The Stability of Topologically Ordered States: Real-Time and Quasi-Adiabatic Evolution

M. B. Hastings, LANL, T-13

Thanks: X.-G. Wen, T. Koma, S. Bravyi, F. Verstraete

M. B. Hastings, PRB 69, 104431 (2004).
M. B. Hastings and X.-G. Wen, PRB 72, 045141 (2005).
M. B. Hastings and T. Koma, CMP in press.
S. Bravyi, M. B. Hastings, and F. Verstraete, quant-ph/0603121.

Why do simple models work for something complicated like topological order?

Topological order: Hamiltonian has multiple ground states such that all matrix elements of local operators are vanishingly small between states and all local operators have close to the same expectation value in each state

$$\langle \Psi_i^0, O\Psi_j^0 \rangle = \text{const} \times \delta_{i,j} + \mathcal{O}(\exp(-L))$$

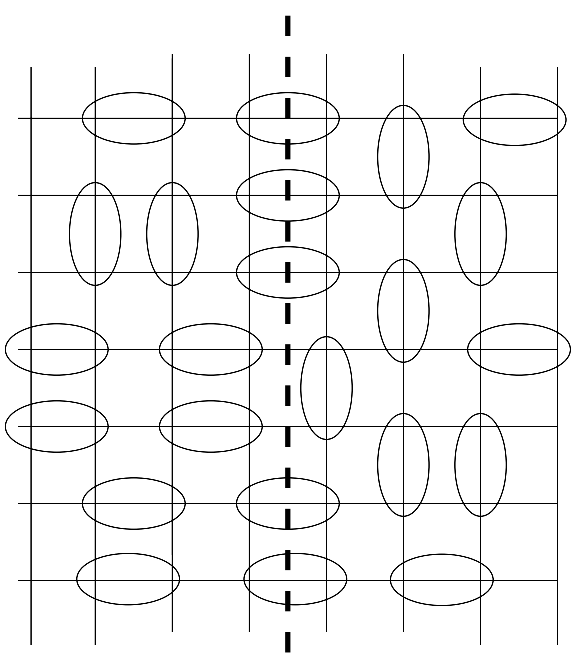
- Fractional Quantum Hall Effect, Laughlin Wavefunction, Haldane Hamiltonian
- Quantum Dimer Model, Equal Amplitude Superposition Wavefunction, Rokhsar-Kivelson Hamiltonian
- Models with emergent gauge symmetry, Wen, Kitaev and others

Each of these model Hamiltonians is a sum of projection operators that exactly annihilate the ground state, what about more realistic models?

How do we make a topologically ordered state? How long does it take? What is the depth of the quantum circuit?

How does a topologically ordered system respond to small or large perturbations?

What is the relation between topological order and "ordinary" order?



$$U\Big(=\rangle\langle= \; + \; ||\rangle\langle|| \Big) + t\Big(=\rangle\langle|| + h.c. \Big)$$

Dimer liquid for Rokhsar-Kivelson Hamiltonian. Not stable to perturbations on square lattice, gapless. Stable on triangular lattice, gapped (Moessner, Sondhi).

2 states, odd vs. even, more on higher genus, both states look the same locally!

Outline:

- Lieb-Robinson bounds: it takes a time of order l/v for information to propagate a distance l under local Hamiltonian evolution.
- Producing order: it takes a time of order l/v to produce correlations on length l. It takes a time of order L/v to produce topological order on a system of size L.
- Quasi-adiabatic evolution: for a gapped system, the ground state evolves under a change in system parameters as if it were undergoing local Hamiltonian evolution
- Combining the above results, topological order is stable if the gap remains open.
- Quasi-adiabatic flux insertion: Lieb-Schultz-Mattis theorem in higher dimensions.

Lieb-Robinson Bounds

Operators A, B separated in space by distance l

$$[A, B] = 0$$

Local Hamiltonian (includes exponentially decaying interactions) implies bound on commutator at different times:

$$t \le l/v \to \|[A(t), B]\| \le \mathcal{O}(\exp(-l))$$

Precise statement:

$$\mathcal{H} = \sum_Z h_Z$$

h_Z acts only on set Z

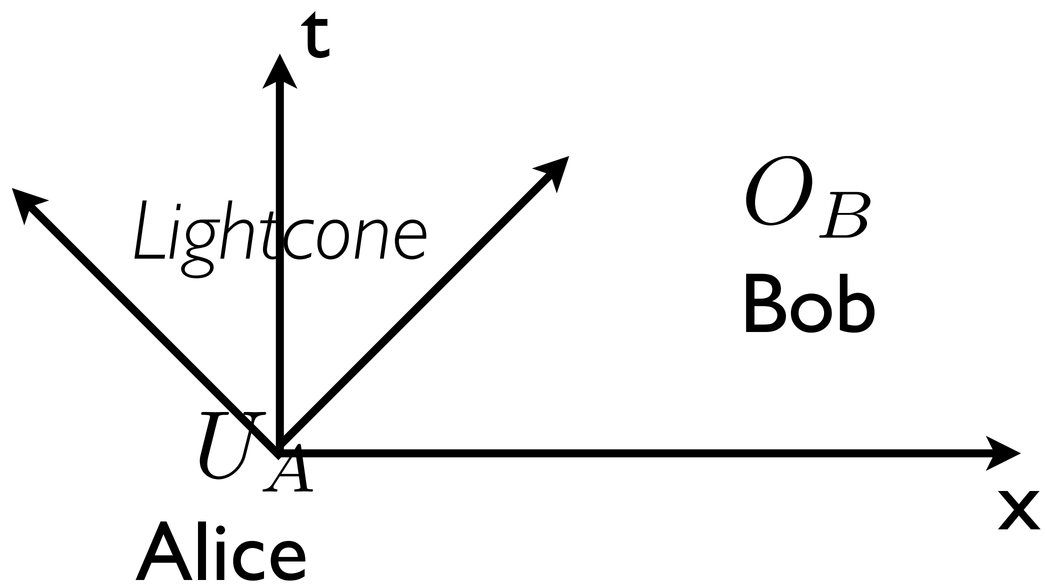
$$\|[A(t), B]\| \le C_B(X, t)$$

A has support on set X, B has support on set Y

Initial conditions: $X \cap Y = \emptyset \to C_B(X, t=0) = 0; \; X \cap Y \neq \emptyset \to C_B(X, t=0) = 2\|A\|\|B\|$

"Wave Equation" for "Lack of Commutation":

$$C_B(X, t) \le C_B(X, 0) + 2 \sum_{Z : Z \cap X \neq \emptyset} \|h_Z\| \int_0^{|t|} ds \, C_B(Z, s)$$



$\langle U_A^\dagger(t) O_B U_A(t)\rangle - \langle O_B\rangle$ is small outside the lightcone

Limit on transmission of information, microcausality.

Lightcone for operators:

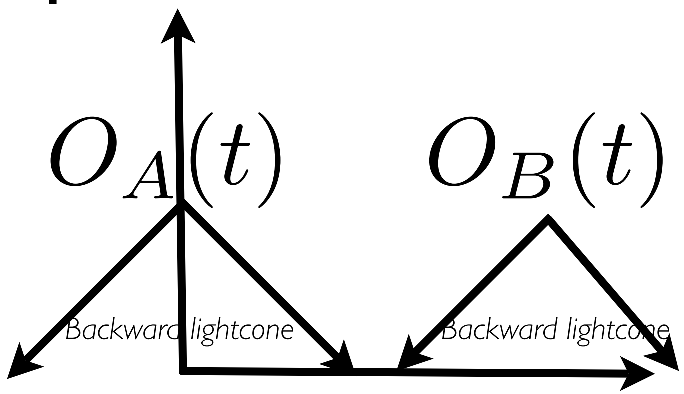
To localize operator O to a lightcone of size l, integrate over unitary rotations over the rest of the system

$$O^l(t) = \int d\mu(U) U O U^\dagger$$

Result: to exponential accuracy, O(t) can be written as an operator which acts only on sites within distance l=vt of O

Producing correlations:

Evolve system for time t under arbitrary time-dependent local Hamiltonian: $\mathcal{H} = \mathcal{H}(t'),\ 0 \leq t' \leq t$



Assume initial state has correlation length small compared to l-vt. Then operators can't be correlated at the final time!

Time to introduce correlations proportional to correlation length!

Producing topological order:

Define a set of states Ψ_i^0 to have topological order to accuracy (ϵ, l)

if for every operator O supported on a set of diameter l, we have $|\langle\Psi_i^0, O\Psi_j^0\rangle - const \times \delta_{i,j}| \leq \epsilon\|O\|$

Topological order as a property of states!

"Topologically ordered" if epsilon is exponentially small in L for l of order L

Suppose we start with states Ψ_1^0, Ψ_2^0
with topological order to given accuracy (ϵ, l)

$U = \exp[\int_0^t \mathcal{H}(t)dt]$ $\quad \langle U\Psi_i^0, OU\Psi_j^0\rangle = \langle\Psi_i^0, O(t)\Psi_j^0\rangle$

O(t) is still a local operator, so states $U\Psi_0^1, U\Psi_0^2$ are topologically ordered with accuracy of order $(\epsilon + \mathcal{O}(exp(-l), l - vt)$

If initial state is "topologically ordered", so is final state! Converse: **takes time t of order L to produce topological order.**

Quantum Phases of Matter

Parameter dependent Hamiltonian: $\mathcal{H}_s, 0 \leq s \leq 1$

Let gap from ground state sector to excited states be at least ΔE for $0 \leq s \leq 1$

"Same phase"

How does ground state evolve?

Adiabatic evolution? $\mathcal{H}(t) = \mathcal{H}_{s=t/t_0}$

$$\Psi_i^0 \to \mathcal{T} \exp[i \int_0^{t_0} \mathcal{H}(t) \mathrm{d}t] \Psi_i^0$$

If $t_0 \Delta E >> 1$ leaves system close to ground state

However, the error scales only as a power of $t_0 \Delta E$, but also scales with system size. Needs long times for macroscopic systems!

Better: Quasi-adiabatic evolution

$$\partial_s \Psi_i^0\rangle = \sum_a \partial_s (\mathcal{H}_s)_{ai} \frac{1}{E_0 - E_a} \Psi_a\rangle$$ Linear perturbation theory

$$= -\int_0^\infty \mathrm{d}\tau [(\partial_s \mathcal{H}_s)^+(i\tau) - (\partial_s \mathcal{H}_s)^-(-i\tau)]\Psi_i^0\rangle$$ An integral representation, positive and negative frequency parts

$$\approx -\int_0^\infty \mathrm{d}\tau [(\partial_s \tilde{\mathcal{H}}_s)^+(i\tau) - (\partial_s \tilde{\mathcal{H}}_s)^-(-i\tau)]\Psi_i^0\rangle$$ A local approximation

$$\tilde{O}^+(i\tau) \equiv \frac{1}{2\pi} \int \mathrm{d}t A(t) \frac{1}{it+\tau} \exp[-(t/t_q)^2/2]$$ A local approximation to positive and negative frequency parts, with exponentially small error

$$\Psi_i^0 \rightarrow \mathcal{S}' \exp[i \int_0^s \mathcal{D}(s')\mathrm{d}s']\Psi_i^0$$ local Hamiltonian evolution

Result: can represent change in ground state wavefunction by local Hamiltonian evolution if gap remains open

Error exponentially small in $t_q \Delta E$ even in thermodynamic limit

Quasi-adiabatic evolution with multiple approximately degenerate ground states:

$\mathcal{S}' \exp[i \int_0^s \mathcal{D}(s') \mathrm{d}s'] \Psi_0^i \approx Q \Psi_0^i(s)$ Q is a unitary matrix

Ground state splitting ϵ

Matrix elements of quasi-adiabatic evolution operator, $\mathcal{D}(s)$, between ground states vanish linearly in $t_q \epsilon$

Hence, the matrix Q agrees with the usual (non-abelian) geometric phase to zeroth order in $t_q \epsilon$

Stability of Topologically Ordered States

$$\Psi_i^0 \rightarrow \mathcal{S}' \exp[i \int_0^s \mathcal{D}(s')\mathrm{d}s]$$

Quasi-adiabatic evolution is local Hamiltonian evolution

From before, this means that topological order cannot be lost under this evolution for short enough times compared to system size. This means that any finite change in parameters which keeps the gap open in the thermodynamic limit keeps the same structure of topologically ordered states!

Example: disorder in FQHE, compare Wen and Niu to first order in perturbation theory

Stability of States with Ordinary Order

$$\Psi_i^0 \rightarrow \mathcal{S}' \exp[i \int_0^s \mathcal{D}(s')\mathrm{d}s]$$

Quasi-adiabatic evolution is local Hamiltonian evolution

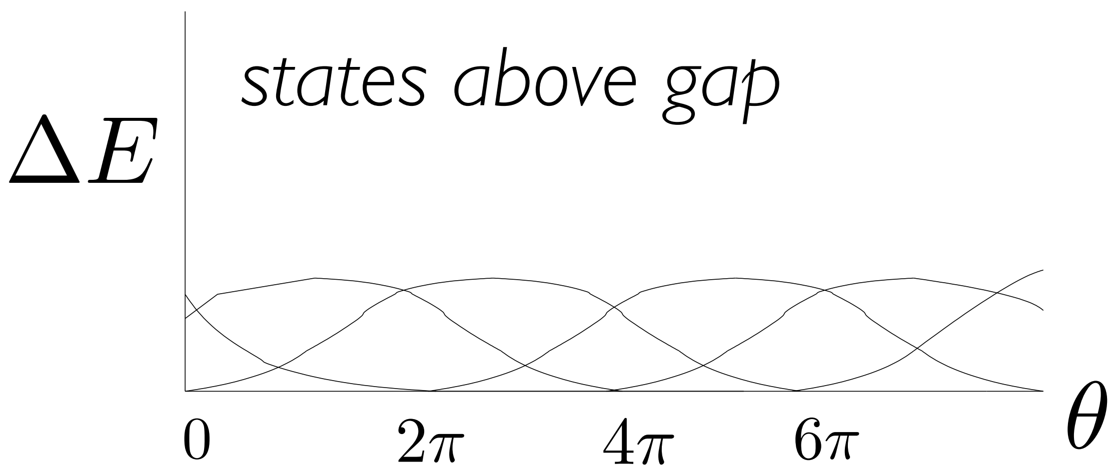
Can always construct dressed operators with the same correlation functions. For example, transverse field Ising model. In ordered phase, long range correlations present. Operators with Ising symmetry cannot break ground state degeneracy if gap remains open.

$$\mathcal{H} = J \sum_{<ij>} S_z^i S_z^j + B \sum_i S_x^i \qquad \langle S_z^i S_z^j \rangle = 1, B = 0$$

Need Ising symmetry of perturbation to make matrix elements vanish at B=0 between ground states. Order is less robust, can be broken by parallel field.

Flux Insertion

Thouless and Gefen in FQHE: insert flux slow enough to avoid exciting local states above gap but fast enough to "shoot through" level crossings.



Inserting flux drives system between different topologically ordered states

Quasi-adiabatic evolution under change in Hamiltonian parameters provides a way to do this.

Due to gauge symmetry, this works even if gap is present only at $\theta = 0$

Lieb-Schultz Mattis Theorem:

- Assume gap at $\theta = 0$
- Apply quasi-adiabatic flux insertion
- Provably create state with different momentum
- Either multiple ground states with topological order or symmetry breaking or $\Delta E \leq \log(L)/L$
- Valid for systems with conserved charge at any non-integer filling

Combining Lieb-Robinson **with a gap:**

- Bounds on propagation of information
- **Locality theorems: gap implies exponential decay of correlations, also results for Fermi systems at non-zero temperature**
- **Stability of phases while a gap remains open**
- **Existence of topological order in certain systems, higher dimensional Lieb-Schultz-Mattis theorems**
- **Local projective Hamiltonian, algorithms for simulating quantum systems**
- Matrix product form for density matrix at non-zero temperature
- Area laws for production of entropy under time evolution

General technique, other extensions: geometric phase for quasi-adiabatic evolution, area laws and related for gapped systems, improved algorithms, LSM theorems for even width systems (dimer liquids), more?