Diagrams for invariant tensors

Greg Kuperberg
UC Davis

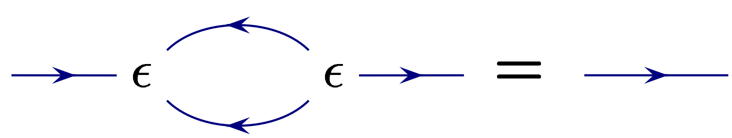
math.QA/9201302
arXiv:q-alg/9712003
(and follow-up and forthcoming papers)

Invariant tensors

If G is a group, you can calculate with its invariant tensors using Feynman-like diagrams, or "spin networks".

Example:
$G = \mathrm{SU}(3)$
$V = \mathbb{C}^3$, the defining irrep.
$\epsilon \in \mathrm{Inv}(V^{\otimes 3}), \epsilon \in \mathrm{Inv}(\bar{V}^{\otimes 3})$

Then, e.g., $\epsilon_{abc}\epsilon^{bcd} = \delta_a^d$. As a spin network:



Note: The "spin" spaces are any irreps V of any group G.

Tensor categories

A *rigid pivotal tensor category* is any abstract calculus of planar "spin" networks. All diagrams evaluate as vertex colors, by some consistent rules. Edges may be colored or oriented.

If the diagrams are on a sphere, it is *spherical*.

If the diagrams are tangled, it is *braided*.

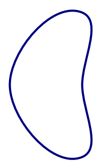
If neither planarity nor tangling matters, it is *symmetric*.

If vertex colors are vector spaces, the category is *additive-linear*.

If all projectors are equivalent to edge colors, the category is *abelian-linear*.

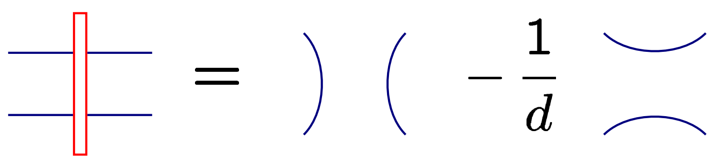
A simple example

The Temperley-Lieb category has a single unoriented edge type, no *ab initio* vertices, and the relation:



$$= d \in \mathbb{C}(\text{or } \mathbb{F})$$

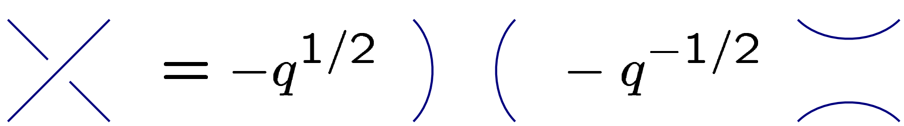
It becomes abelian-linear, if we add projectors as edge colors, e.g.:



(These are called *Jones-Wenzl* projectors.) When $d = -2$, T-L is $\cong$ to the invariants of SU(2).

Quantum groups and a famous polynomial

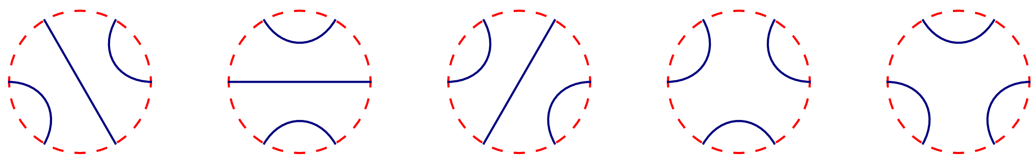
If $d = -q - q^{-1}$, we get $U_q(\mathrm{sl}(2))$ or $\mathrm{SU}(2)_q$. We can also define crossings:



This yields the *Jones polynomial* of a knot.

The basis result

The $2n$-endpoint spin network space has dimension C_n:



The crossingless matchings are a special basis.

The isomorphism says that these must also be a basis of $\mathrm{Inv}_{\mathrm{SU}(2)}(V^{\otimes 2n})$.

This is an old result of Rumer, Teller, and Weyl (1930) that can be proved in three steps:

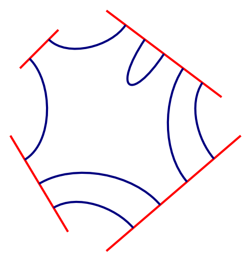
- All $(2n-1)!!$ matchings together span.
- Crossings are not needed.
- $\dim \mathrm{Inv}(V^{\otimes 2n}) = C_n$.

More bases

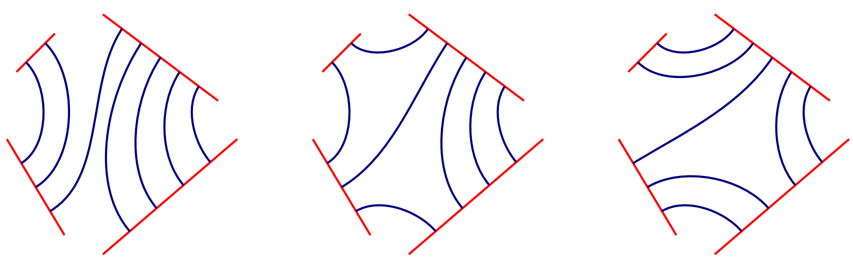
The basis result extends to any invariant space of spins

$$\mathrm{Inv}(V_{j_1} \otimes V_{j_2} \otimes \cdots \otimes V_{j_n}).$$

If we add (half) projectors, some basis vectors vanish:

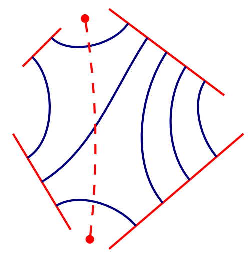
 $= 0$

The remaining vectors form a basis:



More bases

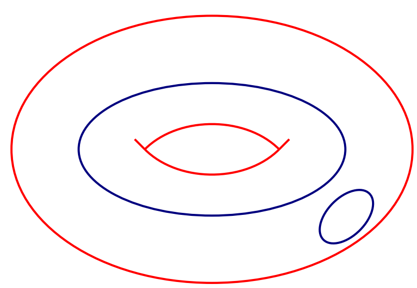
The proof is based on *minimal cut paths*:



...and the fact that $\dim \mathrm{Inv}(V_{j_1} \otimes V_{j_2} \otimes V_{j_3}) \leq 1$.

Even more bases

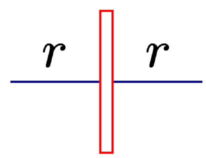
A pivotal category induces relations for graphs on a surface Σ:



These form an algebra if we stack networks in $\Sigma \times I$. The algebra acts on another state space $V(\Sigma)$, which is the "best" space to assign to Σ. (But it has a central "charge".)

What we want, and wishful thinking

I have said nothing about unitarity, or even $\mathbb{C}$ at all. If $q = e^{\pi i/r}$ and $r \in \mathbb{N}$, then the T-L category has a unitary quotient, given by killing the rth projector:



$= 0.$

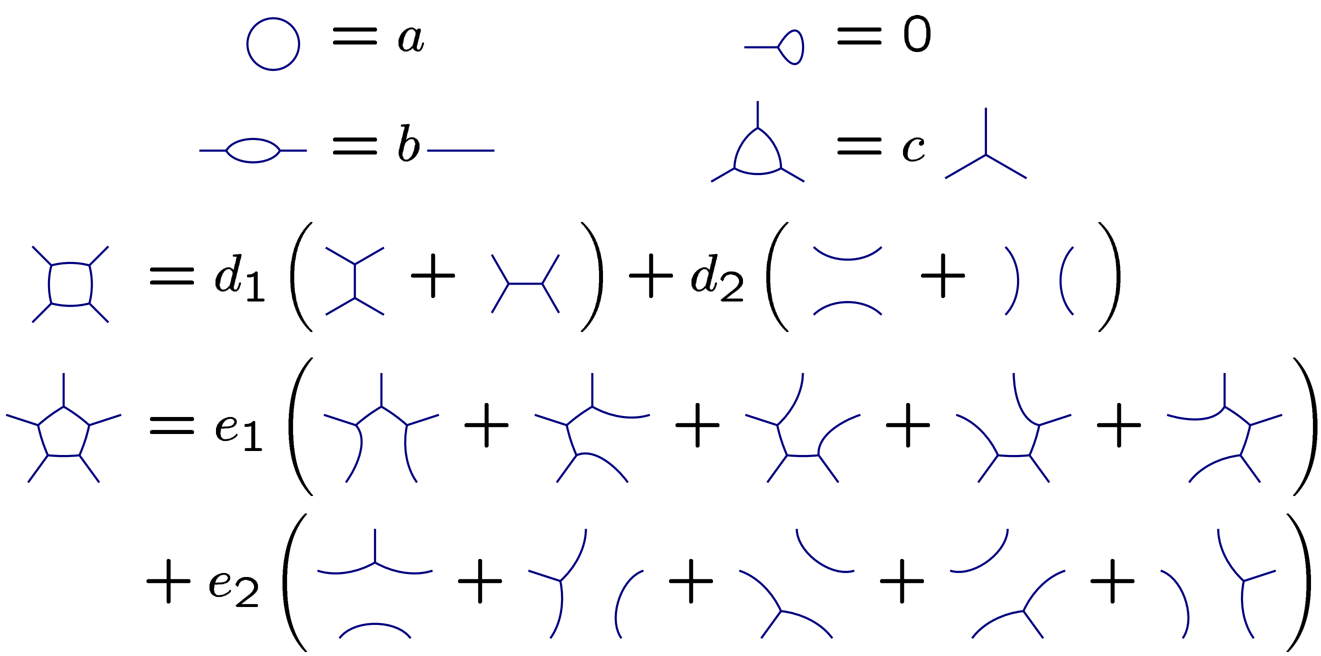
This is great for quantum computation. For example, when $r = 5$, there are 4 surviving edge colors (= projectors). We get the Fibonacci category times a vestigial binary coloring.

(Actually it is the Yang-Lee category. But Fibonacci is a more marketable name.)

The bad news: The nice basis properties are destroyed by the quotient. But the T-L category is still great for calculations, e.g., the F matrix or $6j$-symbol. (C.f. Masbaum-Vogel.)

Where do we go from here?

To generalize, we need some new inductive principle. One idea: We allow trivalent planar graphs, but we suppose that "positively curved" faces can be reduced:



We also suppose that:

- The right sides are l.i.
- The relations are *confluent*.

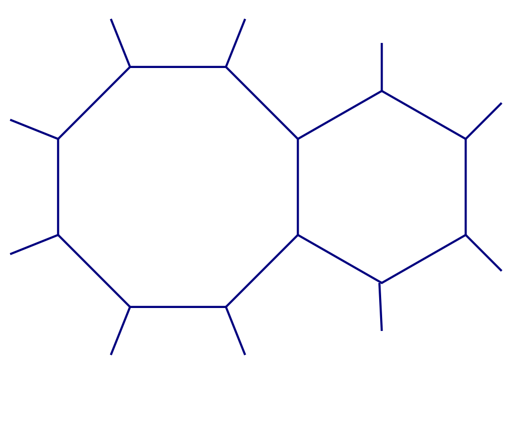
(The first condition fails for $j = 1$ spin networks.)

It works!

Surprisingly, it all works. The confluence equations have these solutions, up to a trivial rescaling:

$$\begin{aligned}
a &= q^5 + q^4 + q + 1 + q^{-1} + q^{-4} + q^{-5} \\
b &= -q^3 - q^2 - q - q^{-1} - q^{-2} - q^{-3} \\
c &= q^2 + 1 + q^{-2} \\
d_1 &= -q - q^{-1} \\
d_2 &= q + 1 + q^{-1} \\
e_1 &= -e_1 = 1
\end{aligned}$$

The state space of a disk with n endpoints has a basis of non-positively curved graphs:



What is it?

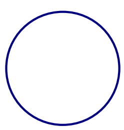
How many non-positively curved graphs with n endpoints are there?

$$1, 0, 1, 1, 4, 10, 35, 120, 455, 1792, 7413, \ldots$$

This equals

$$\dim \mathrm{Inv}_{G_2}(V_7^{\otimes n}).$$

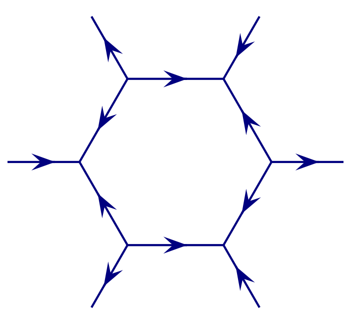
Furthermore,



$$= q^5 + q^4 + q + 1 + q^{-1} + q^{-4} + q^{-5}$$

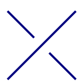
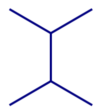
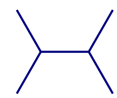
is the quantum dimension of V_7. The category is the invariant theory of G_2.

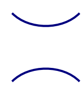
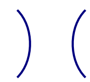
There are similar (but simpler) categories for $A_2 = \mathrm{SU}(3)$ and $B_2 = \mathrm{SP}(4) = \mathrm{Spin}(5)$. The one for A_2 has oriented trivalent graphs:



What else works

The rank 2 theories also have crossings:

 $= \dfrac{q^{-1/2}}{q^{1/2} + q^{-1/2}}$  $+ \dfrac{q^{1/2}}{q^{1/2} + q^{-1/2}}$ 

$+ \dfrac{q^{-3/2}}{q^{1/2} + q^{-1/2}}$  $+ \dfrac{q^{3/2}}{q^{1/2} + q^{-1/2}}$ 

You also get network algebras on surfaces, projectors, and unitary reductions at principal roots of unity. All of these exist by general algebra (Drinfel'd et al) or by TQFT theory (Witten et al). The direct skein relations for graphs are "new".

Proving the basis results makes essential use of minimal cut paths.

What doesn't work

The structure of elliptic confluence and bases does not exist for higher-rank Lie algebras. There is recent progress in finding semi-confluent relations (D. Kim, B. Westbury).

What could work

It may be possible to compute the rank 2 F-matrix, generalizing the famous Racah formula. It amounts to evaluating a tetrahedron of projectors: