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"From string nets to Nonabelions"

Q

Fidkowski, F., Nayak, Walker, Wang

Variation on a theme:

Kitaev - Kupperberg,

Fendley - Fradkin,

Levin - Wen

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Motivation:

Learn something which will help us find / build Doubled Fibonacci theory

(The simplest TQFT with dense braid representations.)

This talk will try to look beyond the FQHE to possible achiral spin-Hamiltonians

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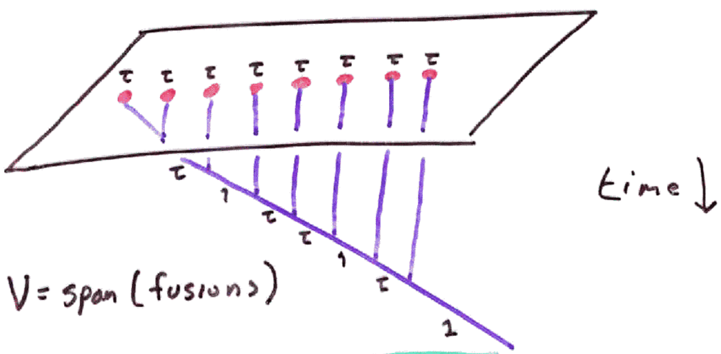
Fibonacci primer

$\exists$ a unique 2+1 D-particle theory with <u>one</u> nontrivial particle τ and fusion rule:

$$\tau \otimes \tau = 1 \oplus \tau$$

Names: Golden theory
Fibonacci theory
CSW $G2_1$
CSW $SO(3)_3$



V = span (fusions)

$$\boxed{\dim V_{n+2} = \dim V_{n+1} + \dim V_n}$$

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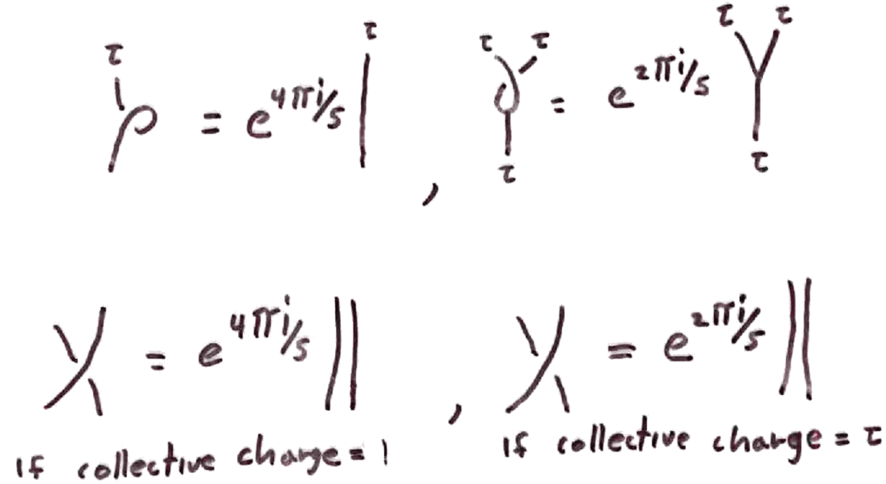
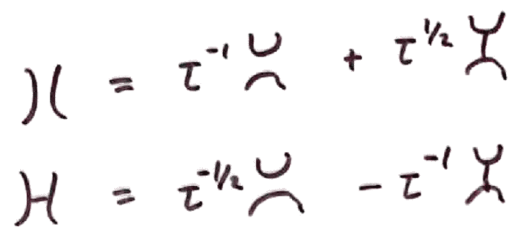
Some data for fib.



$$S\text{-matrix} = \frac{1}{\sqrt{\tau+2}} \begin{vmatrix} 1 & \tau \\ \tau & -1 \end{vmatrix}$$

$$S^{\tau}_{\tau\tau} = e^{3\pi i/10}$$

F-matrix
(6j-symbol)



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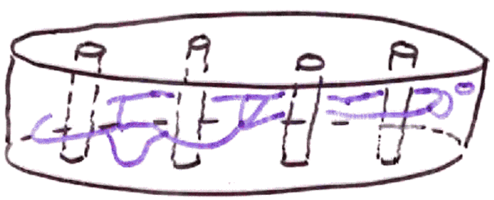
DF = doubled Fibonacci

= Matrices (Fib) = $F^* \otimes F$

= Operators (Fib)

= Wilson loops in Fib

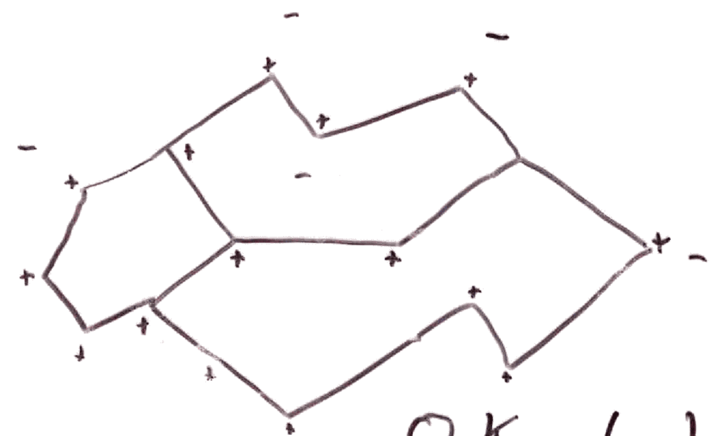
Thus a state of DF looks like a closed particle history in Fib :



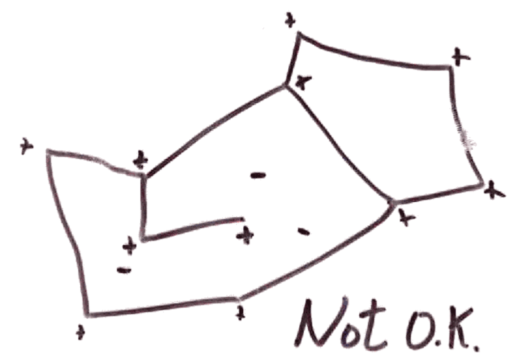
Since DF is a left $\otimes$ right copy of Fib, DF is achiral and a candidate for a spin Hamiltonian.

Don't worry, DF may be easier to "make" than Fib. To start we need: Hilbert space of "string nets".

5.5



O.K. (in $\mathcal{H}$)



Not O.K. $\notin \mathcal{H}$

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How to build $\mathcal{H}$

spin = ½ particle on each site {−, +}.

"Draw" bonds with − on red end and + on black end

To prevent "dead ends" put in 3-body terms Π  and Π .

(forces: black + to join 2 or more red −
red − to join 2 or more black +)

Note: We may "encrypt" this as a 2-body Hamiltonian (order = 4 in S_i^z and S_j^z) on spin = 3/2 particles living on bonds.

6.5

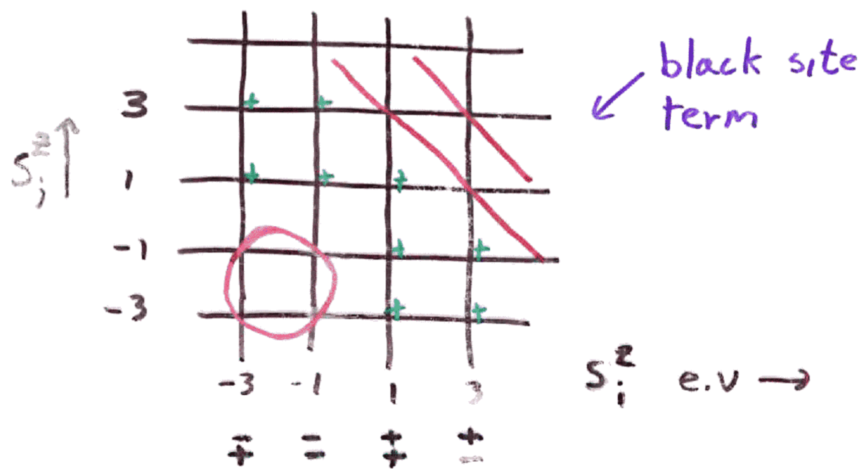
H has terms

$$\left((S_i^z+2)^2+(S_j^z+2)^2-2\right)(S_i^z+S_j^z-4)(S_i^z+S_j^z-6)$$

for pairs of bonds meeting at black sites, and

$$\left((S_i^z-2)^2+(S_j^z-2)^2-2\right)(S_i^z+S_j^z+4)(S_i^z+S_j^z-6)$$

for pairs of bonds meeting at red sites



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"Axioms" for $H_0 : \mathcal{H} \to \mathcal{H}$

1. makes nets fluctuate
2. imposes isotopy:

$$\Psi_{g.s.}(\;) = \Psi_{g.s.}(\;)$$

(possibly with a bond fugasity)

3. kills tadpoles:

$$\Psi_{g.s.}(\;) = 0$$

I will argue the "inevitability" of DF for g.s.m. of $H = H_0 + V$

Key idea: "Nature abhors a degeneracy" (e.g. eigenvalue repulsion)

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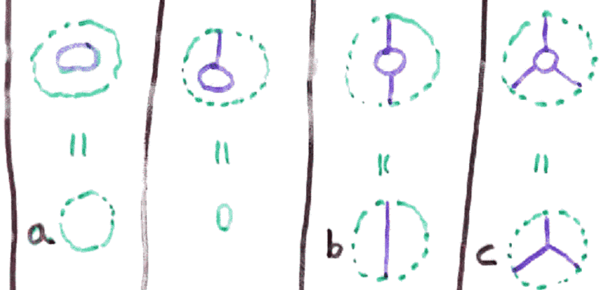
Assume lowest (nontrivial) dimensions for the "disk spaces":

	✓ $\mathcal{H}(D,0)$	✓ $\mathcal{H}(D,1)$	✓ $\mathcal{H}(D,2)$	✓ $\mathcal{H}(D,3)$	✓ $\mathcal{H}(D,4)$
dim	1	0 tadpole	1	1	2 smallest consistant with "tadpole" ~ growth exponential here after!
	= a	= 0	= b	= c	



Let's try to compute which relations in $\mathcal{H}(D,4)$ are consistant with the earlier dimensions:

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Suppose :

are not independent. Set:

$R = h \; + i \; + x \; + y \; = 0$

$= h \; + i \; + x \; + y \; = 0$

$= bh \; + 0 \; + x \; + ay \; = 0$

$= bh + x + ay = 0$

$= bi + ax + y = 0$

$= ch + bi + x = 0$

$= bh + ci + y = 0$

$\left.\begin{array}{l} x = -ch - bi \\ y = -bh - ci \end{array}\right\}$ substitute above

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$\Downarrow$

$(b - c - ab)\, h + (-b - ac)\, i = 0$

$(-b - ac)\, h + (b - c - ab)\, i = 0$

So largest possible relation space is 2-D, occurs for $b - c - ab = 0$ and $-b - ac = 0$

$\Rightarrow \; -ac - c + a^2 c = 0$, or

$a^2 = a + 1$ (Gold)

Specialize:

$h=0, i=1 \quad -b \; -c \; = 0 \Rightarrow \; = \frac{1}{a} \; + \frac{1}{b}$

$h=1, i=1/a, x=0 \Rightarrow \; + \frac{1}{a} \; + (-b - \frac{c}{a}) \; = 0$

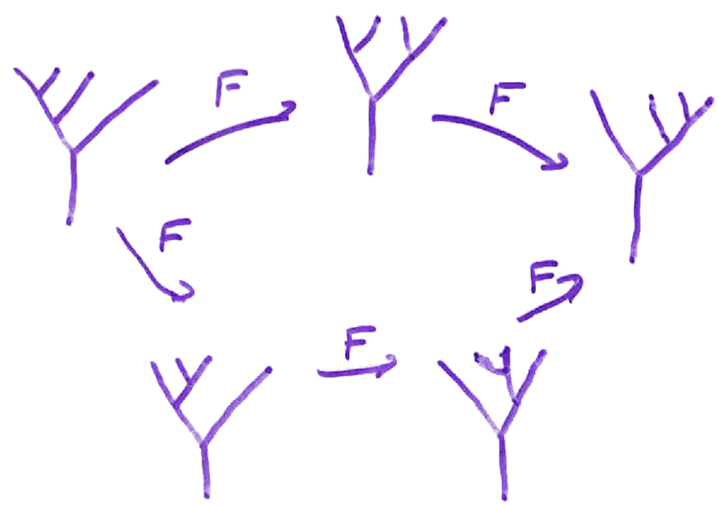
$\Rightarrow \; = (-b + \frac{b}{a^2}) \; - \frac{1}{a}$

Thus $F = \begin{vmatrix} 1/a & 1/b \\ -b + \frac{b}{a^2} & -1/a \end{vmatrix}$

now unitarity + $a^2 = a + 1 \Rightarrow$

$F = \begin{vmatrix} \tau^{-1} & \tau^{-1/2} \\ \tau^{-1/2} & -\tau^{-1} \end{vmatrix}, \quad \tau = \frac{1+\sqrt{5}}{2}$

Remarkably, the 6j-symbol F for Fib is already determined on dimensional grounds. F satisfies the ⬠-equ. (Eliot-Biedenharn), but the theory is already specified on the "1-skeleton"



These rules for string nets:

$$a = \tau$$
$$b = \tau^{1/2}$$
$$c = -\tau^{-1/2}$$
$$F = \begin{vmatrix} \tau^{-1} & \tau^{-1/2} \\ \tau^{-1/2} & -\tau^{-1} \end{vmatrix}$$

allow us to compute all Hilbert spaces for DF.

E.g. $DF(\text{Torus}) \cong \mathbb{C}^4$.

We can also compute all particle types. We of course find:

$$1 \otimes 1 \quad 1 \otimes \tau \quad \tau \otimes 1 \quad \text{and} \quad \tau \otimes \tau$$

but in graphical disguise!

12.5

Remark: If $V(D,4)$ is allowed to have dimension = 3, a generic d solution exists: Yamada polynomial. For each k odd >3 the Yamada polynomial has a specialization – corresponding to a drop in $\dim(V(D,2k))$ – which yields Double $(SO(3)_k)$.

If $V(D,4)$ has dim = 4, a relation in $V(D,5)$ realizes Kupperberg's G2-spider. Presumably additional relations in $\tilde{V}(D,n)$ at special d-values yield Double $(G2_k)$

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What the Q state Potts model tells us about the statistics of multi-loops

high temperature expansion (h.t.e.)

and the statistics of string nets

low temperature expansion (l.t.e.)

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h.t.e. (cluster expansion)

$$Z = \sum_{\sigma \in \text{spin states}} e^{\beta J \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}} = \sum_{\sigma} \prod_{\langle i,j \rangle} e^{\beta J \delta_{\sigma_i \sigma_j}}$$

$$= \sum_{\sigma} \prod_{\langle i,j \rangle} (1 + \underbrace{\gamma}_{e^{\beta J} - 1} \delta_{\sigma_i \sigma_j}) = \sum_{\text{conf}} Q^{\text{clusters}} \gamma^{\text{Edges}}$$

Self dual: $Q^{C^*} \gamma^{E^*}$ "=" $Q^C \gamma^E$

Euler: C^* "=" $C + E$, and E^* "=" $-E$

$$\Rightarrow Q^{C+E} \gamma^{-E} = Q^C \gamma^E \Rightarrow \boxed{\gamma = \sqrt{Q}}$$

We now relate the Gibbs weight, d^2 per loop on the "surround loops" of configurations to the Self-dual Potts

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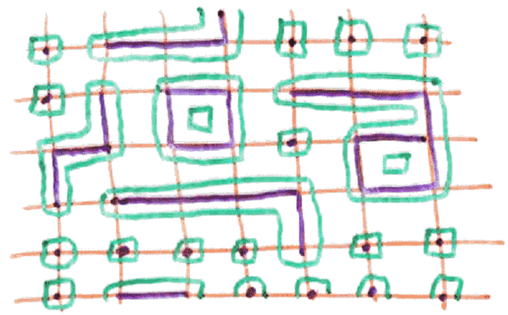

Surround loops

$$\text{weight} = (d^2)^L = (d^2)^{C + C^*} = (d^4)^C (d^2)^E$$

Recall $C^* = C + E$ and note $L = C + C^*$

This self-dual model is critical when $d^4 = Q \leq 4$, i.e. $d \leq \sqrt{2}$.

(Each loop is outer-most boundary of either a cluster or dual-cluster.)

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l.t.e.

$$Z = \sum_{G \in \text{"Nets"}} \chi_{\hat{G}}(Q)\,(e^{-\beta J})^{L}$$

↑ Chromatic ↑ bond fugasity

Recall: $e^{\beta J} - 1 = \gamma = \sqrt{Q}$, at criticality.

$$\overset{c}{=} \sum_{G} \chi_{\hat{G}}(Q)\left(\frac{1}{\sqrt{Q}+1}\right)^{L}$$

The rules: a, b, c and F which we solved for, allow us to compute a (real) "amplitude" for any net G. Call it $\langle G \rangle_\tau$, with $(\langle G \rangle_\tau)^2$ the corresponding unnormalized probability.

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String net evaluation:

$$\langle G \rangle_\tau = \tau^{-5}\,(\tau^{3/2})^{V(\hat{G})}\,\chi_{\hat{G}}(\tau^2)$$

Tutte Golden thm:

$$(\chi_{\hat{G}}(\tau+1))^2 (\tau^3)^{V(\hat{G})} (\tau+2)(\tau^{-10}) = \chi_{\hat{G}}(\tau+2)$$

use $\tau^2 = \tau + 1$ and combine:

$$\boxed{(\langle G \rangle_\tau)^2 = \frac{1}{\tau+2}\,\chi_{\hat{G}}(\tau+2)}$$

Note: $\tau + 2 \approx 3.6 < 4$ and $\frac{1}{\sqrt{\tau+2}+1} \approx .35 < 1$

So topological weighting is high temperature limit above a critical Potts transition.

This is analogous to Toric Code as high temp. limit above Ising critical point.

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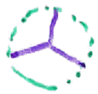
conclusions

I. Given a Hilbert space of "string nets" and Hamiltonian $H: \mathcal{H} \to \mathcal{H}$ which:

1. generates isotopy
2. kills Tadpoles  $= 0$

   but allows branching  $\neq 0$
3. minimizes degeneracy: $\dim V(D; i)$ minimal for $1 \le i \le 4$

will have Double Fib. as ground states.

II For parameters near their golden values: $a \approx \tau$ $b \approx \tau^{1/2}$ $c \approx -\tau^{-1/2}$ we expect H_0 satisfying 1. and 2. to be in plasma analogy to a stat. mech. system in same class as

$$(\langle G \rangle_\tau)^2 = \frac{1}{\tau+2} \chi_G(\tau+2)$$

It seems reasonable that a large class of perturbations would impose 3. and yield Dfib.