Probing Non-Abelian Statistics with Quasiparticle Interferometry

Kirill Shtengel
University of California, Riverside

Collaborators: **Parsa Bonderson** and **Alexei Kitaev** (Caltech)
Joost Slingerland (Microsoft Project Q)

- Parsa Bonderson, Alexei Kitaev, KS, Phys. Rev. Lett. 96, 016803 (2006);
- A. Stern and B. I. Halperin, Phys. Rev. Lett. 96, 016802 (2006)
- Parsa Bonderson, KS, J. K. Slingerland, cond-mat/0601242

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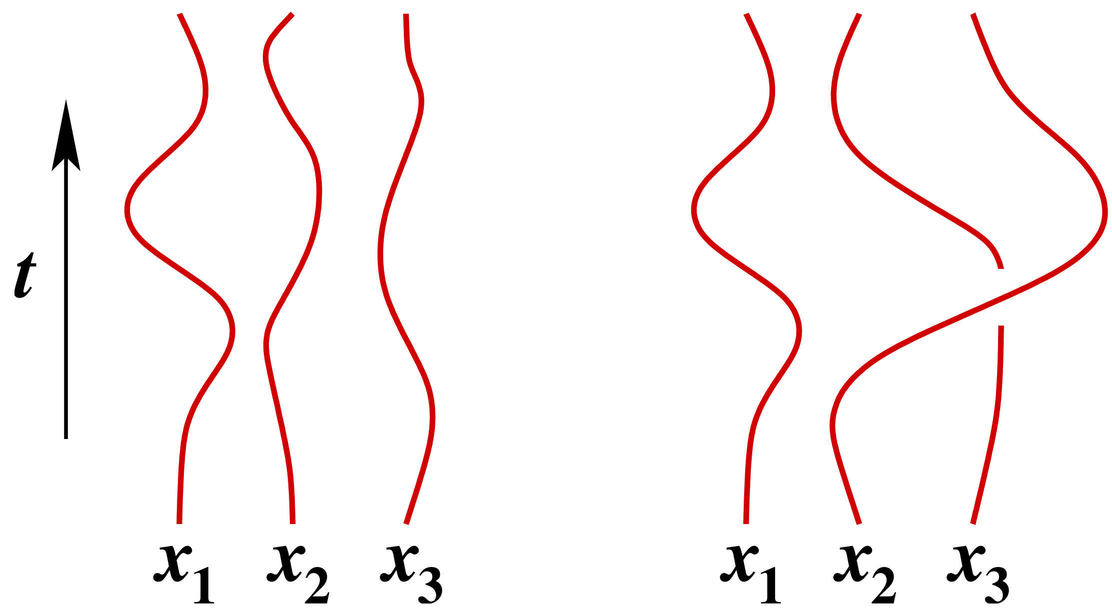
CALTECH

Exchange Statistics in (2+1)D

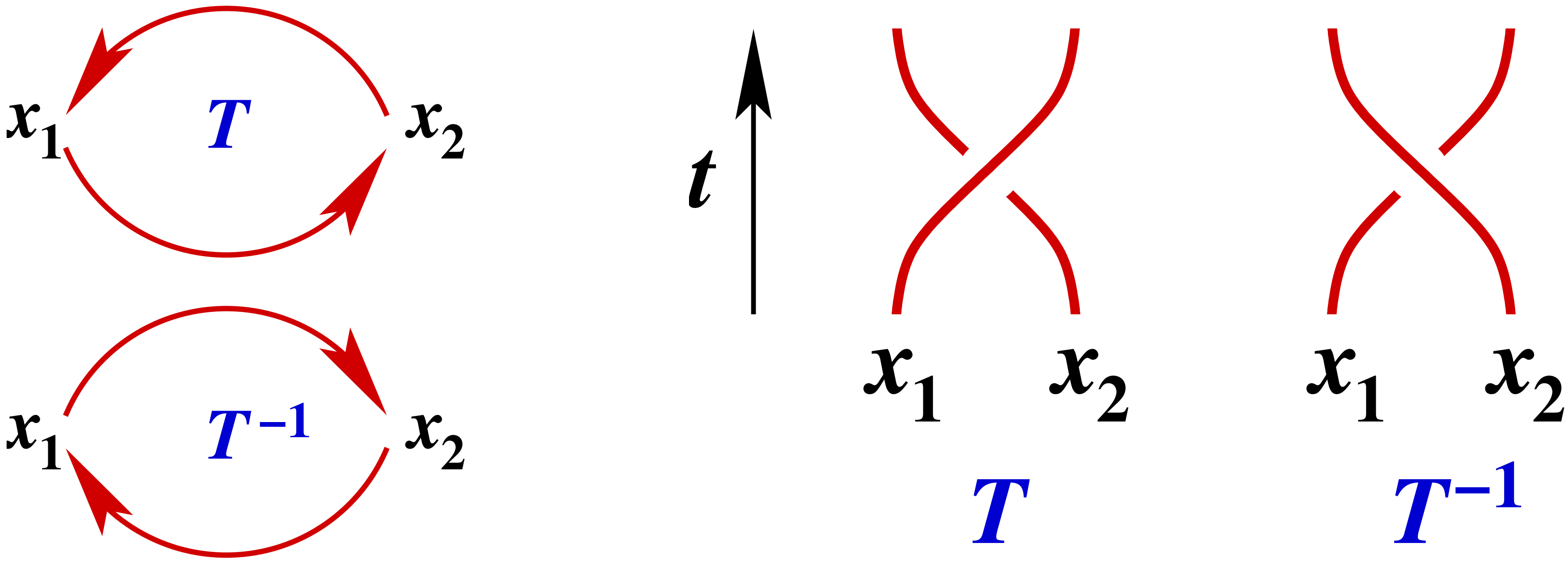
Quantum-mechanical amplitude for particles at $x_1, x_2, \ldots, x_n$ at time t_0 to return to these coordinates at time t.

Feynman: Sum over all trajectories, weighting each one by e^{iS}.

Exchange statistics: What are the relative amplitudes for these trajectories?



Exchange Statistics in (2+1)D

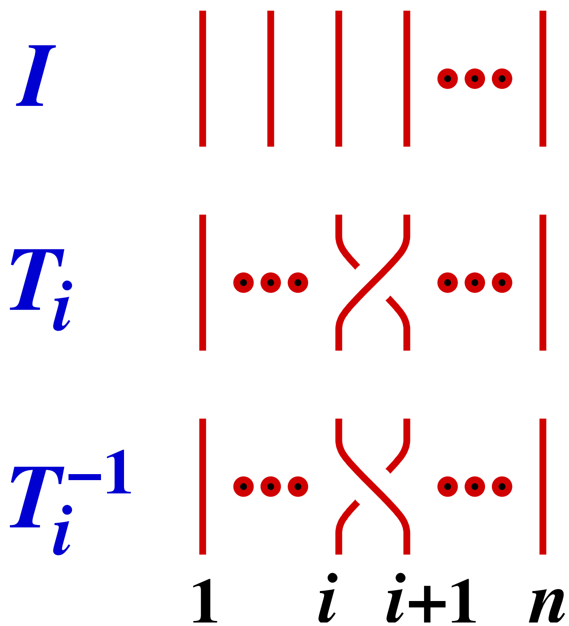


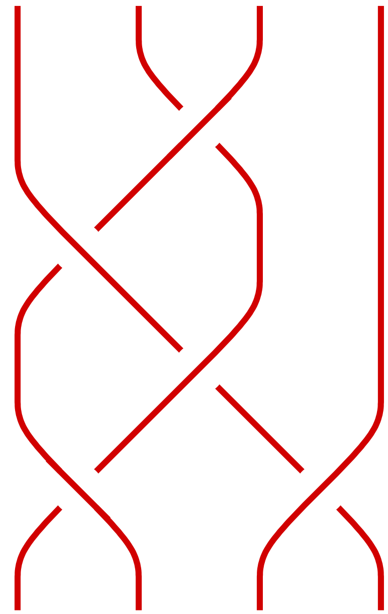
In (3+1)D, $T^{-1} = T$, while in (2+1)D $T^{-1} \neq T$!

If $T^{-1} = T$ then $T^2 = 1$, and the only types of particles are bosons and fermions.

Exchange Statistics in (2+1)D: The Braid Group

Another way of putting this is to say that in (3+1)D the particle statistics correspond to representations of the group of *permutations*. In (3+1)D, we should consider the *braid group* instead:

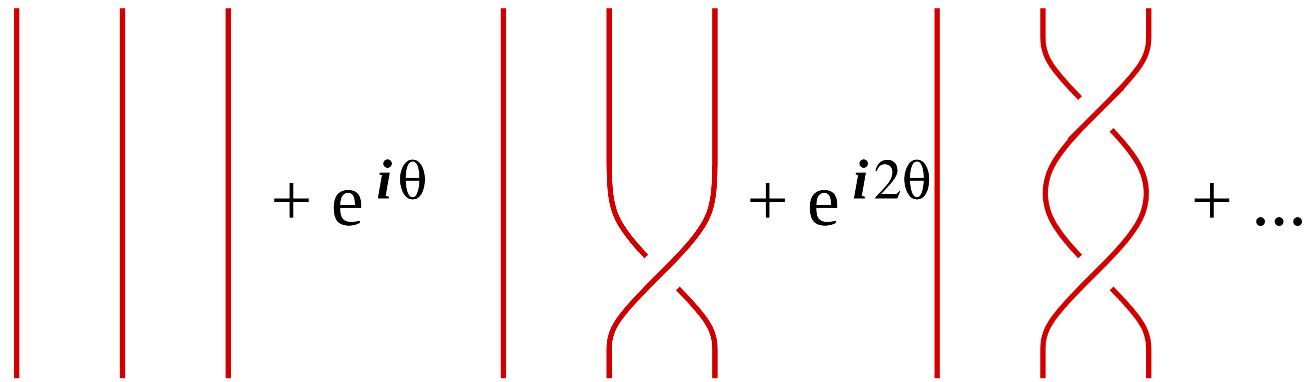




$T_1^{-1} T_3 T_2 T_1^{-1} T_2$

Anyons

• Different elements of the braid group correspond to disconnected subspaces of trajectories in space-time.
Possible choice: weight them by different overall phase factors (Leinaas and Myrrheim, Wilczek, . . .).

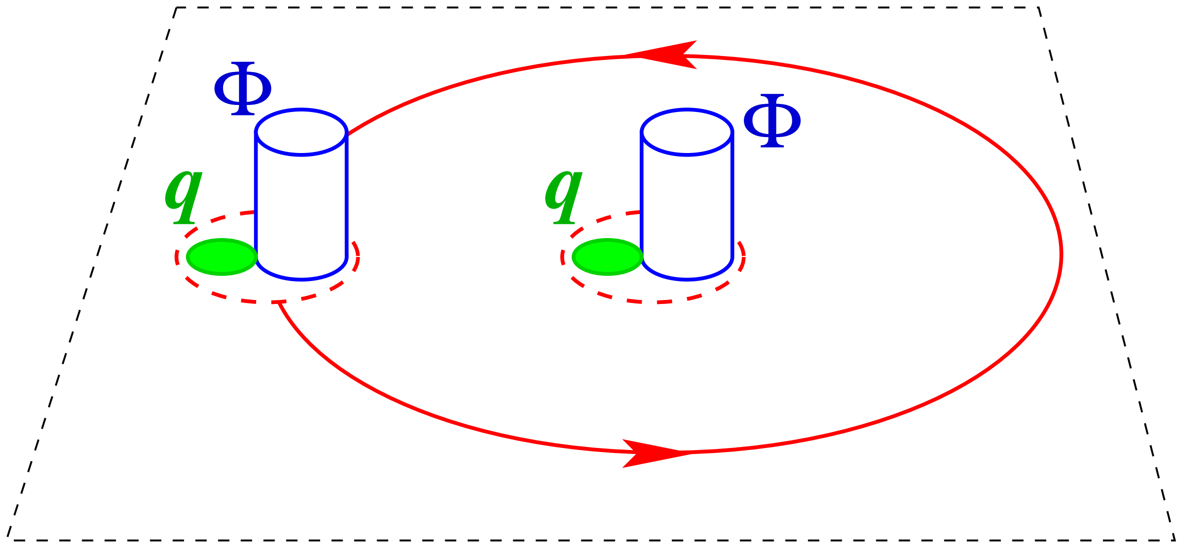


• These phase factors realise an Abelian representation of the braid group.
E.g $\theta = 2\pi/m$ for a Fractional Quantum Hall state at a filling factor $\nu = 1/m$.

• Topological Order is manifested in the ground state degeneracy on higher-genus manifolds (e.g. a torus):
m-fold degenerate ground states for FQHE (Haldane, Rezayi '88, Wen '90).

Abelian Anyons

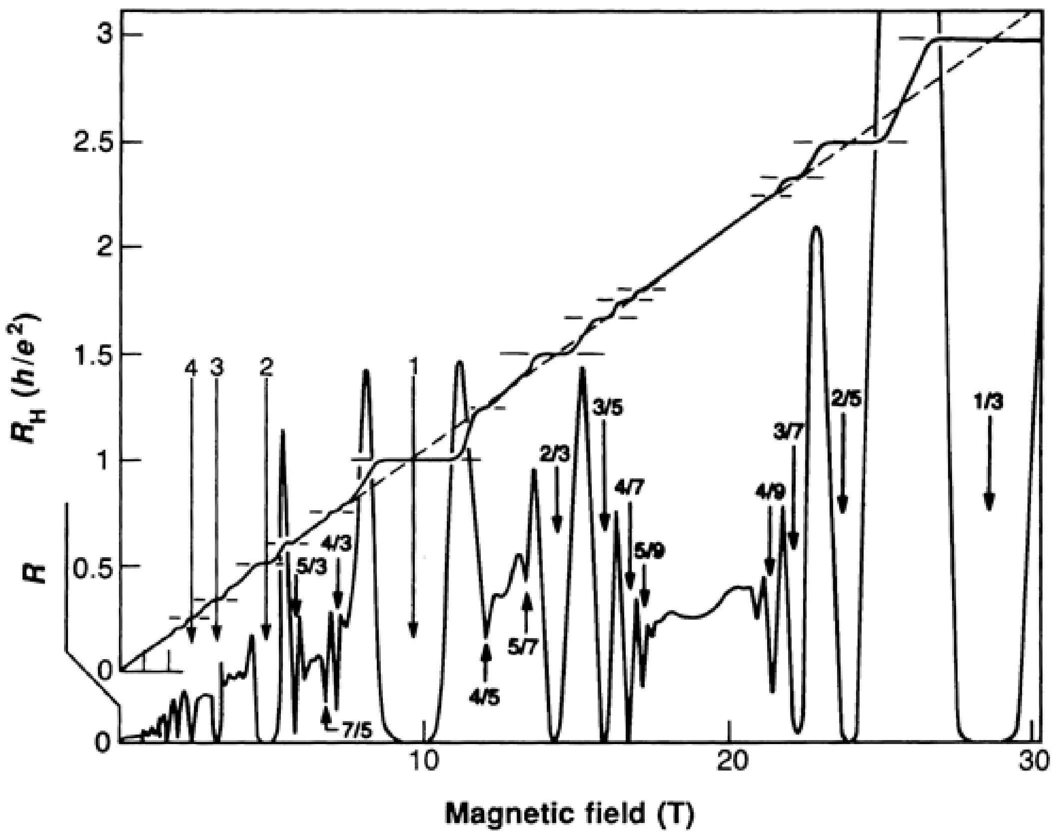
- Toy model (Wilczek 1982):



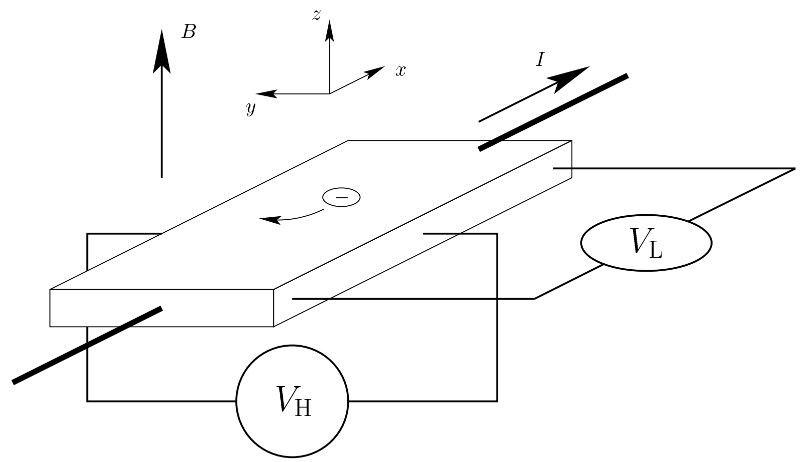
The Aharonov-Bohm phase $2\theta = q\Phi + q\Phi = 2q\Phi$ (in the units $\hbar = c = 1$)

E.g. for $q = e$, $\Phi = 2\pi/ne$, the statistical angle $\theta = 2\pi/n$.

Abelian anyons in real life: FQHE



Eisenstein and Störmer, 1990



Filling factor:

$$\nu \equiv \frac{N_e}{N_\Phi} = \frac{p}{q}$$

where $N_\Phi = BA/\Phi_0$

Abelian anyons in real life: FQHE

Laughlin states: $\nu = N_e/N_\Phi = 1/q$ with q - odd.

In particular, at $\nu = 1/3$, the ground state wavefunction is

$$\Psi_{\text{GS}} = \prod_{j<k} (z_j - z_k)^3 \prod_j e^{-|z_j|^2/4}$$

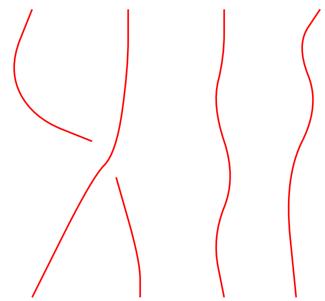
while the wavefunction for a state with a quasihole at z_0 is

$$\Psi_{\text{qh}} = \prod_{j<k} (z_j - z_k)^3 \prod_j (z_j - z_0) e^{-|z_j|^2/4}$$

$$e^* = e/3, \quad \theta = 2\pi/3$$

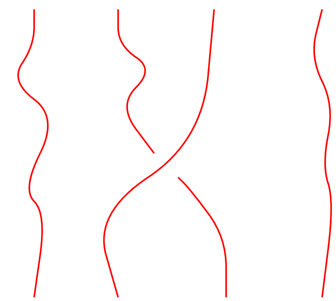
Non-Abelian Statistics

Exchanging particles 1 and 2:



$$\psi_a \longrightarrow M^{12}_{ab}\,\psi_b$$

Exchanging particles 2 and 3:



$$\psi_a \longrightarrow M^{23}_{ab}\,\psi_b$$

M^{12}_{ab} and M^{23}_{ab} need not commute, hence **Non-Abelian Statistics**.

- Matrices M form a higher-dimensional representation of the braid-group.

- For fixed particle positions, we have a non-trivial multi-dimensional Hilbert space where we can store information.

Non-Abelian Statistics

A (relatively) simple model: Unpaired zero-energy Majorana modes in the vortex cores in a $p + ip$-wave superconductor (Read and Green, 1999; Ivanov, 2001; Stern, Mariani and von Oppen, 2004; Stone, Chung, 2005).

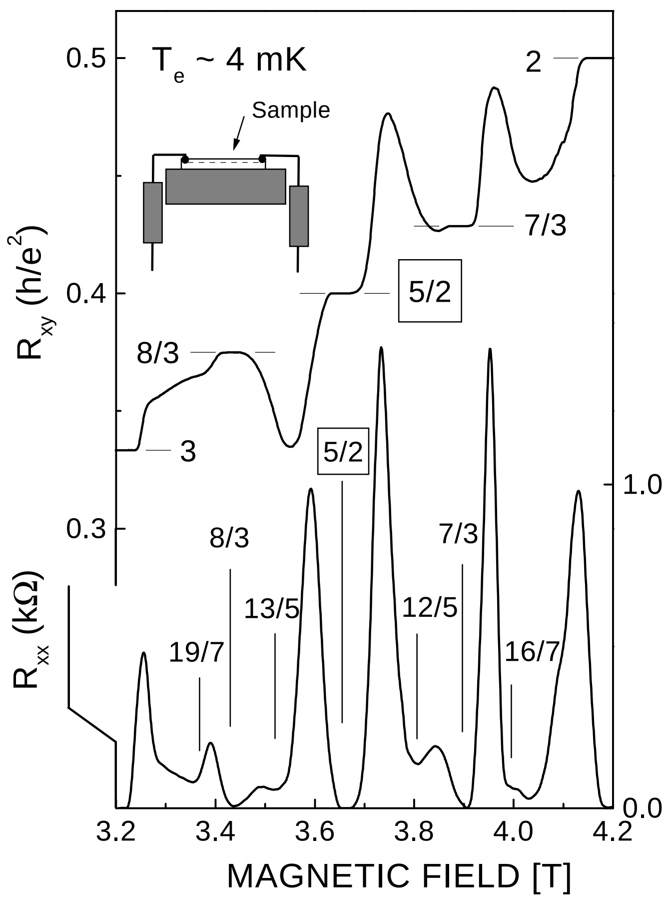
• To see the non-Abelian nature of excitations, look at 4 such modes (corresponding to 2 complex fermions). E.g., choose the basis $(|0\rangle,\ \Psi_1^\dagger|0\rangle,\ \Psi_2^\dagger|0\rangle,\ \Psi_1^\dagger\Psi_2^\dagger|0\rangle)$ where $\Psi_1 = (\gamma_1 + i\gamma_2)/2$, $\Psi_2 = (\gamma_3 + i\gamma_4)/2$.
The braiding matrices are (Ivanov, 2001):

$$T_1 = \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{i\pi/4} & & \\ & & e^{-i\pi/4} & \\ & & & e^{i\pi/4} \end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{-i\pi/4} & & \\ & & e^{i\pi/4} & \\ & & & e^{i\pi/4} \end{pmatrix}$$

Non-Abelian anyons in real life: FQHE?

Fig. 1, Pan et al



Non-Abelian anyons in real life: FQHE?

The prime candidate for non-Abelian statistics of excitations is the Moore-Read (*aka* Pfaffian) state at $\nu = 5/2 = 2 + 1/2$. This state is believed to be a spin-polarised superconductor of p-wave paired composite fermions (Read and Green, 1999).

The ground state wavefunction is (Moore and Read, 1991):

$$\Psi_{\mathrm{GS}} = \prod_{j<k} (z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \mathrm{Pf}\left(\frac{1}{z_j - z_k}\right)$$

where $\mathrm{Pf}(M) \equiv \sqrt{\mathrm{Det}(M)}$ for an antisymmetric matrix M, in this case $M_{jk} = (z_j - z_k)^{-1}$, $j \neq k$, while $M_{jj} = 0$.

Probing Abelian Statistics in FQHE

Experimental confirmation of charge $e^* = e/3$: Goldman and Su (1995).

Experimental confirmation of statistics $\theta = 2\pi/3$: Camino, Zhou and Goldman (2005).

What took so long?

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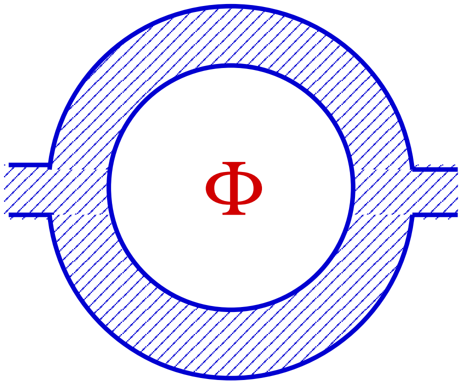
What took so long?

It is very non-trivial to tell charge from statistics!

Consider a "simple" interferometric experiment.
Q: What is the expected periodicity in Φ?
A:



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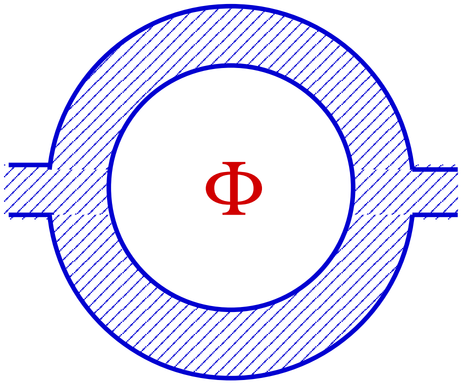

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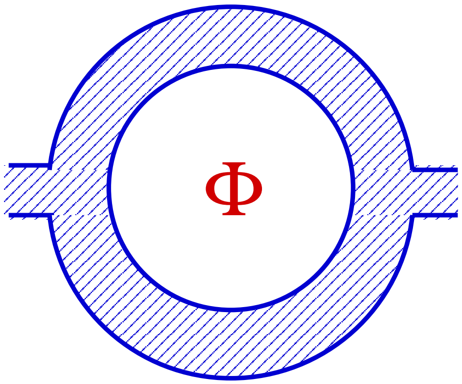
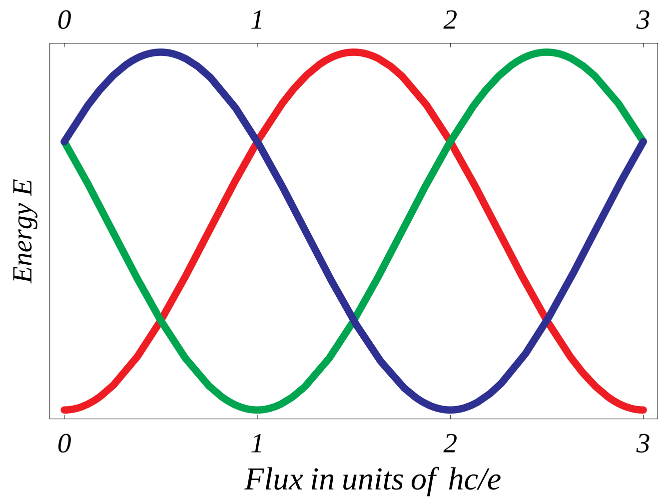
Wrong!
$\Phi^* = \Phi_0$
(Goldman, Liu and Zaslavsky, 2005)



Probing Abelian Statistics in FQHE

The reason for $\Phi^* = \Phi_0$ is gauge invariance. At the end, the system is "made" of real electrons, hence adiabatically adding Φ_0 should bring the system back to its ground state (Byers and Yang, 1961).

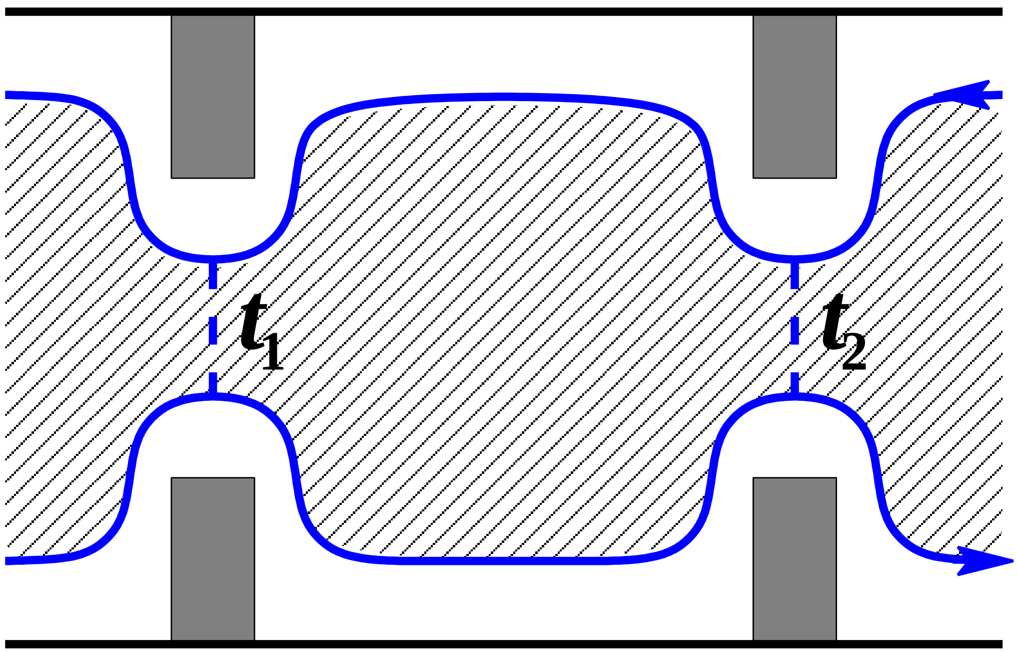
But this needs not be the same ground state (Thouless and Gefen, 1991)! A transition between these ground states is accompanied by tunnelling a quasihole between the inner and the outer edge of a ring.



So, such experiment would pick *both* charge and statistical contributions.

Probing Abelian Statistics in FQHE

A different, non-annular setup:



Probing Abelian Statistics in FQHE

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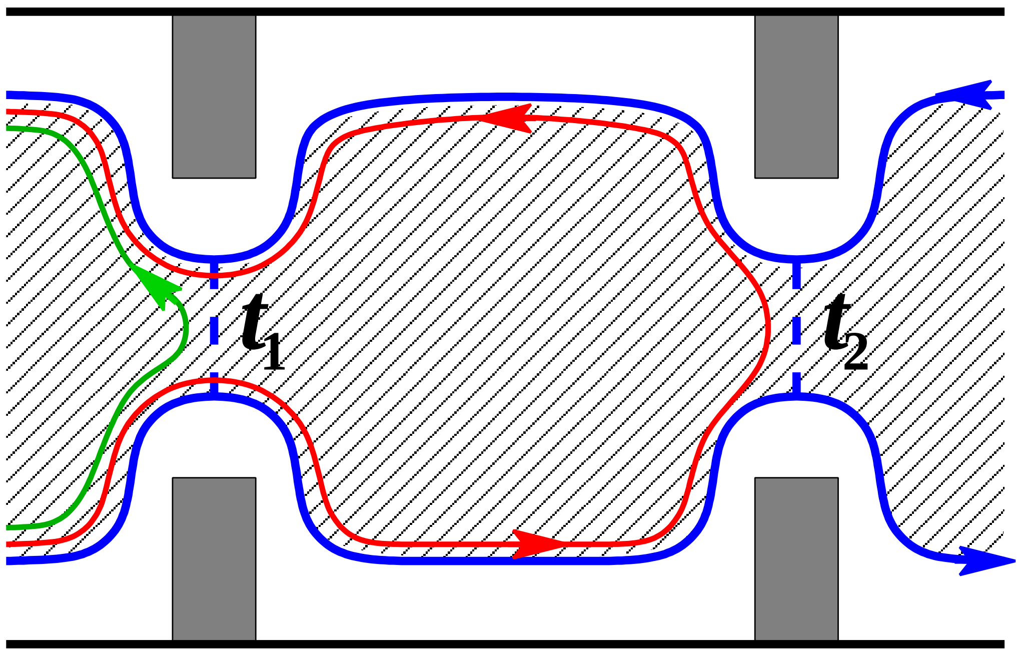
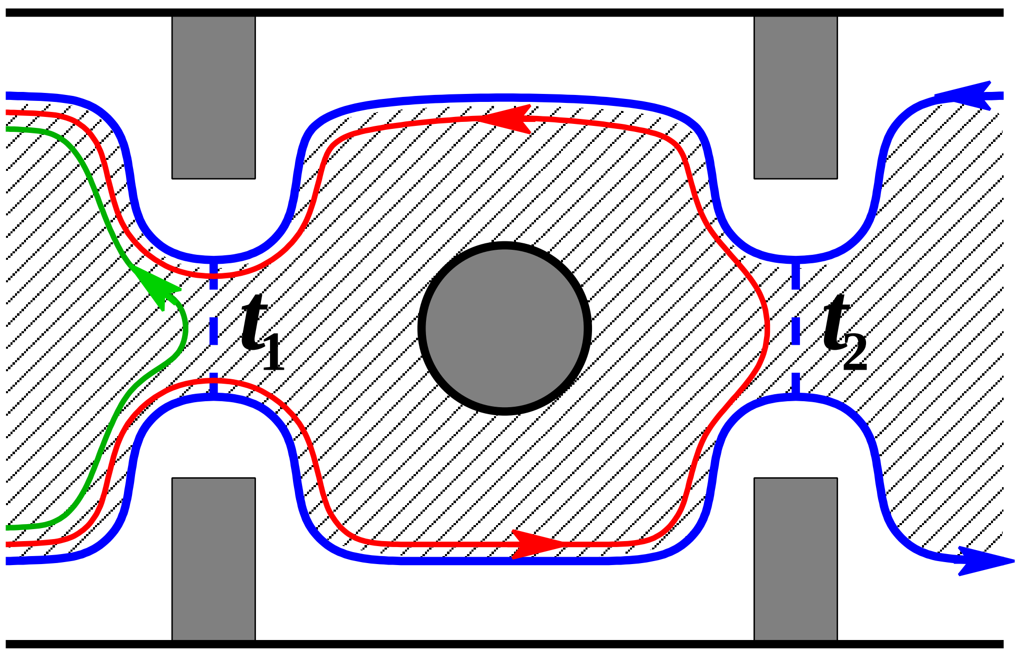


"Leaking back" results in deviation of σ_{xy} from its quantised value (and in the appearance of $\sigma_{xx} \neq 0$). The two back-scattering channels interfere with each other.

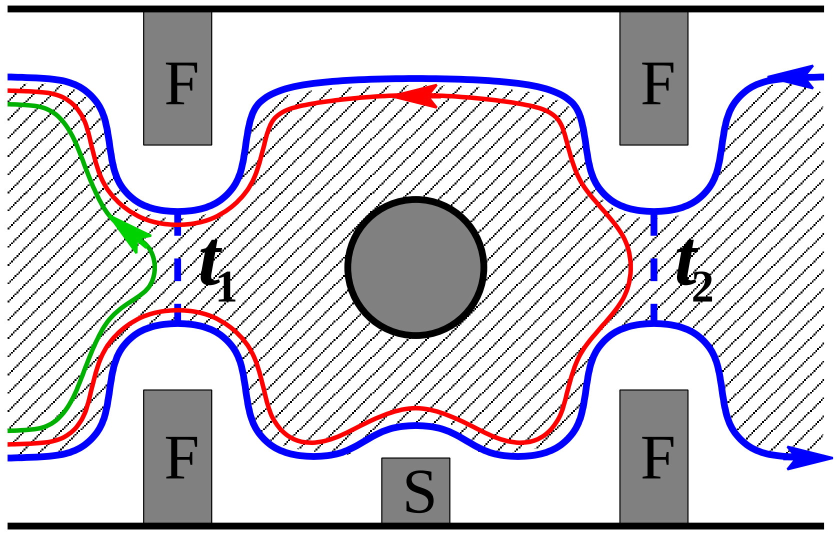
Quasiparticle interferometer

Chamon, Freed, Kivelson, Sondhi, Wen. 1997; Fradkin, Nayak, Tsvelik, Wilczek 1998.



A two point-contact interferometer for measuring the quasiparticle statistics. The hatched region contains an incompressible FQH liquid. The front gates are used to bring the opposite edge currents close to each other to form two tunneling junctions. Applying voltage to the central gate creates an antidot in the middle and controls the number of quasiparticles contained there.

Quasiparticle interferometer



$$\begin{aligned} \sigma_{xx} &\propto \left|(t_1 U_1 + t_2 U_2)\left|\Psi\right\rangle\right|^2 \\ &= |t_1|^2 + |t_2|^2 + 2\text{Re}\left\{t_1^* t_2 \left\langle\Psi\right| U_1^{-1} U_2 \left|\Psi\right\rangle\right\} \\ &= |t_1|^2 + |t_2|^2 + 2\text{Re}\left\{t_1^* t_2 e^{i\alpha} \left\langle\Psi\right| M_n \left|\Psi\right\rangle\right\} \end{aligned}$$

Here $\left\langle\Psi\right| M_n \left|\Psi\right\rangle$ is the statistical contribution from n quasiholes on an antidot.

Calculating the statistical contribution.

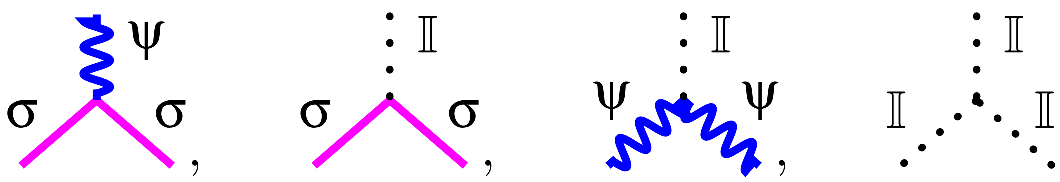
There are simple formal rules originating from CFT that make calculations simple.

• Non-Abelian anyons carry "labels" - analog of non-Abelian charge. For Moore-Read state, these labels are $\mathbb{I}$, σ and ψ.

• These quasi-particles can be combined according to the fusion rules:

$$\begin{array}{ccc} \mathbb{I}\times\mathbb{I}=\mathbb{I}, & \mathbb{I}\times\sigma=\sigma, & \mathbb{I}\times\psi=\psi, \\ \sigma\times\sigma=\mathbb{I}+\psi, & \sigma\times\psi=\sigma, & \psi\times\psi=\mathbb{I}. \end{array}$$

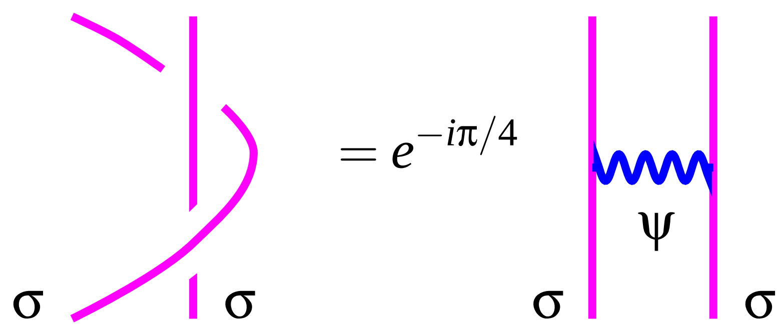
or, graphically



Braiding rules graphically

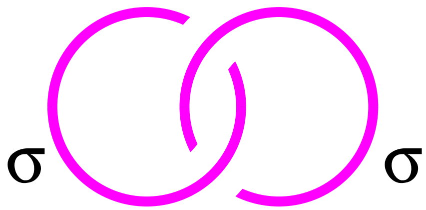
• There are several consistency conditions, such as *associativity* of fusion rules that have to be satisfied.

• Using these conditions, one can easily derive the following braiding rules:

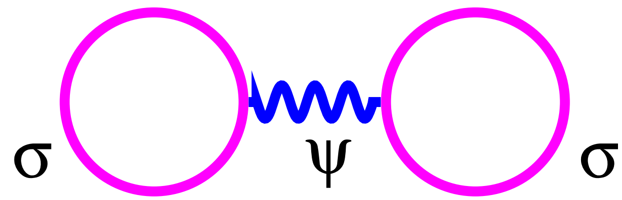


Quasiparticle interferometer

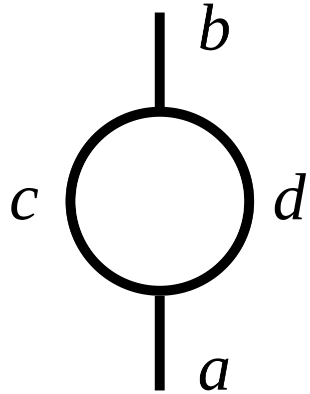
Use the graphical rules to evaluate the statistical contribution, $\langle\Psi|\, M_n \,|\Psi\rangle$. E.g. for a single σ-particle on an antidot it is equal to



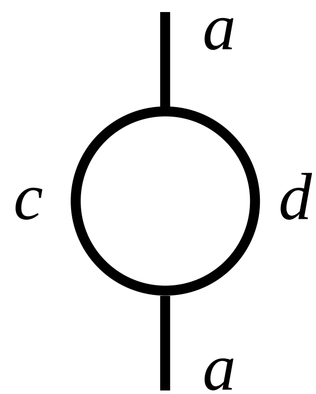
$$= e^{i(n-1)\frac{\pi}{4}}$$



But this contribution vanishes because of a general rule:

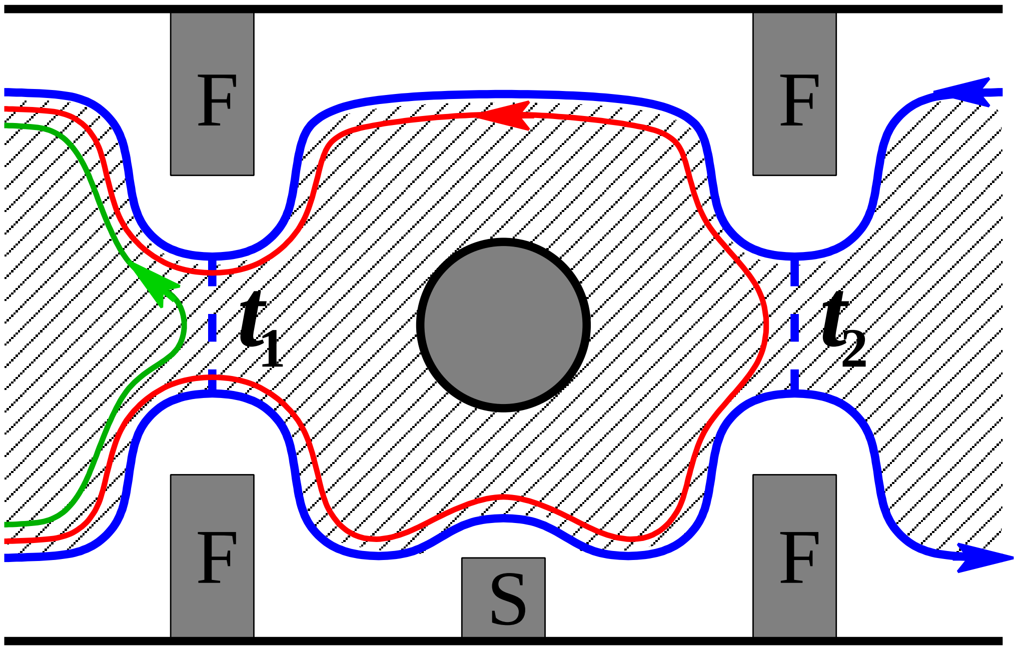


$$= \delta_{a,b}$$



i.e. a particle can't change its type without changing something else!

Quasiparticle interferometer



For an odd number of quasiholes on an antidot, we have *no* interference!

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 , \qquad n \text{ odd}$$

The interference pattern for an even number of quasiholes is

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + (-1)^{N_\psi} 2 |t_1| |t_2| \cos\left(\beta + n\frac{\pi}{4}\right) , \quad n \text{ even}$$

This even-odd effect is drastically different from the Abelian case!

Extension to $\nu = 12/5$

P. Bonderson, KS, J. K. Slingerland, cond-mat/0601242
S. B. Chung, M. Stone, cond-mat/0601594

The observed $\nu = 12/5$ state is expected to be particle-hole symmetric to the Read-Rezayi $\nu = 13/5$ state described by the $Pf_3 \times U(1)$ theory. The Pf_3 theory has six fields: $\mathbb{I}$, ε, $\sigma_{1,2}$, $\psi_{1,2}$, with the (non-trivial) fusion rules:

$$\begin{array}{ccc} \sigma_1 \times \sigma_1 = \psi_1 + \sigma_2, & \sigma_1 \times \sigma_2 = \mathbb{I} + \varepsilon, & \sigma_1 \times \psi_1 = \varepsilon, \\ \sigma_1 \times \psi_2 = \sigma_2, & \sigma_1 \times \varepsilon = \psi_2 + \sigma_1, & \sigma_2 \times \sigma_2 = \psi_2 + \sigma_1, \\ \sigma_2 \times \psi_1 = \sigma_1, & \sigma_2 \times \psi_2 = \varepsilon, & \sigma_2 \times \varepsilon = \psi_1 + \sigma_2, \\ \varepsilon \times \psi_1 = \sigma_2, & \varepsilon \times \psi_2 = \sigma_1, & \varepsilon \times \varepsilon = \mathbb{I} + \varepsilon, \\ \psi_1 \times \psi_1 = \psi_2, & \psi_1 \times \psi_2 = \mathbb{I}, & \psi_2 \times \psi_2 = \psi_1 \end{array}$$

Quasiholes carry anyonic charge: $(e/5, \sigma_1)$.
Electrons carry anyonic charge: $(-e, \psi_1)$.

Extension to $\nu = 12/5$

The Bratteli diagram helps to keep track of the allowed Pf_3 charge for n quasiholes:

$$
\begin{array}{c|ccccccc}
 & & \psi_2 & & \mathbb{I} & & \psi_1 & \\
 & \varepsilon & & \sigma_2 & & \sigma_1 & & \varepsilon \\
 & & \sigma_1 & & \varepsilon & & \sigma_2 & \\
 & \mathbb{I} & & \psi_1 & & \psi_2 & & \mathbb{I} \\
\hline
n & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

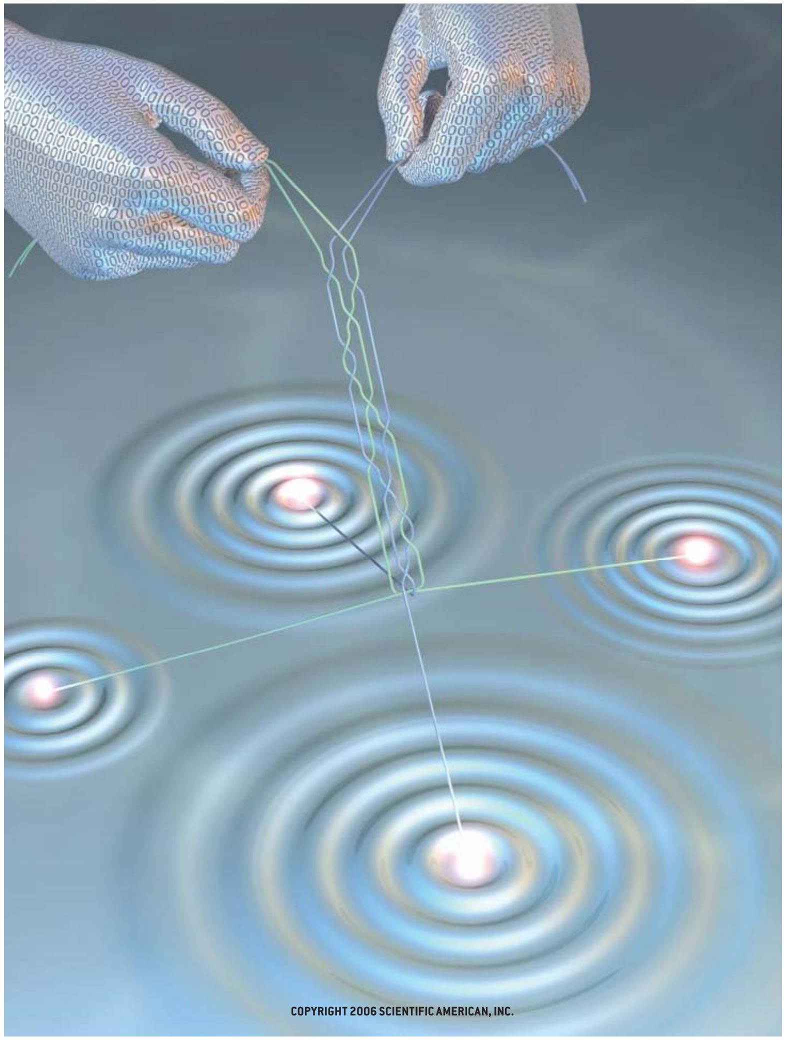
The longitudinal conductivity in the interferometry experiment will be

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + 2\,|t_1 t_2| \left(-\phi^{-2}\right)^{N_\phi} \cos\left(\beta + n\frac{4\pi}{5}\right)$$

$N_\phi = 0$ if the n quasihole composite on the antidot has Pf_3 charge with quantum dimension 1 (i.e. $\mathbb{I}$, ψ_1, or ψ_2).

$N_\phi = 1$ if the composite has quantum dimension $\phi = \frac{1+\sqrt{5}}{2}$ (i.e. σ_1, σ_2, or ε).

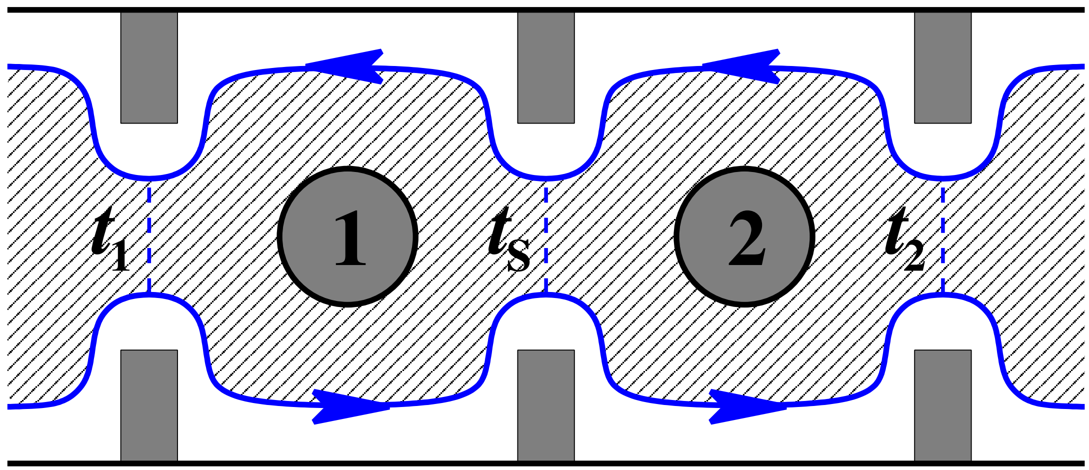
Topological Quantum Computation



KITP, May 15, 2006

Reading a Topologically Protected Qubit

(Das Sarma, Freedman and Nayak, 2005)



One (or any *odd* number) of quasiholes per antidot. Their combined state can be either $\mathbb{I}$ or ψ, we can measure the state by doing interferometry:

$$\sigma_{xx} \propto |t_1|^2 + |t_2|^2 + (-1)^{N_\psi} 2 \, |t_1| \, |t_2| \cos\left(\beta + n\frac{\pi}{4}\right), \quad n \text{ even}$$

(The state can be switched by sending another quasihole through the middle constriction.)