Braiding and Fusion in $p_x + ip_y$ Superconductors

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Work done in collaboration with

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Outline

- $p_x + ip_y$ superconductors
- Edge states
- Vortex-core states
- Fusion
- Braiding
- Conclusions

$p_x + ip_y$ Superconductors

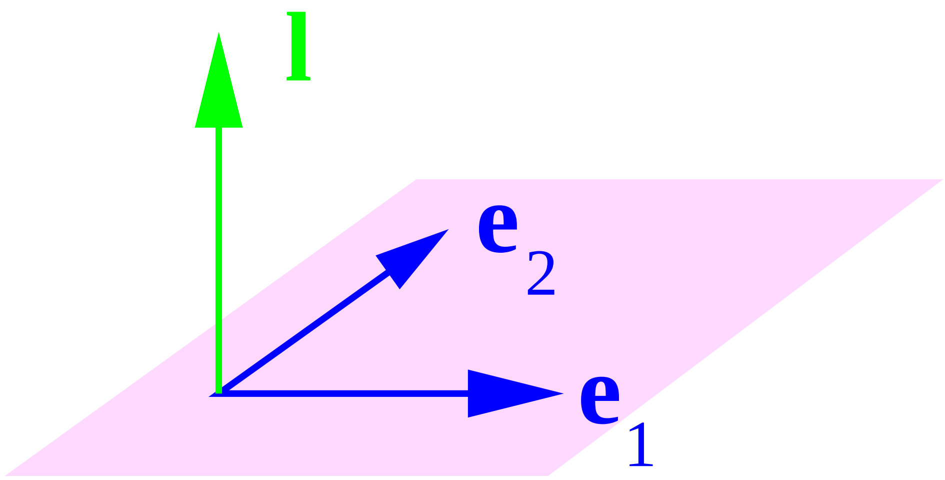
- $L = 1$, $S = 1$ pairing
- Two-dimensional analogue of ^{3}He-A phase
- Breaks time-reversal symmetry
- Candidate material: Sr_2RuO_4, $T_c = 1.5\mathrm{K}$. (Maeno *et al,* Rice, Sigrist, Baskaran)

$p_x + ip_y$ Superconductors

Triplet Order Parameter

$$\langle\hat{\psi}_\alpha(x)\hat{\psi}_\beta(y)\rangle \propto (i\sigma_2 \mathbf{d}\cdot\boldsymbol{\sigma})_{\alpha\beta}\,\{(\mathbf{e}_1 + i\mathbf{e}_2)\cdot\boldsymbol{\nabla}\}\,\delta^3(x-y)$$

$$\mathbf{e}_1\cdot\mathbf{e}_2 = 0, \quad \mathbf{e}_1\times\mathbf{e}_2 = \mathbf{l}$$



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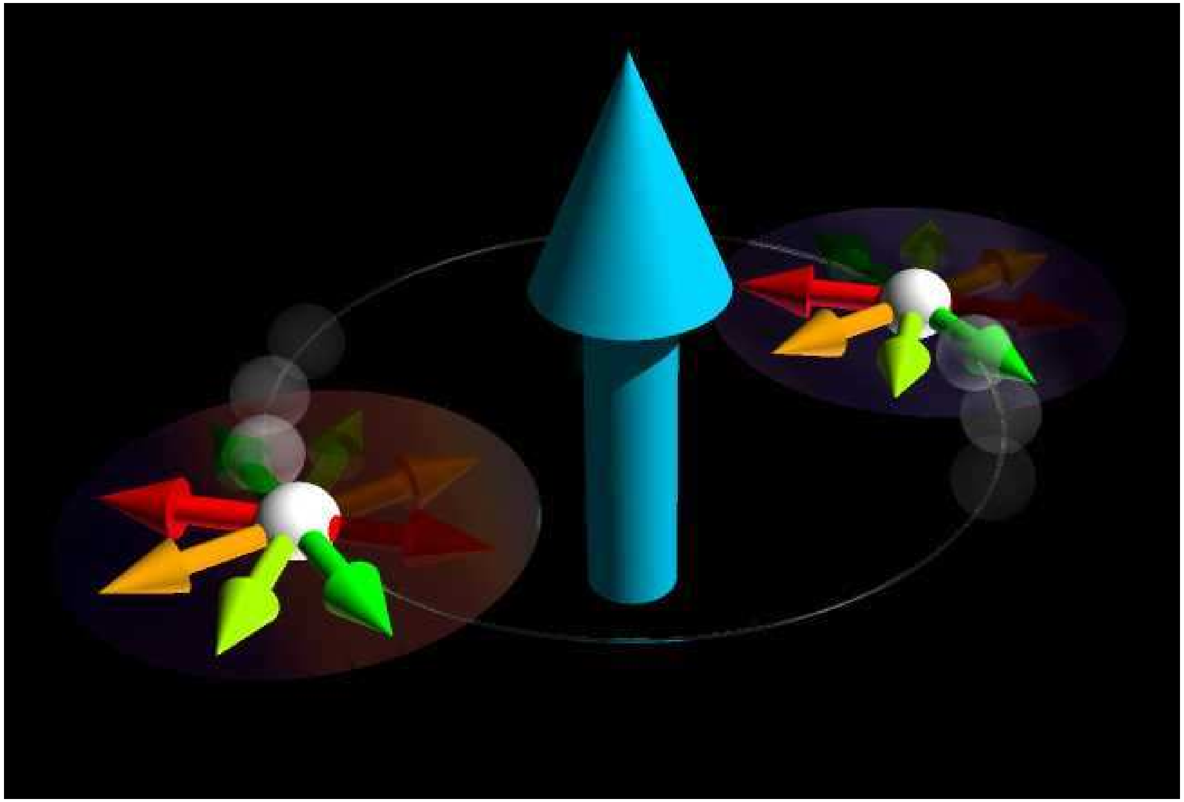
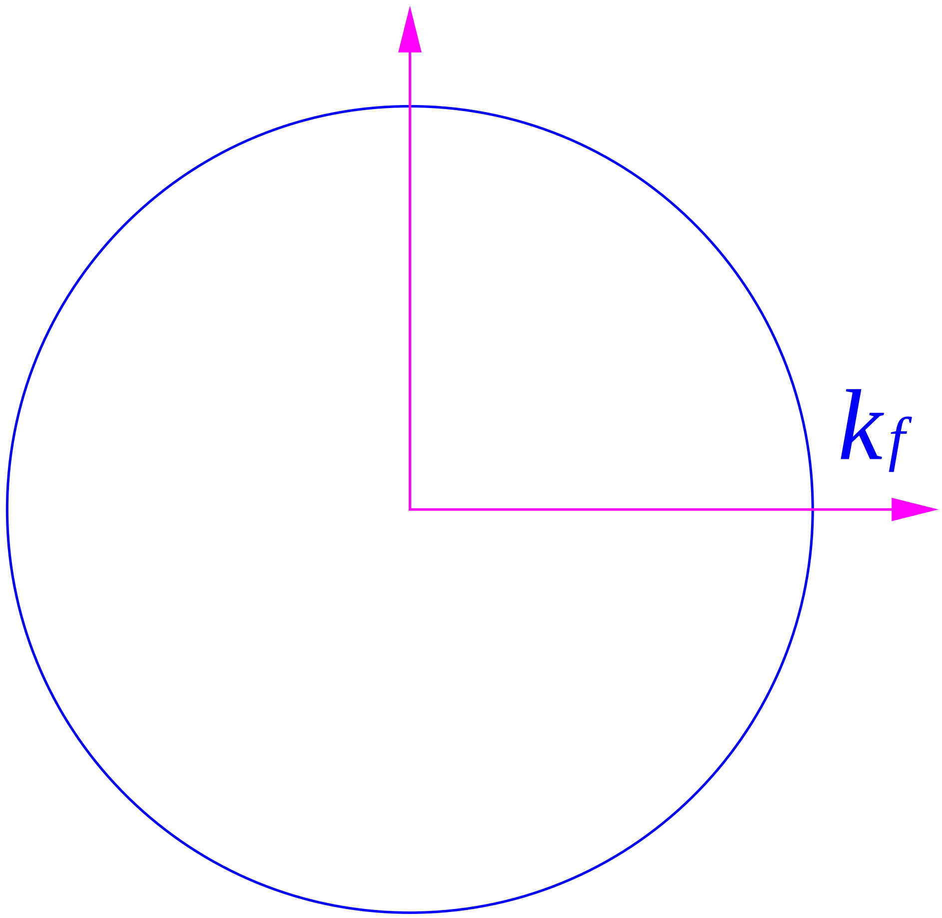


(Y.Maeno, K.Deguchi)

- Vector $\mathbf{l} = \pm\hat{z}$: orbital angular momentum vector. Perpendicular to material plane
- Vector $\mathbf{d}$: direction in which condensate preserves spin.

Fermi surface of Sr_2RuO_4

Theorist's Fermi surface:



Fermi surface of Sr_2RuO_4

Actual Fermi surface:

Topological Features

- Majorana-Weyl edge modes
- Half-quantum vortices

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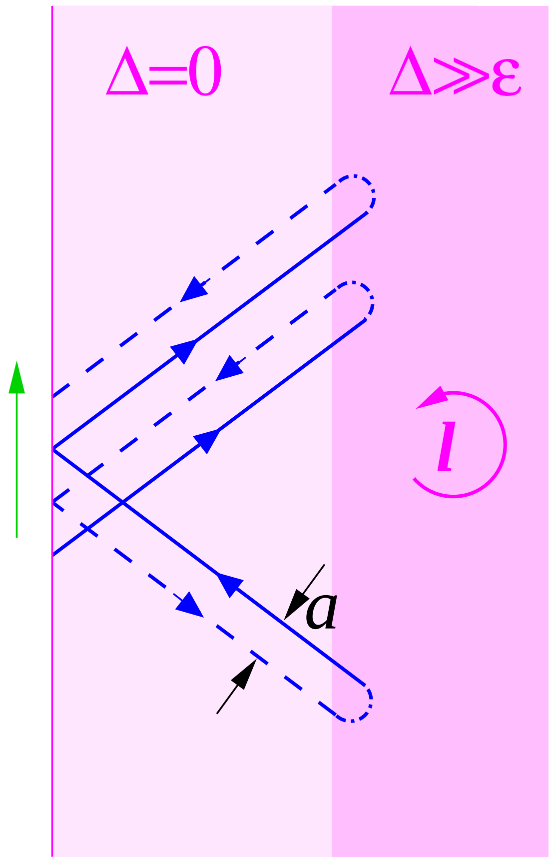
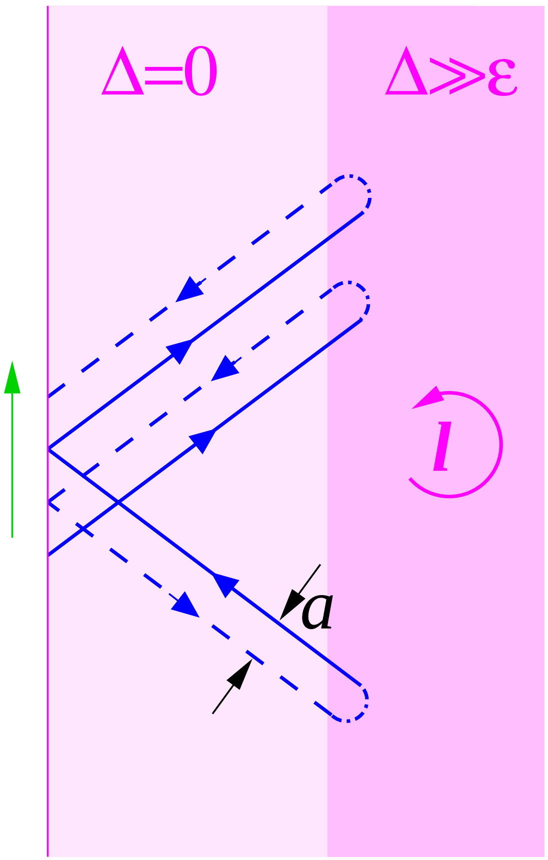
  - Majorana Core states
  - Ising fusion and braiding rules

Majorana-Weyl edge mode



Spin-triplet $p_x + ip_y$ SC has chiral Majorana edge-mode.

Majorana-Weyl edge mode



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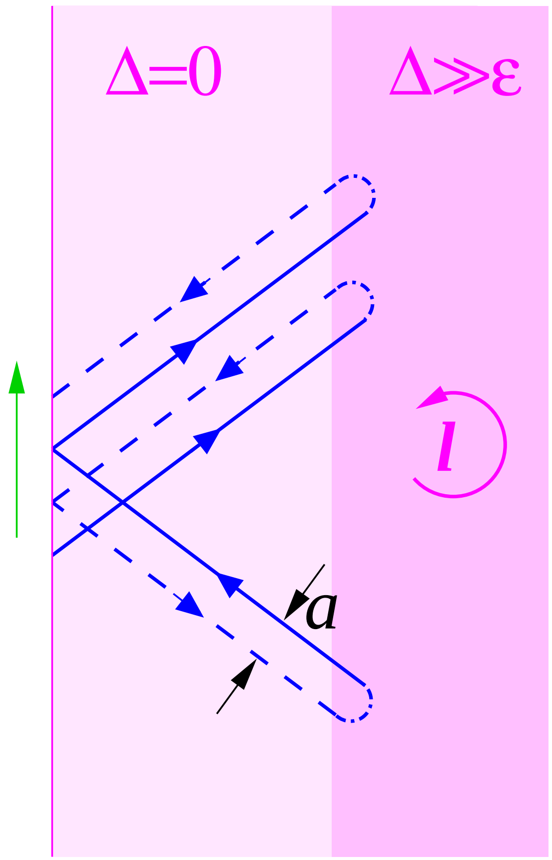
- Why *Weyl*?

Majorana-Weyl edge mode



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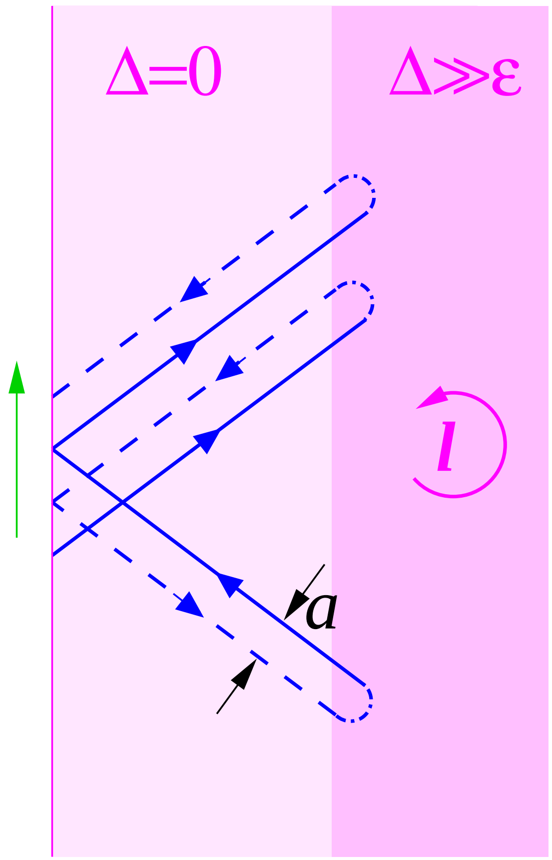

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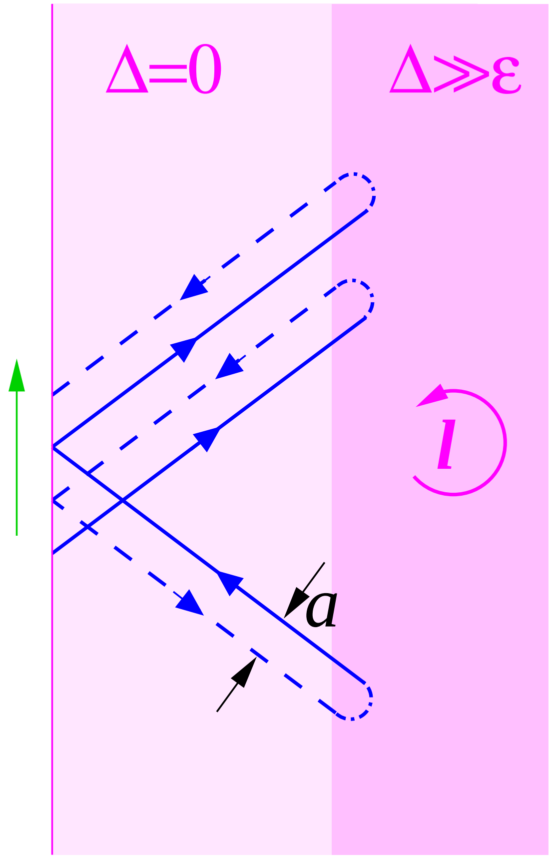
Majorana-Weyl edge mode



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- Why *Weyl*?
- Cooper pair has $l = \hbar$
- $\Rightarrow$ Andreev reflection offset $k_{\text{Fermi}} a = \hbar$.
- $\Rightarrow$ one-way edge creep

$$\varepsilon_k = ck$$

Why Majorana?

S-wave, S=0, superconductor

$$\hat{b}_{\uparrow,k} = \hat{a}_{\uparrow,k} + \hat{a}^{\dagger}_{\downarrow,-k}$$

$$\boxed{\hat{b}_{\uparrow,k} = \hat{b}^{\dagger}_{\downarrow,-k}}$$

distinct anti-particle $\Rightarrow$ not Majorana

Why Majorana?

P-wave, S=1, superconductor

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own anti-particle $\Rightarrow$ Majorana

Half-quantum Vortices

In ordinary vortex:

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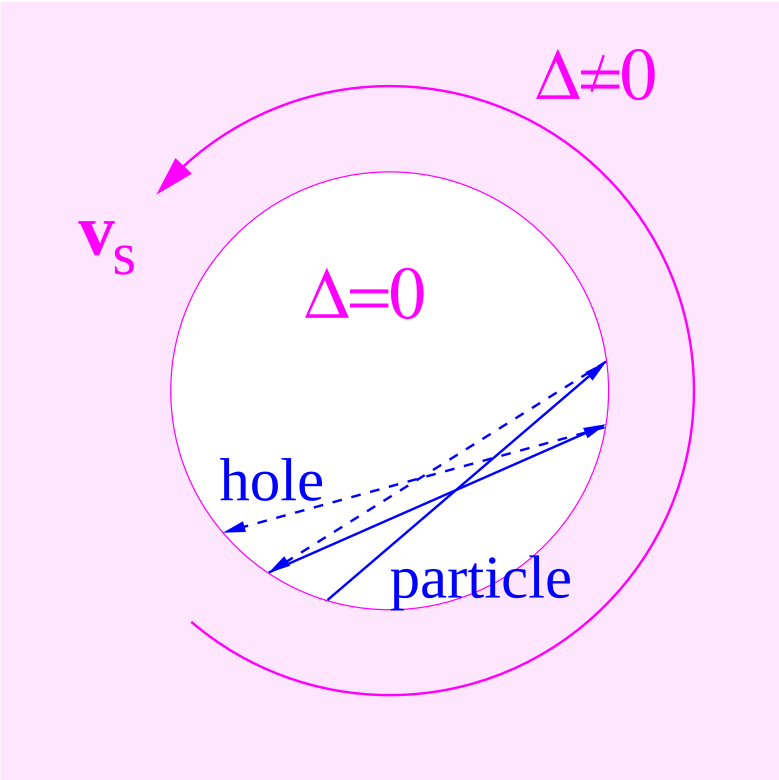
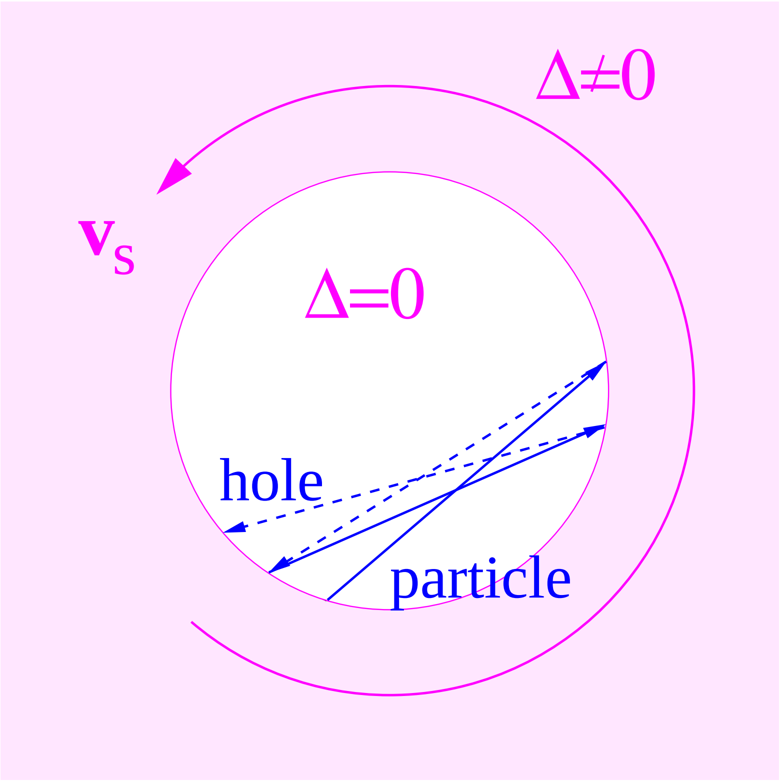
We can therefore treat fermions as polarized

vortex core states



Andreev bound state

vortex core states



Andreev bound state

- Andreev reflection not *quite* retro-reflective

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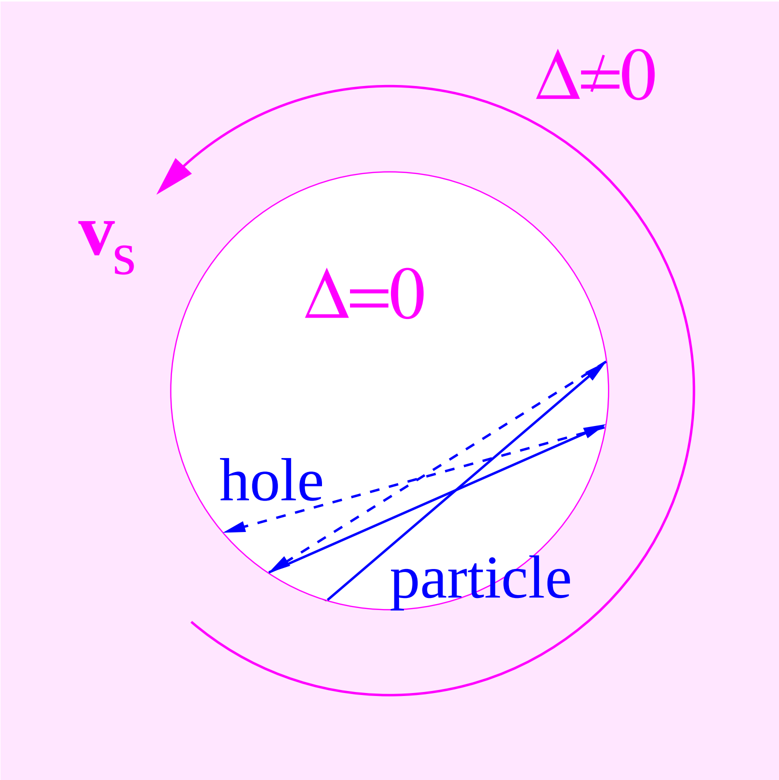

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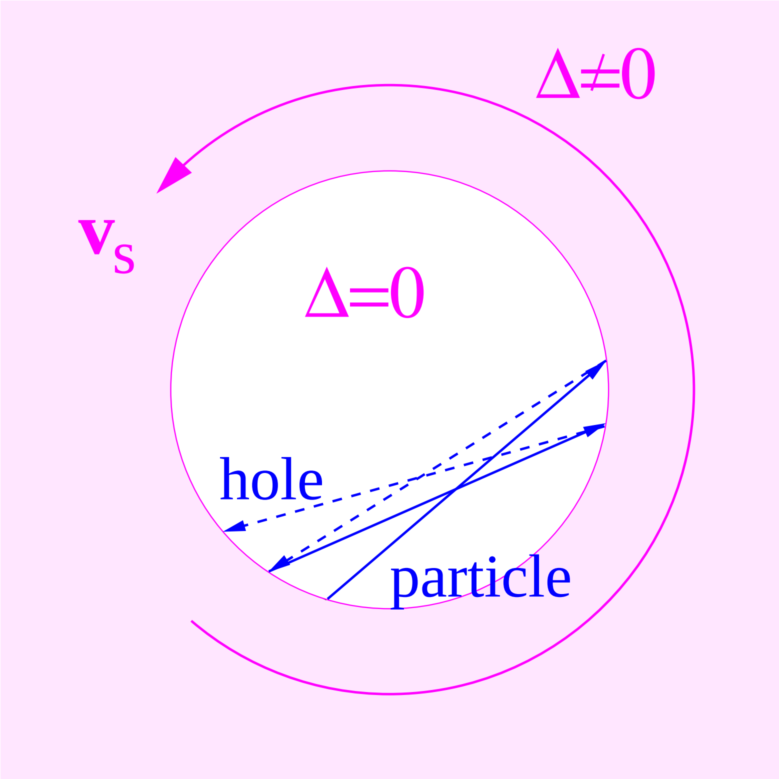

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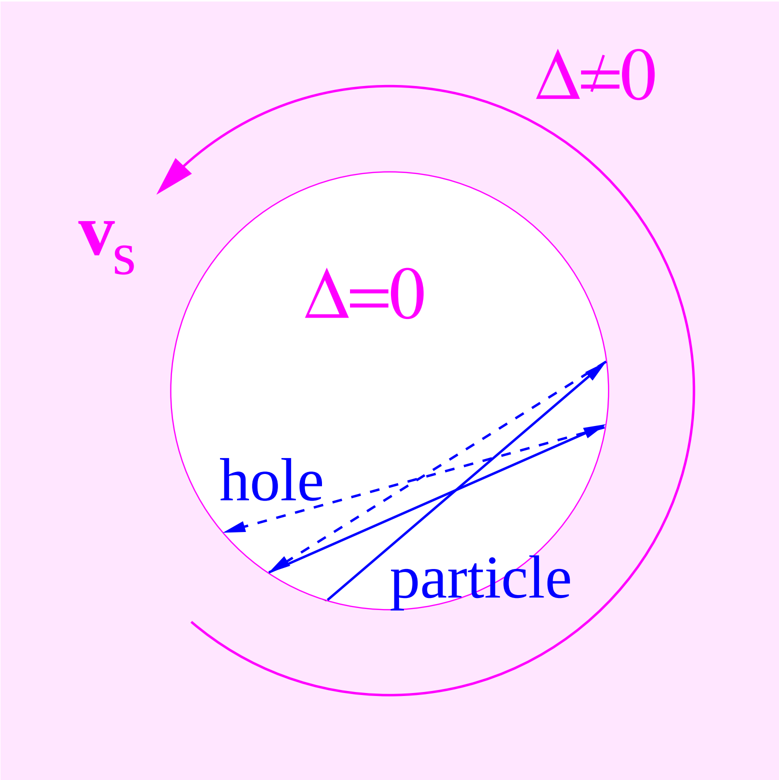

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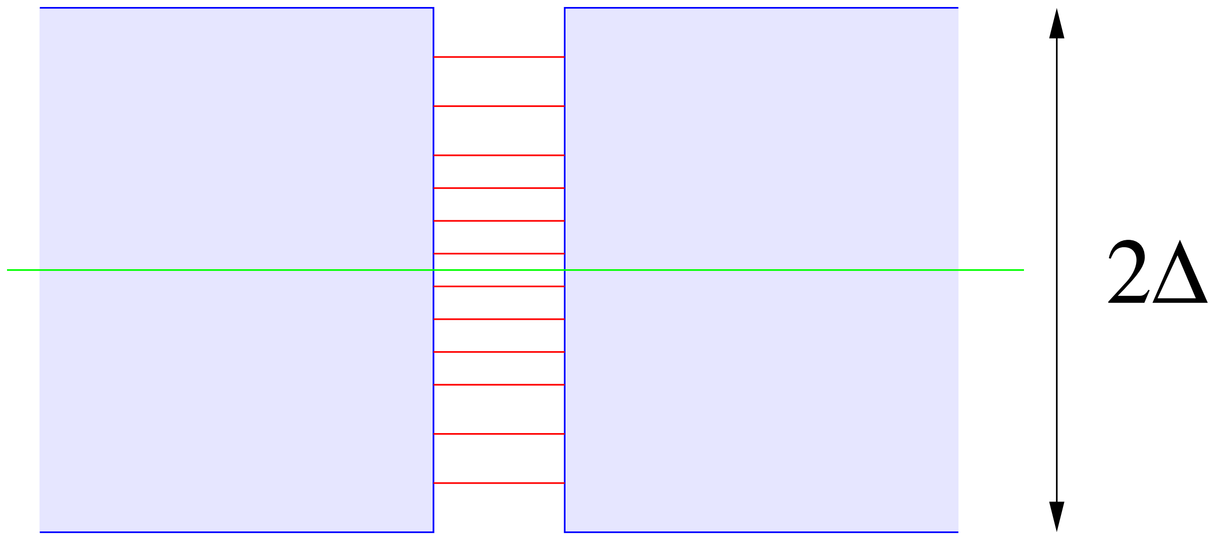
- α?

Core spectrum

Vortex-core bound-state spectrum always has $\boxed{\varepsilon \longrightarrow -\varepsilon}$
BdG symmetry $\Rightarrow \alpha = 0, \frac{1}{2}$.



S-wave bound states $\alpha = \frac{1}{2} \Rightarrow$ no zero mode

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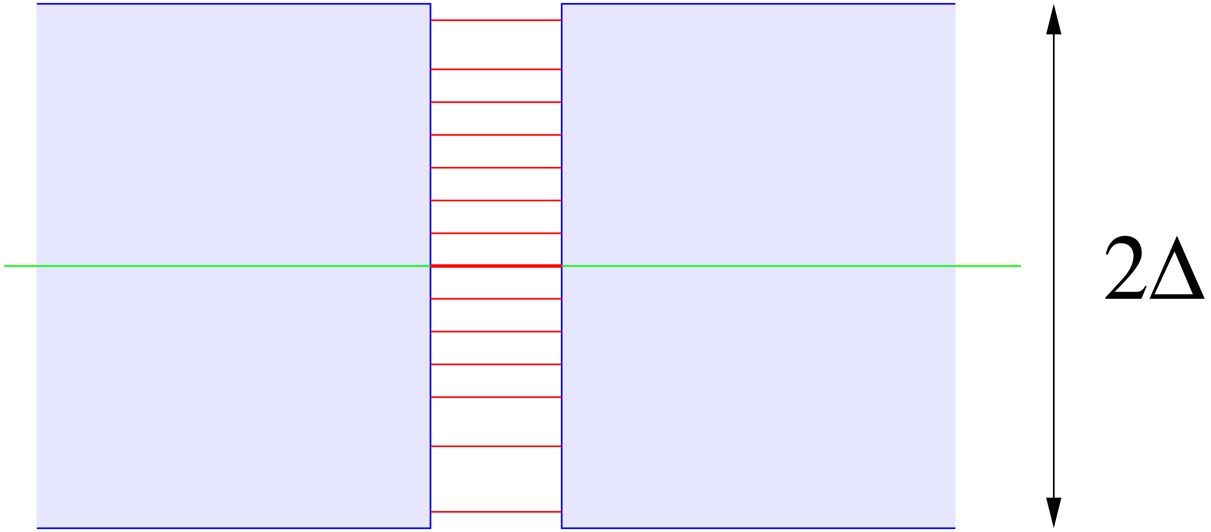
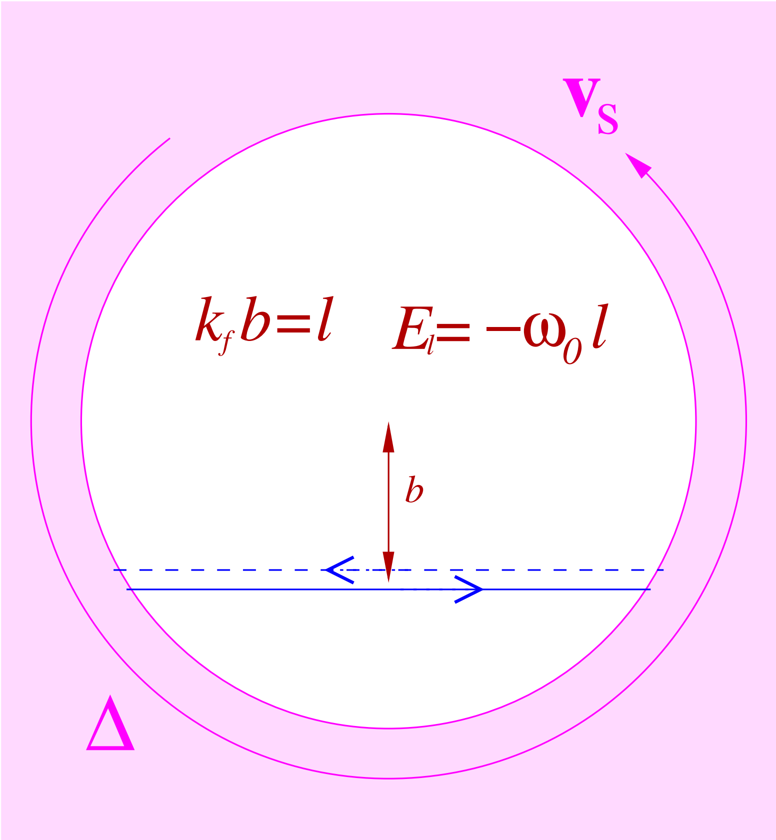


P-wave bound states $\alpha = 0 \Rightarrow$ exact zero mode

Core bound wavefunctions



- Electron is Andreev reflected:
  - electron $\rightarrow$ hole
  - hole $\rightarrow$ electron
- Energy is determined by phase difference at reflection points
- $E_l = -\omega_0 l$

Core bound states

$$\begin{pmatrix} \hat{\psi}(r,\theta) \\ \hat{\psi}^\dagger(r,\theta) \end{pmatrix} = \sum_l \hat{b}_l \begin{pmatrix} e^{i\theta} u_l(r) \\ e^{-i\theta} v_l(r) \end{pmatrix} e^{il\theta} + \text{higher-energy modes.}$$

$$\begin{pmatrix} u_{-l}(r) \\ v_{-l}(r) \end{pmatrix} = \begin{pmatrix} v_l^*(r) \\ u_l^*(r) \end{pmatrix}$$

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$$\begin{aligned} \hat{\psi} &\sim \hat{b}_l e^{i\theta} u_l + \hat{b}_{-l} e^{i\theta} v_l^* \\ \hat{\psi}^\dagger &\sim \hat{b}_l e^{-i\theta} v_l + \hat{b}_{-l} e^{i\theta} u_l^* \end{aligned}$$

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$$\boxed{\Rightarrow \hat{b}_0 = \hat{b}_0^\dagger}$$

Completeness

$$\delta^2(x-y) = \{\hat{\psi}(x), \hat{\psi}^\dagger(y)\}$$

$$\Rightarrow \hat{b}_0^2 = 1/2; \quad \{\hat{b}_0, \hat{b}_l\} = \{\hat{b}_0, \hat{b}_l^\dagger\} = 0, \;\; l \neq 0.$$

For i-th vortex set $\gamma_i = \sqrt{2}\hat{b}_0$

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- Clifford algebra
  $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$

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- pair up γ's:
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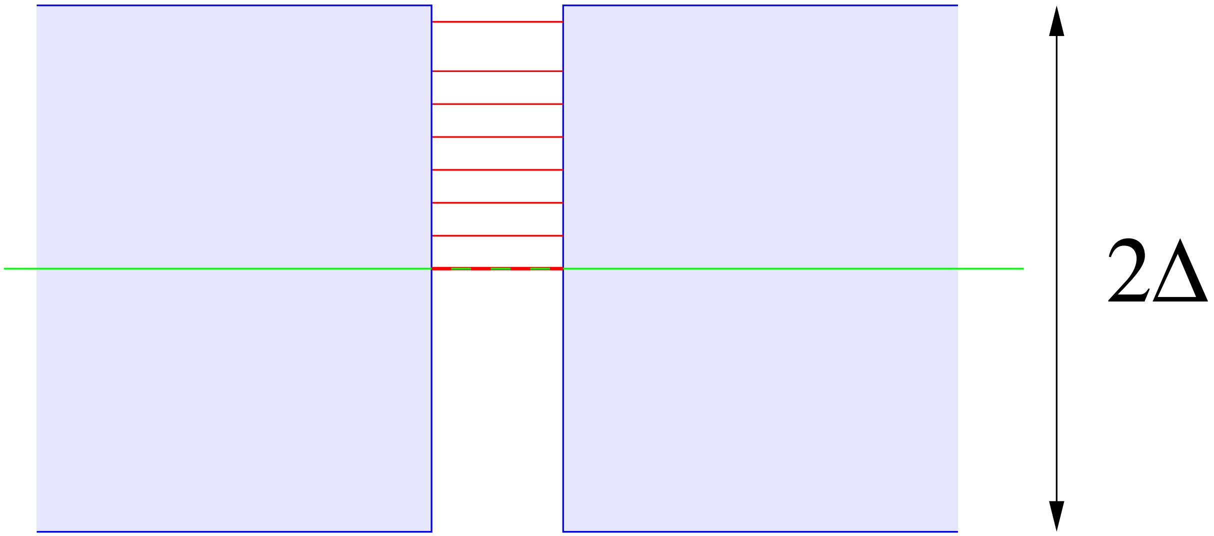
- $\Rightarrow 2^{N_\gamma/2}$ dimensional.

Core spectrum II

Only positive energy states:



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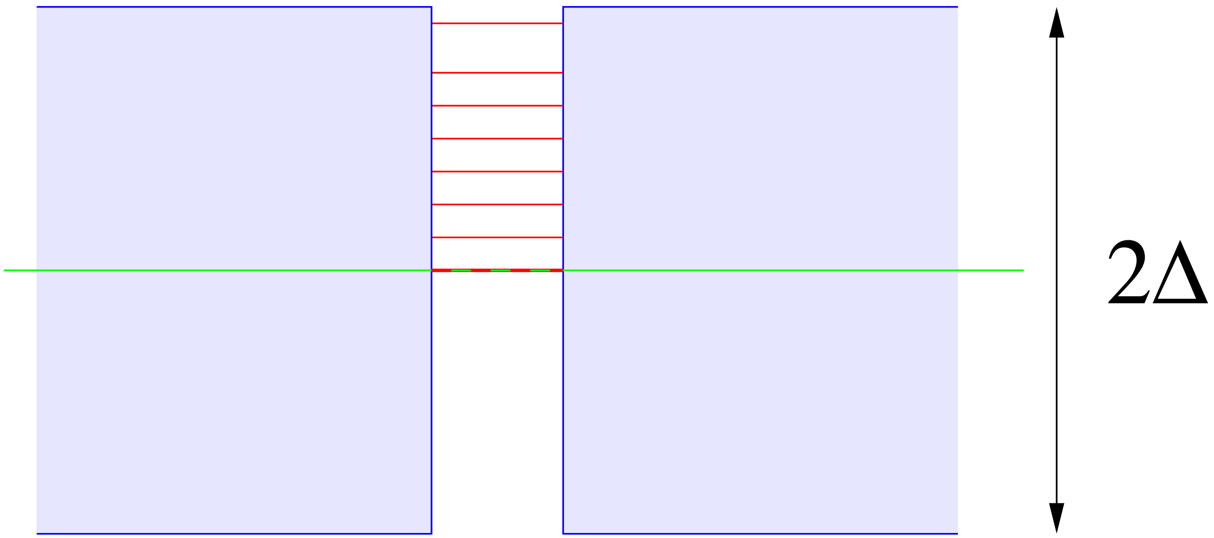

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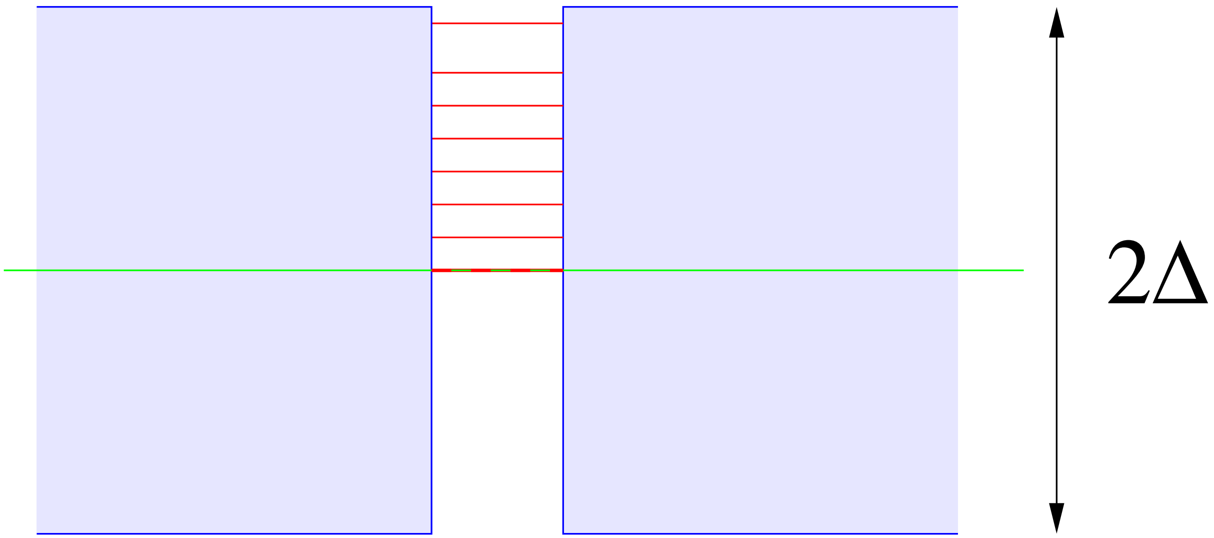


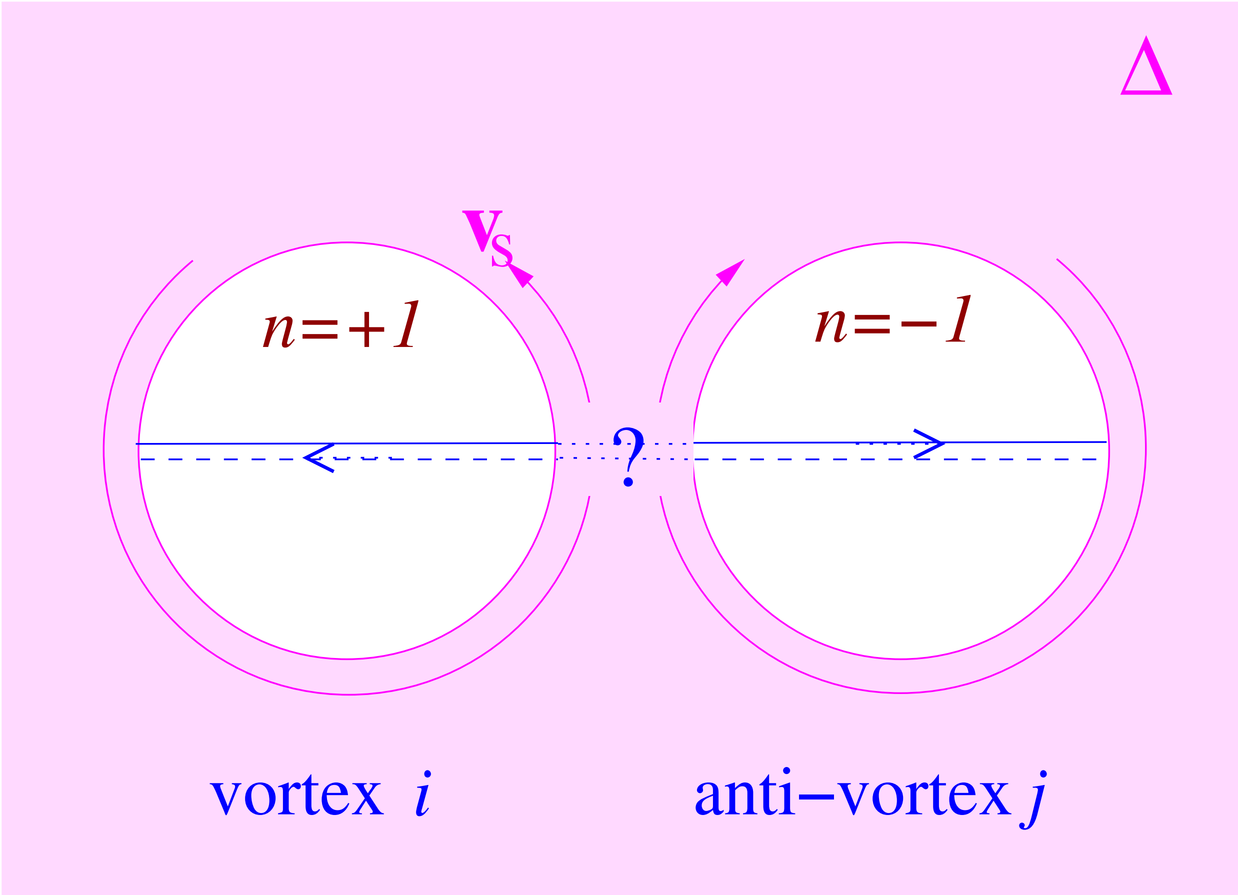
- Need two vortices to have one zero mode that can be occupied or not.
- Ground state degeneracy: $2^{N_{\text{vortex}}/2-1}$

Ising Fusion Rules

Seek to understand

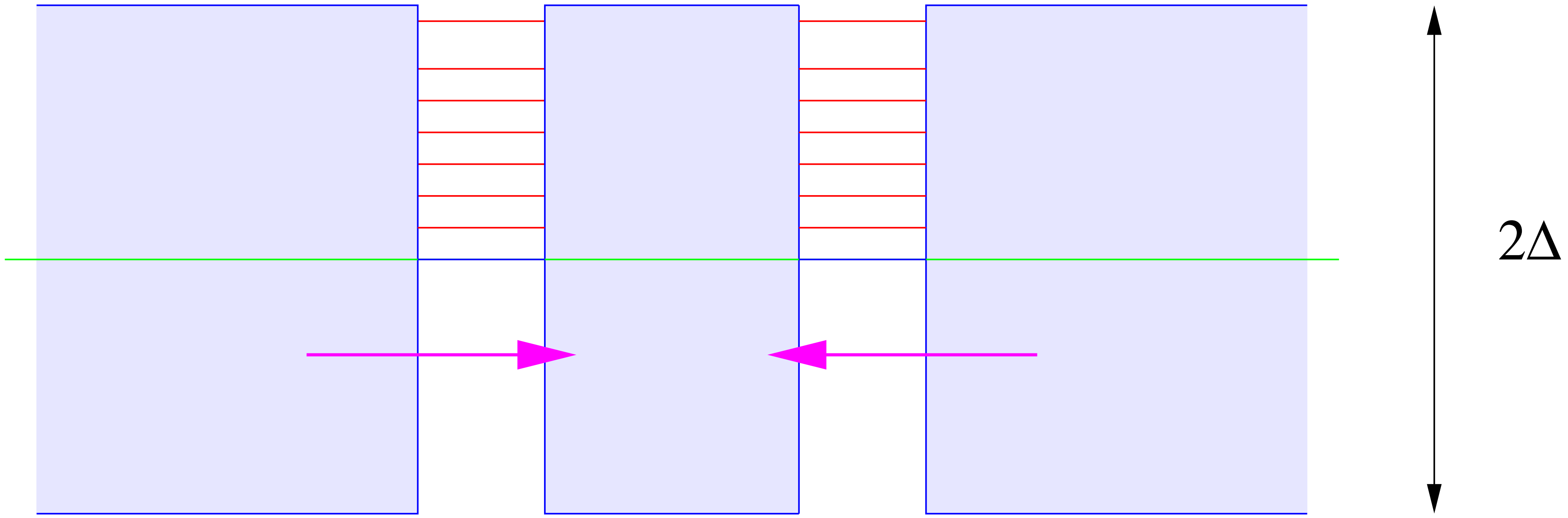
$$\psi \times \psi = \mathbb{I}, \quad \sigma \times \sigma = \mathbb{I} + \psi, \quad \psi \times \sigma = \sigma.$$

Tunnel Splitting



$$\hat{H}_{\text{tunnel}} = E_{\text{split}} \left(\frac{1}{4i}[\gamma_i, \gamma_j] + \frac{1}{2} \right), \quad E_{\text{split}} \sim e^{-C|\mathbf{r}_i - \mathbf{r}_j|}$$

Fusion



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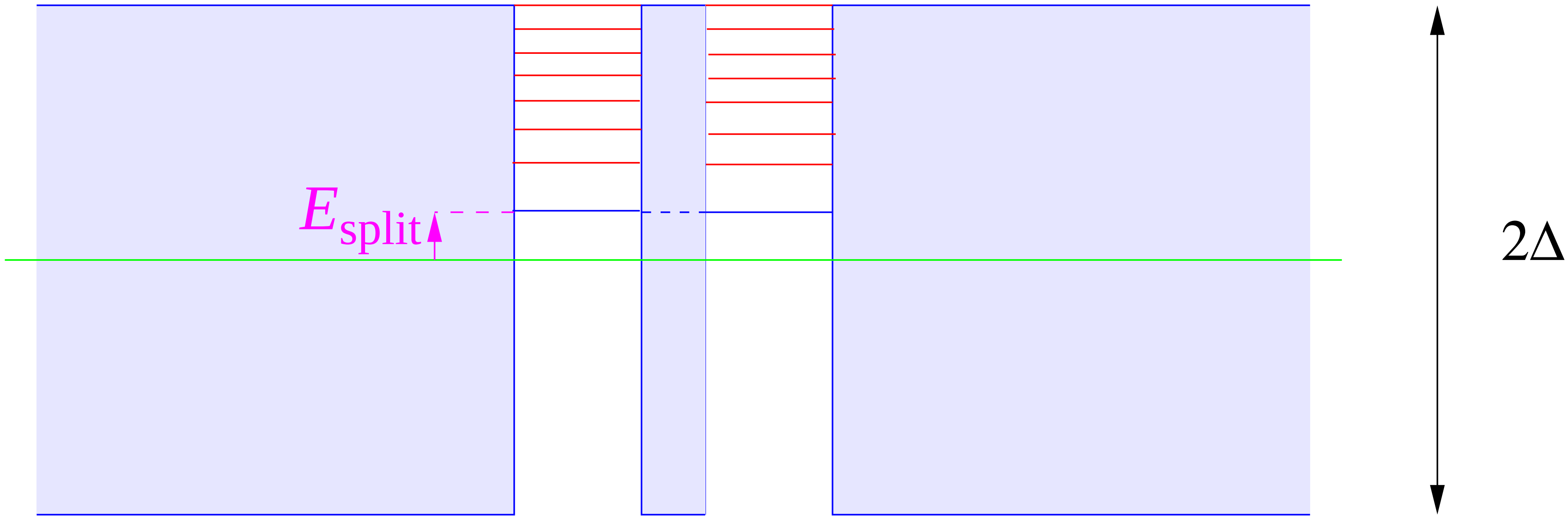

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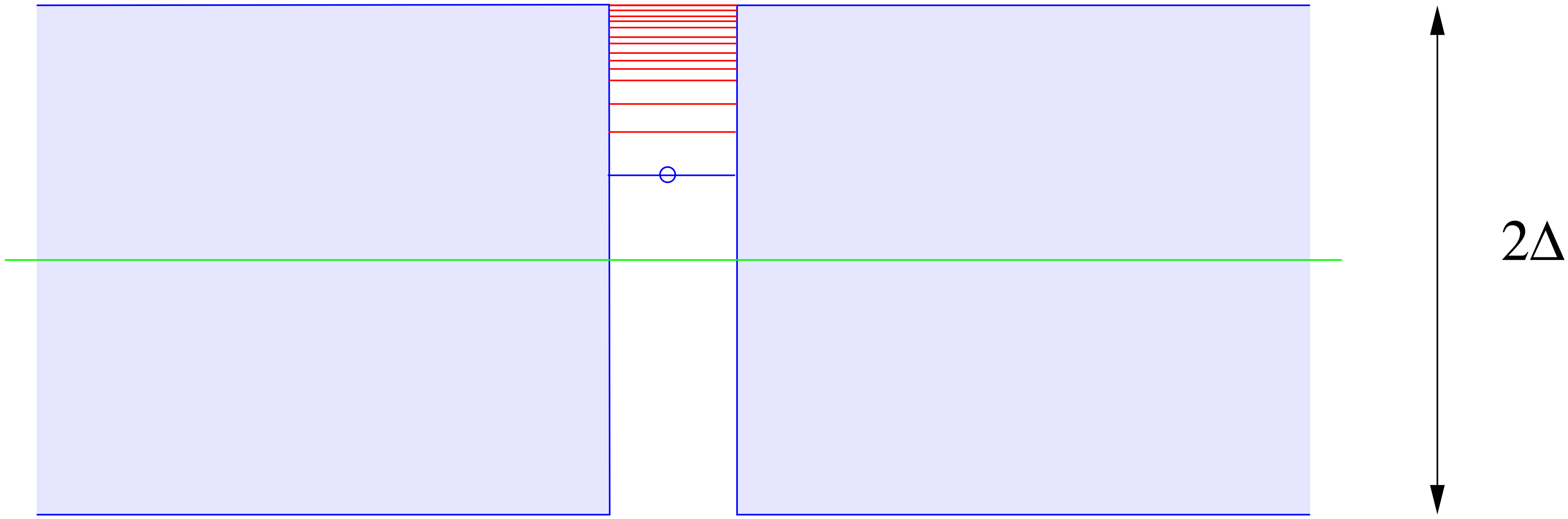

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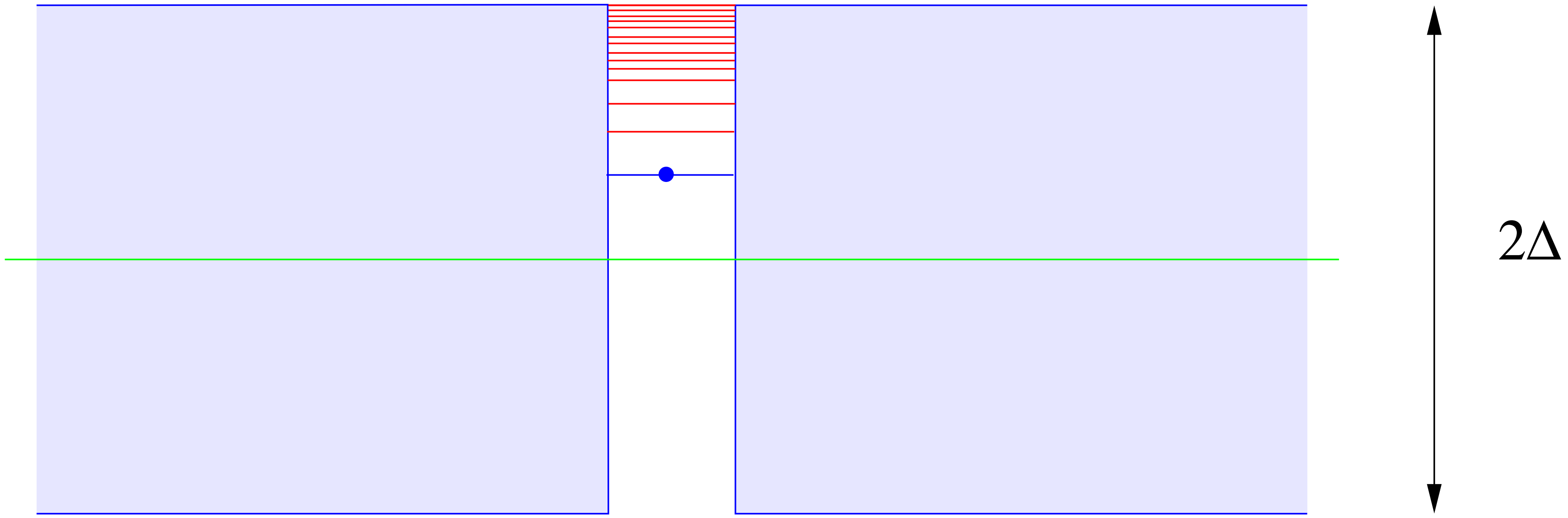

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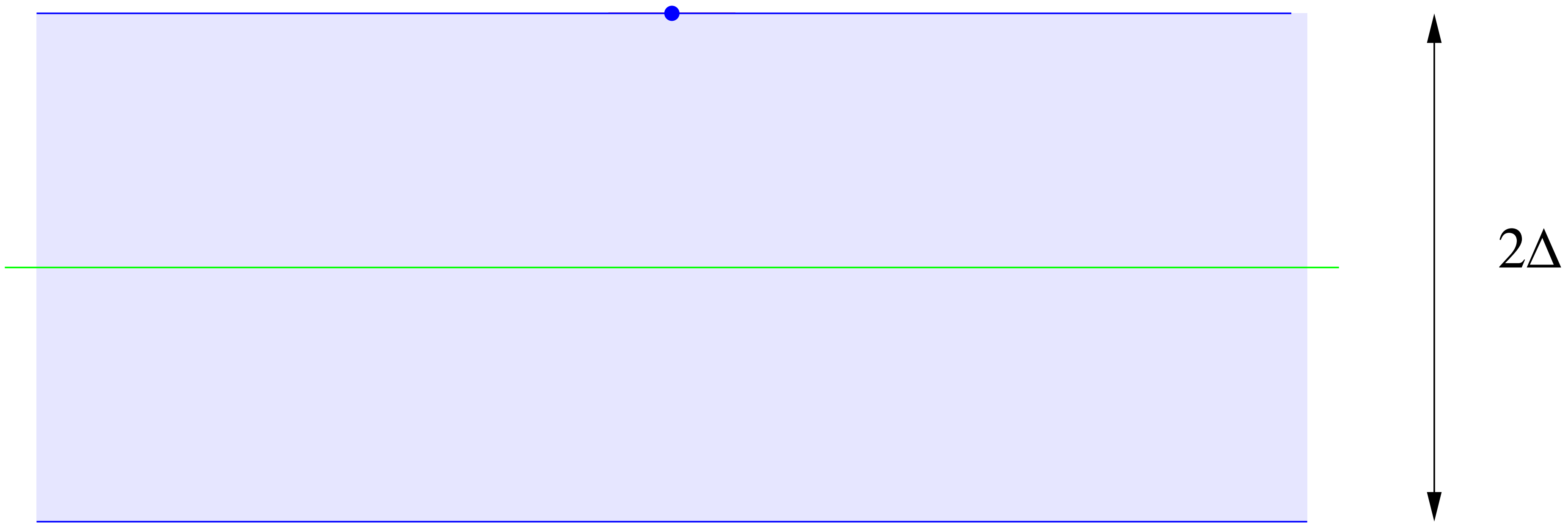
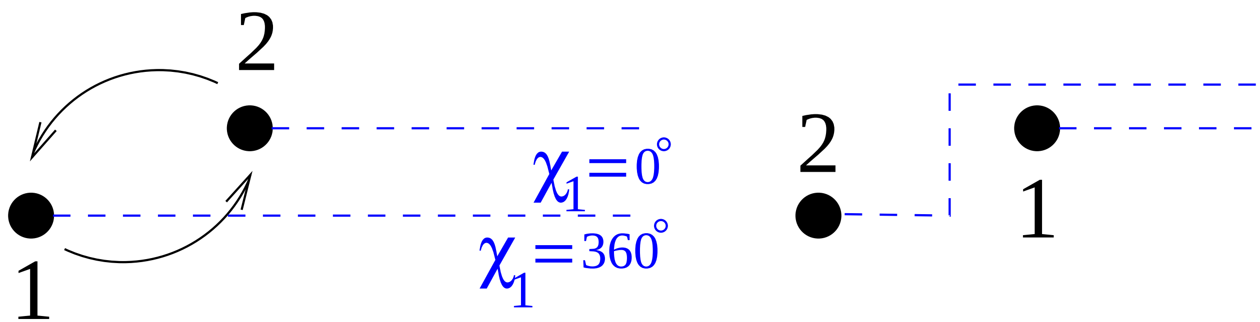


Monodromy

$$\Delta \to e^{i\chi}\Delta \Rightarrow \begin{pmatrix} u_l(r,\theta) \\ v_l(r,\theta) \end{pmatrix} \to \begin{pmatrix} e^{i\chi/2}u_l(r,\theta) \\ e^{-i\chi/2}v_l(r,\theta) \end{pmatrix}.$$



$$T_i : \begin{cases} \gamma_i \to \quad \gamma_{i+1} \\ \gamma_{i+1} \to -\gamma_i . \end{cases}$$ (Ivanov)

Braiding

Monodromy $\Rightarrow$ Braiding?

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Need to be sure that there is no additional Berry transport

Bogoliubov transformation

$$\begin{aligned} a_i &= u_{i\alpha} b_\alpha + v^*_{i\alpha} b^\dagger_\alpha \\ a^\dagger_i &= v_{i\alpha} b_\alpha + u^*_{i\alpha} b^\dagger_\alpha . \end{aligned}$$

$$\begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix} = E_\alpha \begin{pmatrix} u_\alpha \\ v_\alpha \end{pmatrix}, \quad E_\alpha \geq 0$$

$$\Rightarrow \begin{pmatrix} H & \Delta \\ \Delta^\dagger & -H^T \end{pmatrix} \begin{pmatrix} v^*_\alpha \\ u^*_\alpha \end{pmatrix} = -E_\alpha \begin{pmatrix} v^*_\alpha \\ u^*_\alpha \end{pmatrix} .$$

$\Rightarrow E \leftrightarrow -E$ symmetry

BCS Ground State

Ground state $b_i|0\rangle_b = 0$,

$$\Rightarrow \quad (a_i + a_k^\dagger v_{k\alpha}^* (u^{*-1})_{\alpha i})|0\rangle_b = 0, \quad i = 1, \ldots N.$$

$$S_{ij} = v_{i\alpha}^* (u^{*-1})_{\alpha j}, \quad S_{ij} = -S_{ji}$$

$$\exp\left\{\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} a_k \exp\left\{-\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} = a_k + a_i^\dagger S_{ik}.$$

$$\Rightarrow |0\rangle_b = \mathcal{N} \exp\left\{\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} |0\rangle_a$$

$2n$-particle Pfaffian wavefunction

Let

$$|S\rangle \stackrel{\text{def}}{=} \exp\left\{\frac{1}{2} a_i^\dagger a_j^\dagger S_{ij}\right\} |0\rangle_a,$$

and

$$|i_1, i_2, \ldots, i_{2n}\rangle = a_{i_{2n}}^\dagger a_{i_{2n-1}}^\dagger \ldots a_{i_1}^\dagger |0\rangle_a.$$

Then

$$\begin{aligned} \langle i_1, i_2, \ldots, i_{2n}|S\rangle &= \frac{1}{2^n n!} \epsilon_{i_1 i_2 \ldots i_{2n}}^{j_1 j_2 \ldots j_{2n}} S_{j_1 j_2} \cdots S_{j_{2n-1} j_{2n}} \\ &= \epsilon_{i_1 i_2 \ldots i_{2n}} \mathrm{Pf}(S). \end{aligned}$$

Ground-State Berry Phase

$$\ln \mathcal{N} = -\frac{1}{4} \ln \det (I + S^\dagger S), \quad \text{(Kähler Potential)}$$

$$\begin{aligned} iA_{\text{Berry}} &= (\langle S | \mathcal{N}) d (\mathcal{N} | S \rangle) \\ &= \sum_{i<j} \left(\frac{\partial \ln \mathcal{N}}{\partial S^*_{ij}} dS^*_{ij} - \frac{\partial \ln \mathcal{N}}{\partial S_{ij}} dS_{ij} \right) \\ &= \frac{1}{2} \sum_{\alpha=1}^{N} \left(v_\alpha \quad u_\alpha \right) d \begin{pmatrix} v^*_\alpha \\ u^*_\alpha \end{pmatrix} + \frac{i}{2} d \left\{ \text{Arg} \left(\det u \right) \right\} \end{aligned}$$

(Remember that it is the (v^*, u^*) that have negative energy, and so are "occupied" in ground state.)

Other-State Berry Phase

$$|\alpha_1,\ldots,\alpha_n\rangle = b^\dagger_{\alpha_1}\cdots b^\dagger_{\alpha_n}|\tilde{S}\rangle,$$

Use

$$\begin{aligned} db^\dagger_\alpha &= (u_{i\beta}b_\beta + v^*_{i\beta}b^\dagger_\beta)\,dv_{i\alpha} + (v_{i\beta}b_\beta + u^*_{i\beta}b^\dagger_\beta)\,du_{i\alpha} \\ &= (v^*_{i\beta}dv_{i\alpha} + u^*_{i\beta}du_{i\alpha})\,b^\dagger_\beta + (u_{i\beta}dv_{i\alpha} + v_{i\beta}du_{i\alpha})\,b_\beta. \end{aligned}$$

To find, for example,

$$\begin{aligned} iA &= \langle\alpha_1,\ldots,\alpha_n|d|\alpha_1,\ldots,\alpha_n\rangle \\ &= \langle\tilde{S}|d|\tilde{S}\rangle + \langle\tilde{S}|b_{\alpha_n}\cdots b_{\alpha_1}d\left(b^\dagger_{\alpha_1}\cdots b^\dagger_{\alpha_n}\right)|\tilde{S}\rangle \\ &= \langle\tilde{S}|d|\tilde{S}\rangle + \sum_{m=1}^{n}\begin{pmatrix} u^*_{\alpha_m} & v^*_{\alpha_m}\end{pmatrix} d\begin{pmatrix} u_{\alpha_m} \\ v_{\alpha_m}\end{pmatrix}. \end{aligned}$$

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- $p_x + ip_y$ provides a paradigm for Non-Abelian statistics.
- Physical understanding of fusion rules, *e.g.* $\sigma \times \sigma = \mathbb{I} + \psi$.
- Easy-to-see Monodromy.
- Can verify no additional non-Abelian Berry transport.

References

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