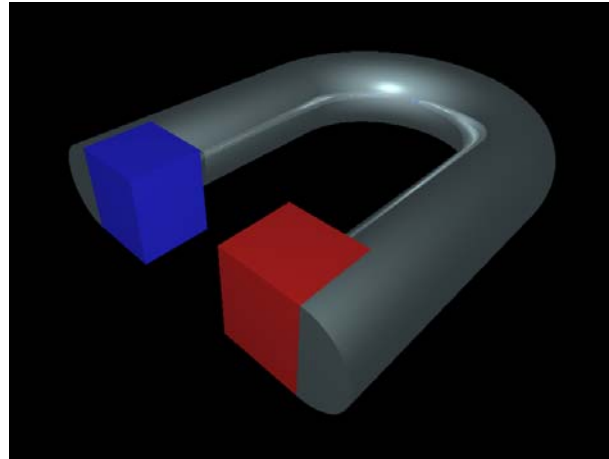
$$H(\mathbf{k}) = \epsilon_0(\mathbf{k})\mathbf{I}_{4\times 4} + \begin{pmatrix} \mathcal{M}(\mathbf{k}) & A_1 k_z & 0 & A_2 k_- \\ A_1 k_z & -\mathcal{M}(\mathbf{k}) & A_2 k_- & 0 \\ 0 & A_2 k_+ & \mathcal{M}(\mathbf{k}) & -A_1 k_z \\ A_2 k_+ & 0 & -A_1 k_z & -\mathcal{M}(\mathbf{k}) \end{pmatrix} + o(\mathbf{k}^2)$$

$$H_{\text{surf}}(k_x, k_y) = \begin{pmatrix} 0 & A_2 k_- \\ A_2 k_+ & 0 \end{pmatrix}$$

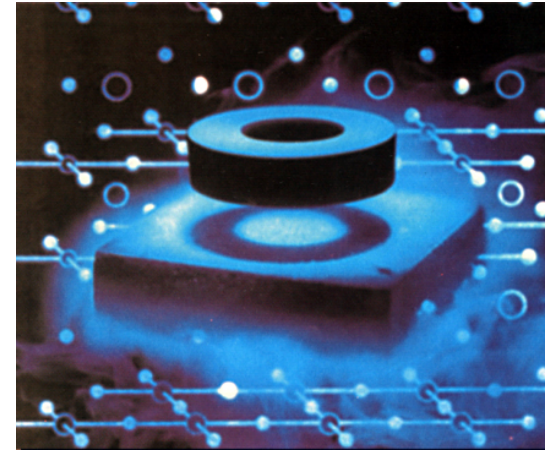
Summary: the search for new states of matter



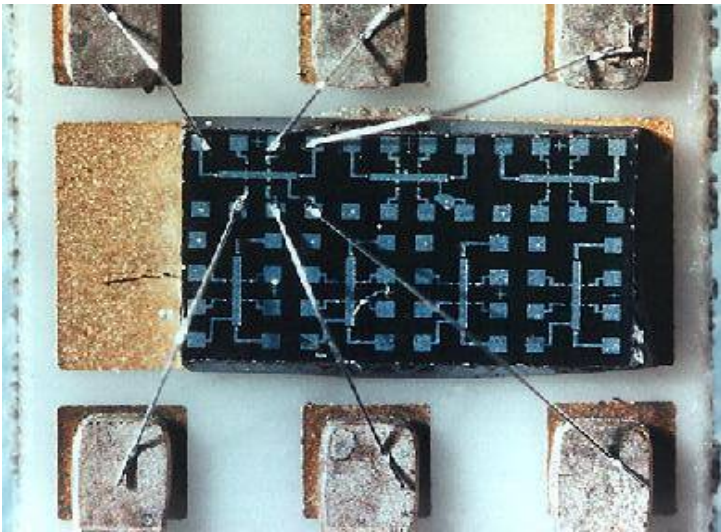
Crystal



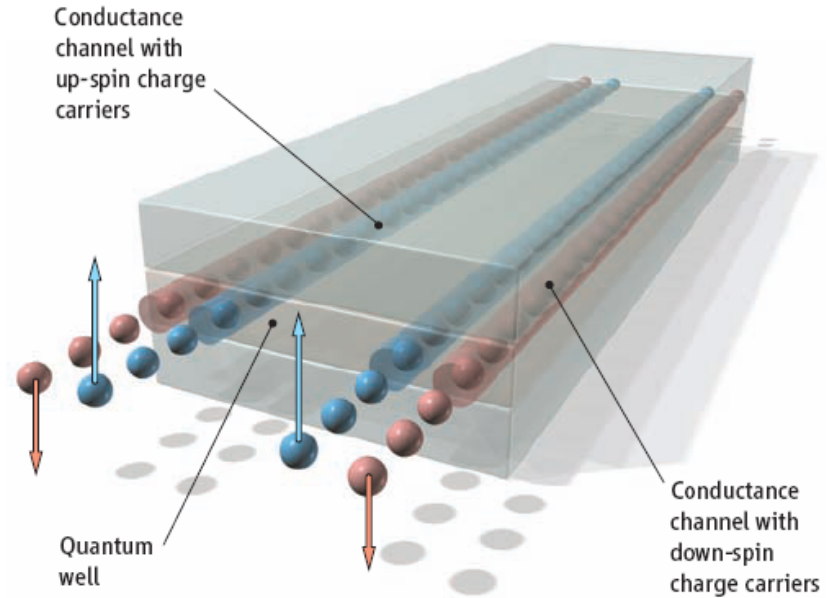
Magnet



s-wave superconductor



Quantum Hall



Quantum Spin Hall