Braid Topologies for Quantum Computation

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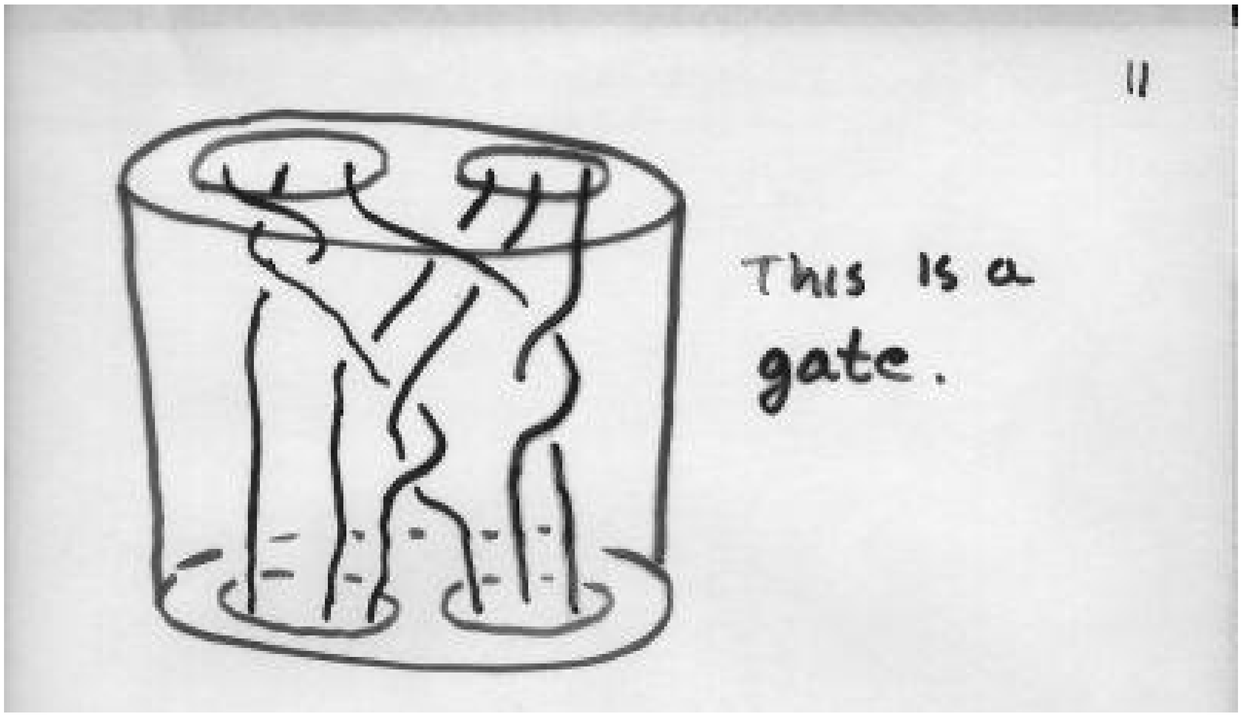
Steven H. Simon

Lucent Technologies

NEB, L. Hormozi, G. Zikos, S.H. Simon, Phys. Rev. Lett. 95 140503 (2005)

Support: US DOE

Inspiration



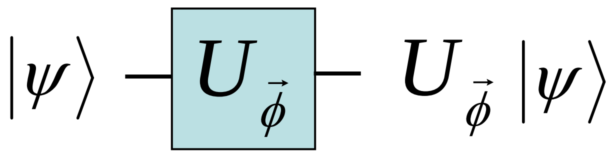
From "A topological modular functor which is universal for quantum computation"

Talk given by Michael Freedman at "Mathematics of Quantum Computation", MSRI, Feb. 2000 (available online).

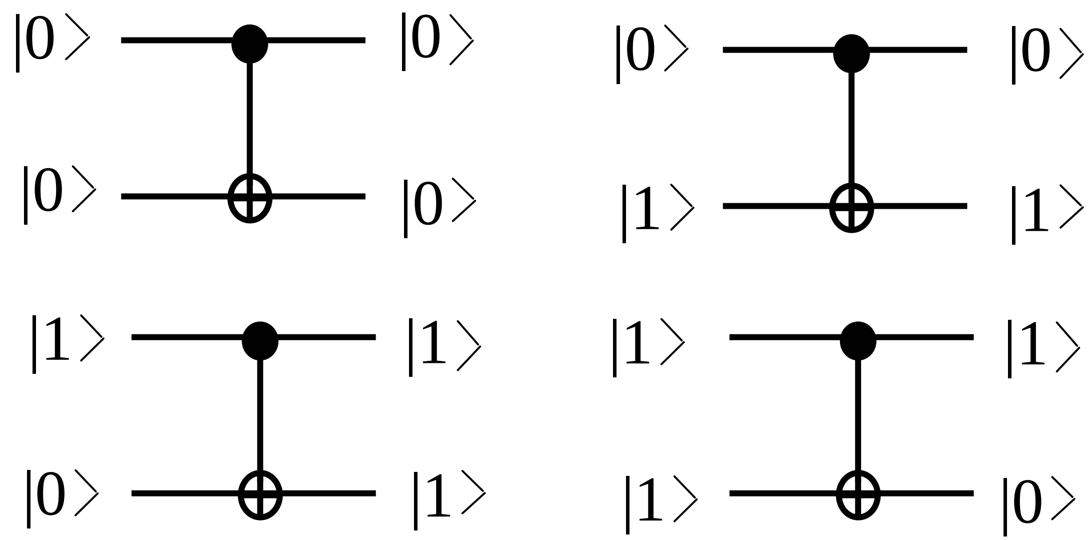
http://www.msri.org/communications/ln/msri/2000/qcomputing/freedman/1/index.html

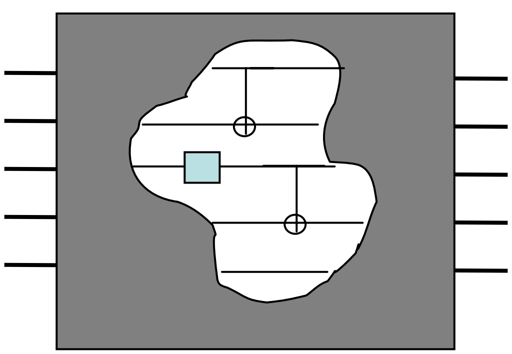
Universal Quantum Gates

Single Qubit Rotation



Controlled Not



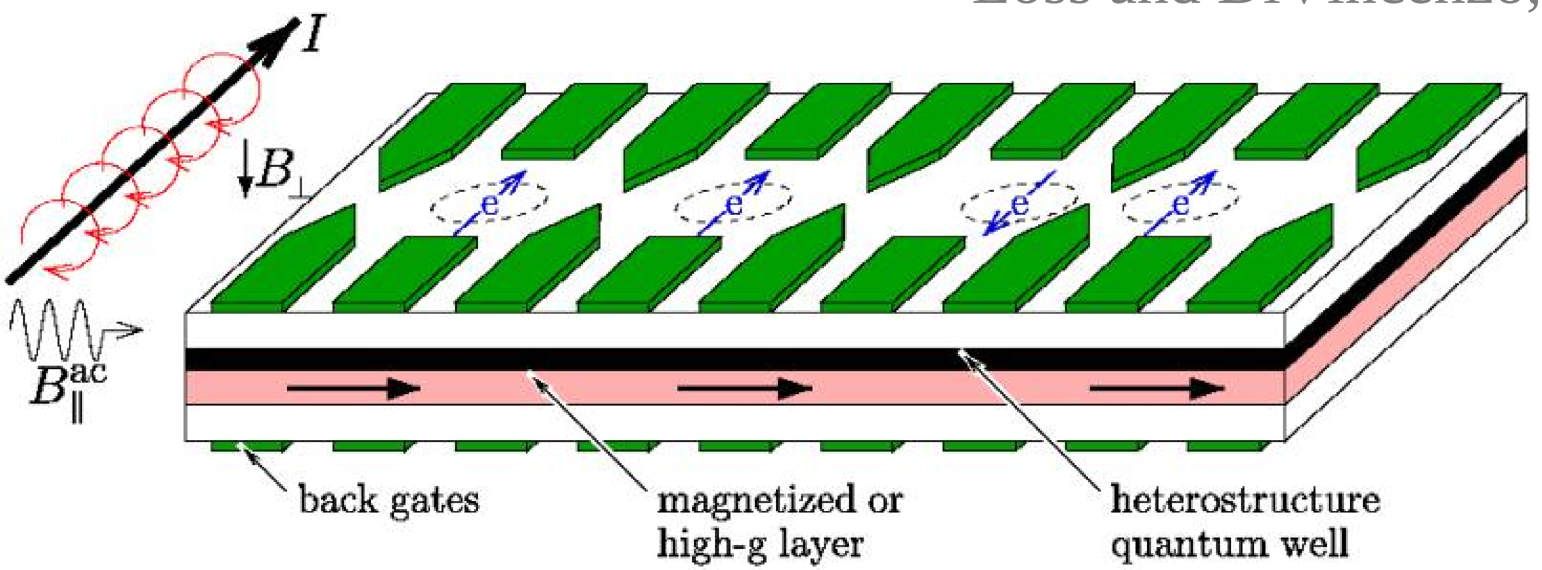


Any N qubit operation can be carried out using these two gates.

$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

One way to go… $|0\rangle = \uparrow$ $|1\rangle = \downarrow$

Loss and DiVincenzo, '98



Manipulate electron spins with electric and magnetic fields to carry out quantum gates.

Problem: Errors and Decoherence! May be solvable, but it won't be easy!

Another way to go…

Fault-tolerant quantum computation by anyons

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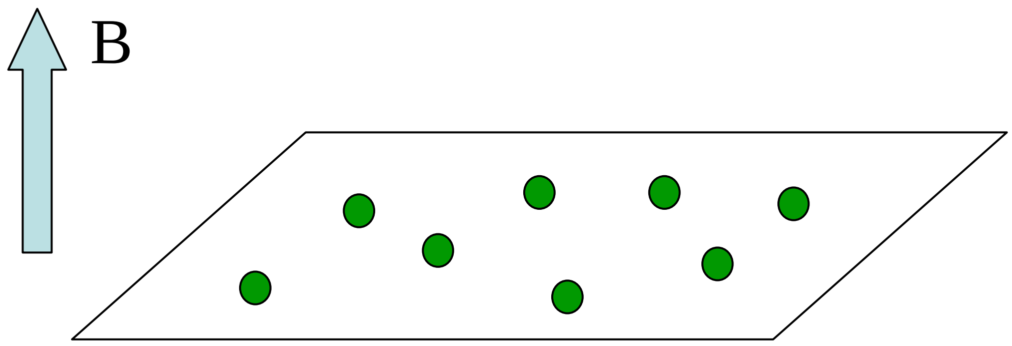
A Modular Functor Which is Universal for Quantum Computation

Michael H. Freedman[1], **Michael Larsen**[2], **Zhenghan Wang**[2]

[1] Microsoft Research, One Microsoft Way, Redmond, WA 98052-6399, USA
[2] Indiana University, Dept. of Math., Bloomington, IN 47405, USA

Fractional Quantum Hall (FQH) States



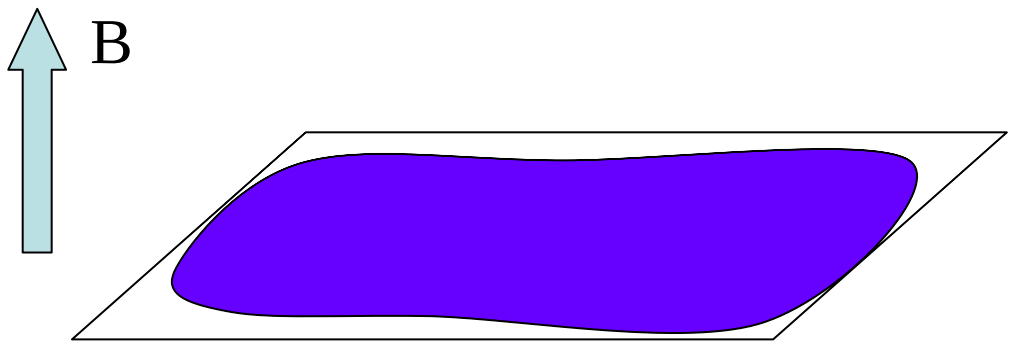
Occurs when a **two-dimensional electron gas** is placed in a magnetic field

An **incompressible quantum liquid** can form when the Landau level filling fraction $\nu = n_{elec}(hc/eB)$ is a rational fraction.

Quasiparticle excitations can have **fractional charge**.

Great stuff, but what does this have to do with quantum computing?

Fractional Quantum Hall (FQH) States



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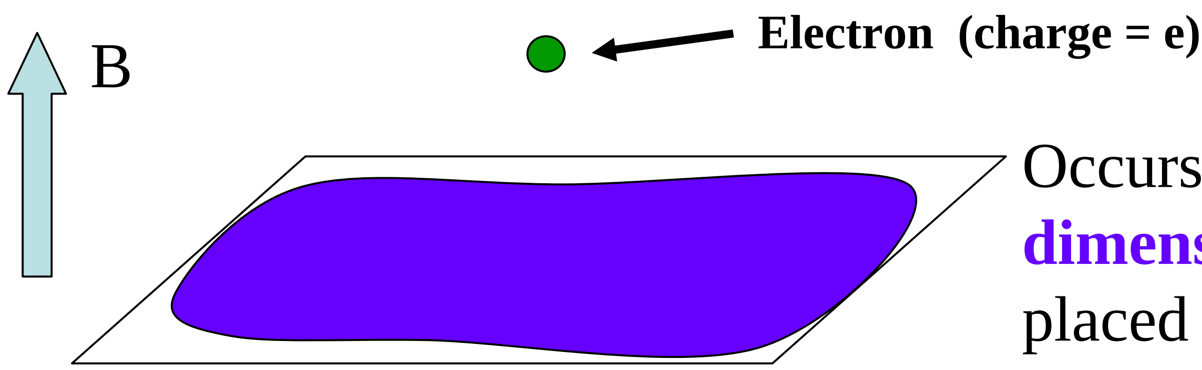

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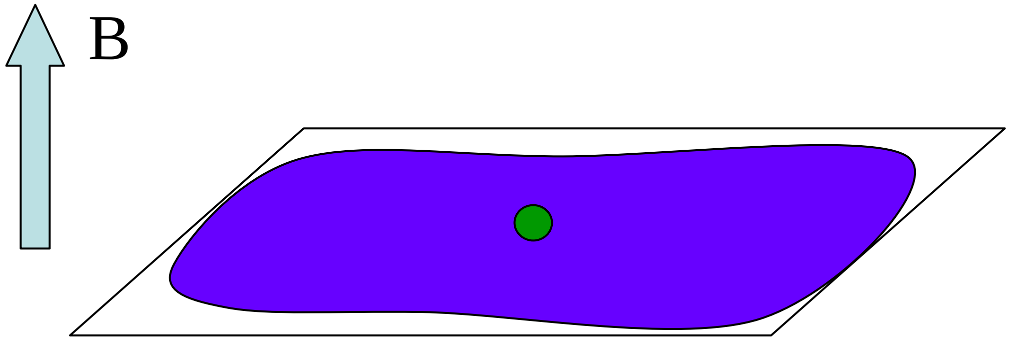

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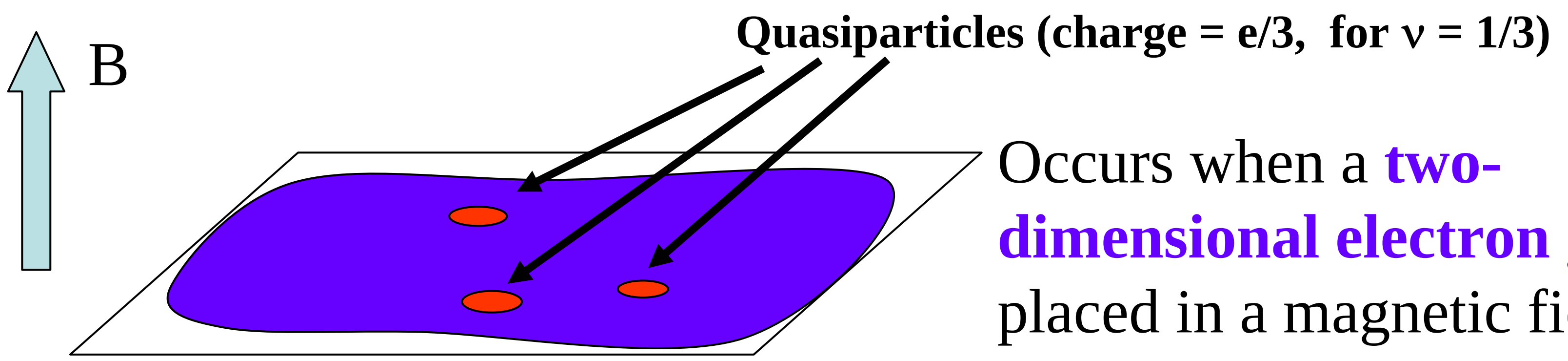

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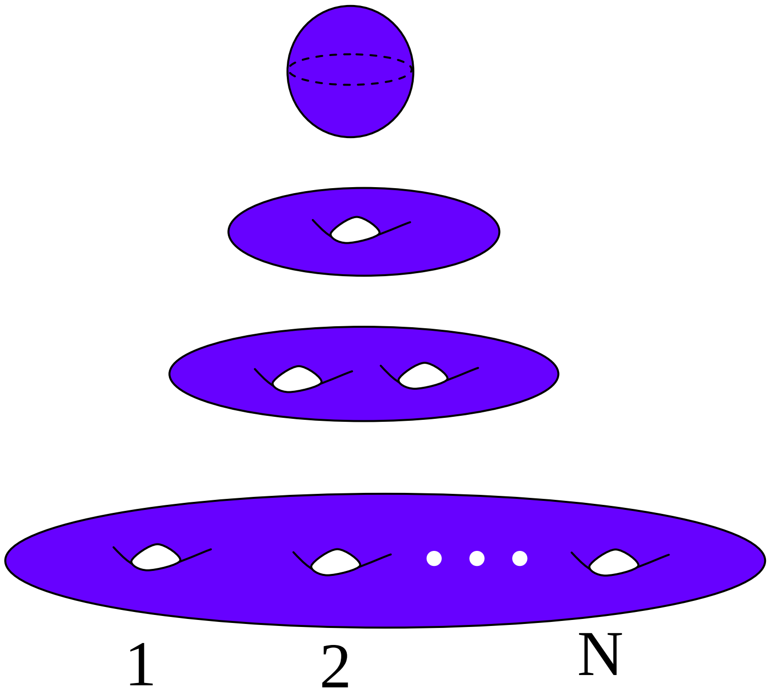
Topological Degeneracy (X.G. Wen)

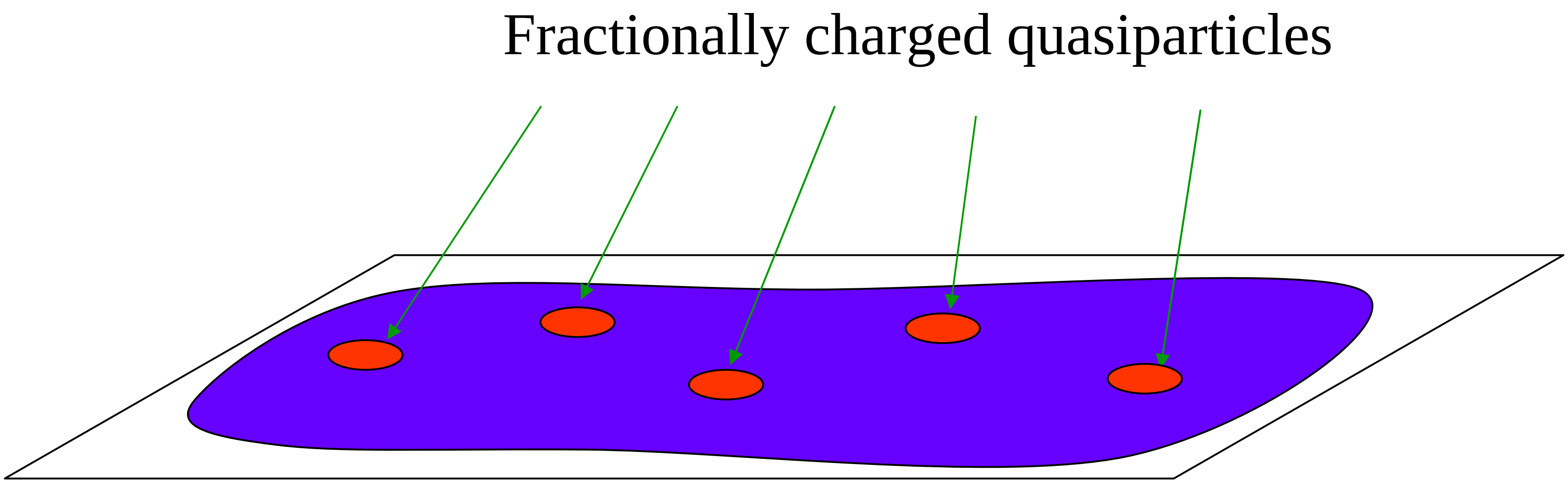
A theoretical curiosity: FQH states on **topologically nontrivial surfaces** have degenerate ground states which **can only be distinguished by global measurements**.

For the ν **= 1/3** state:



Degeneracy
1
3
9
3^N

Non-Abelian FQH States (Moore, Read '91)



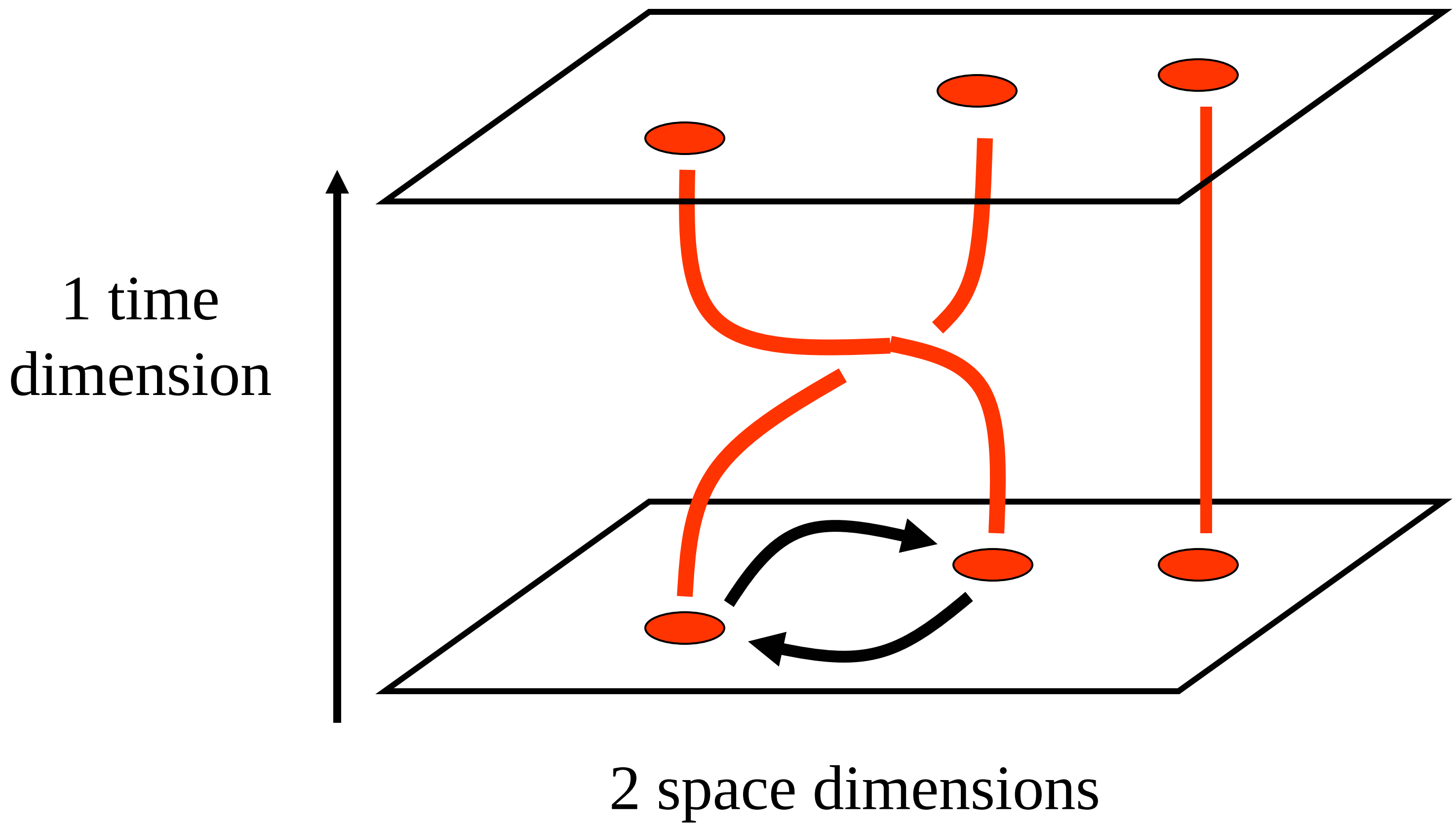
Essential features:

A degenerate Hilbert space whose dimensionality is **exponentially large in the number of quasiparticles**.

States in this space **can only be distinguished by global measurements** provided quasiparticles are far apart.

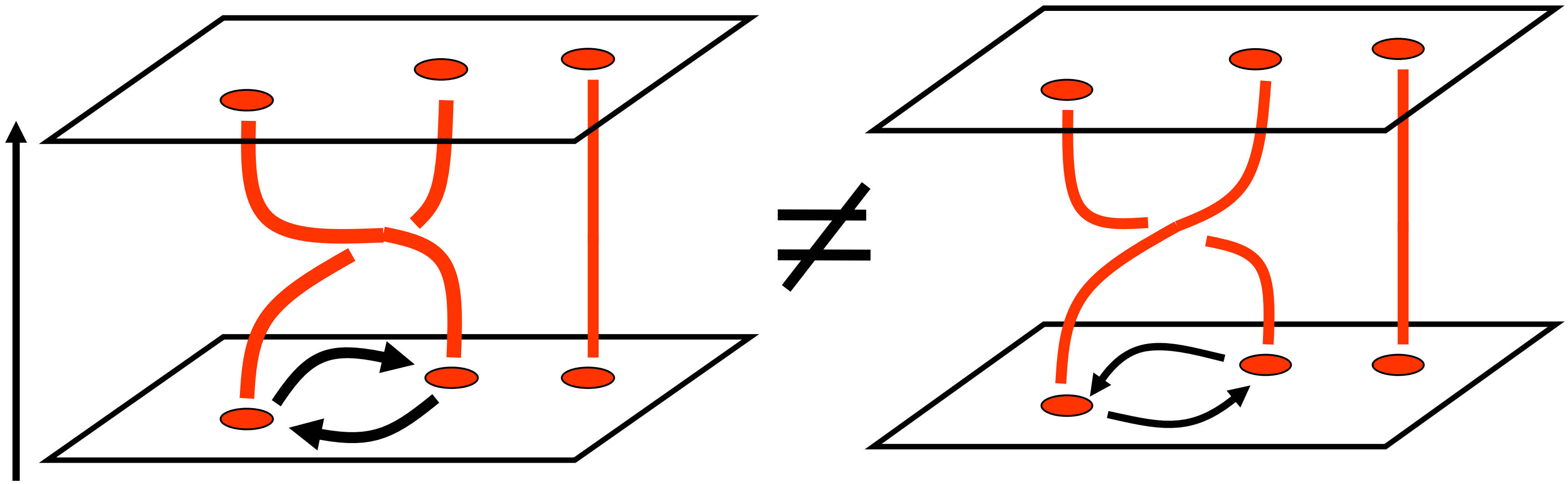
→ ***A perfect place to hide quantum information!***

Exchanging Particles in 2+1 Dimensions



Particle “world-lines” form **braids** in 2+1 (=3) dimensions

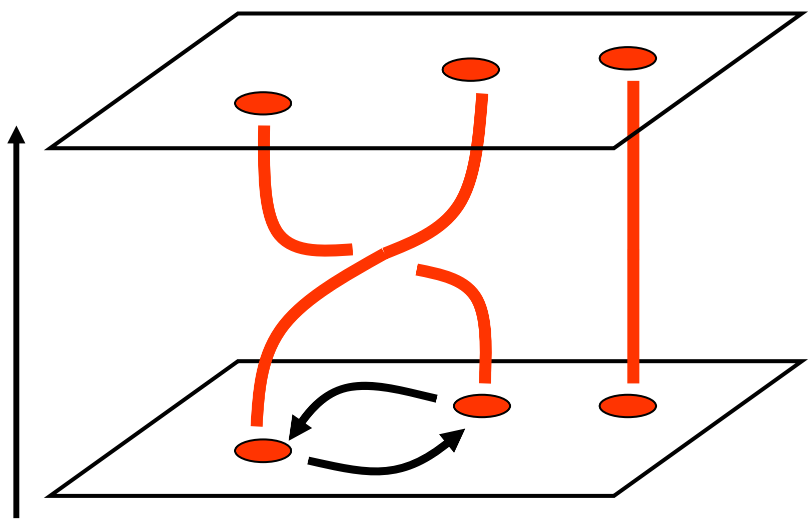
Exchanging Particles in 2+1 Dimensions



Clockwise exchange ≠ **Counterclockwise exchange**

Particle "world-lines" form **braids** in 2+1 (=3) dimensions

Fractional (Abelian) Statistics



$$|\psi_f\rangle = e^{i\vartheta} |\psi_i\rangle$$

Phase

$$|\psi_i\rangle$$

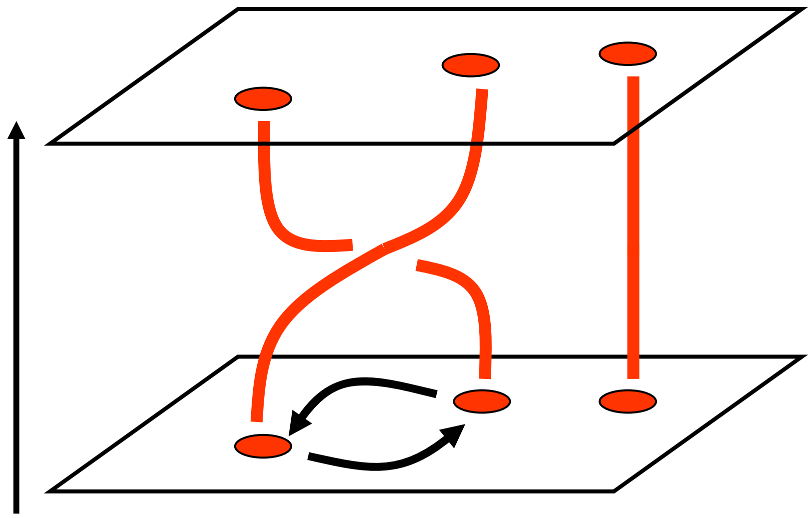
$\theta = 0$ Bosons

$\theta = \pi$ Fermions

$\theta = \pi/3$ ν=1/3 quasiparticles

Only possible for particles in 2 space dimensions.

Non-Abelian Statistics (Moore, Read '91)

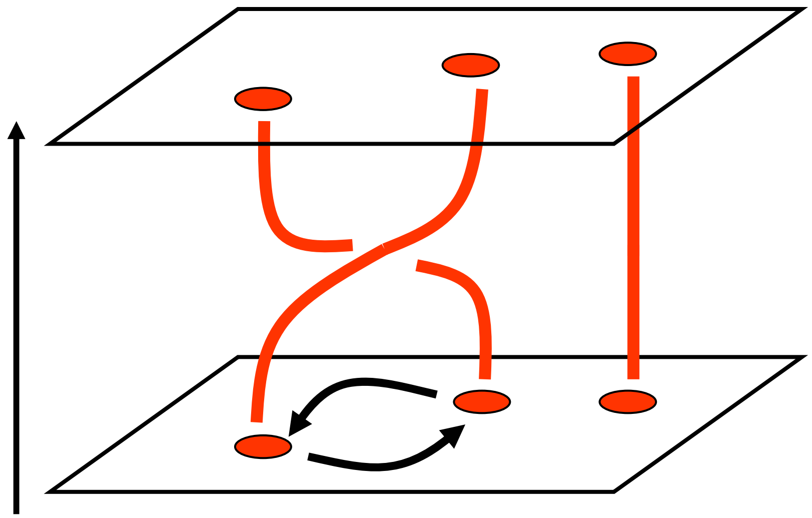


$$|\psi_f\rangle = \tilde{\alpha}|\psi_0\rangle + \tilde{\beta}|\psi_1\rangle$$

$$|\psi_i\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle$$

degenerate states

Non-Abelian Statistics (Moore, Read '91)



$$|\psi_f\rangle = \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

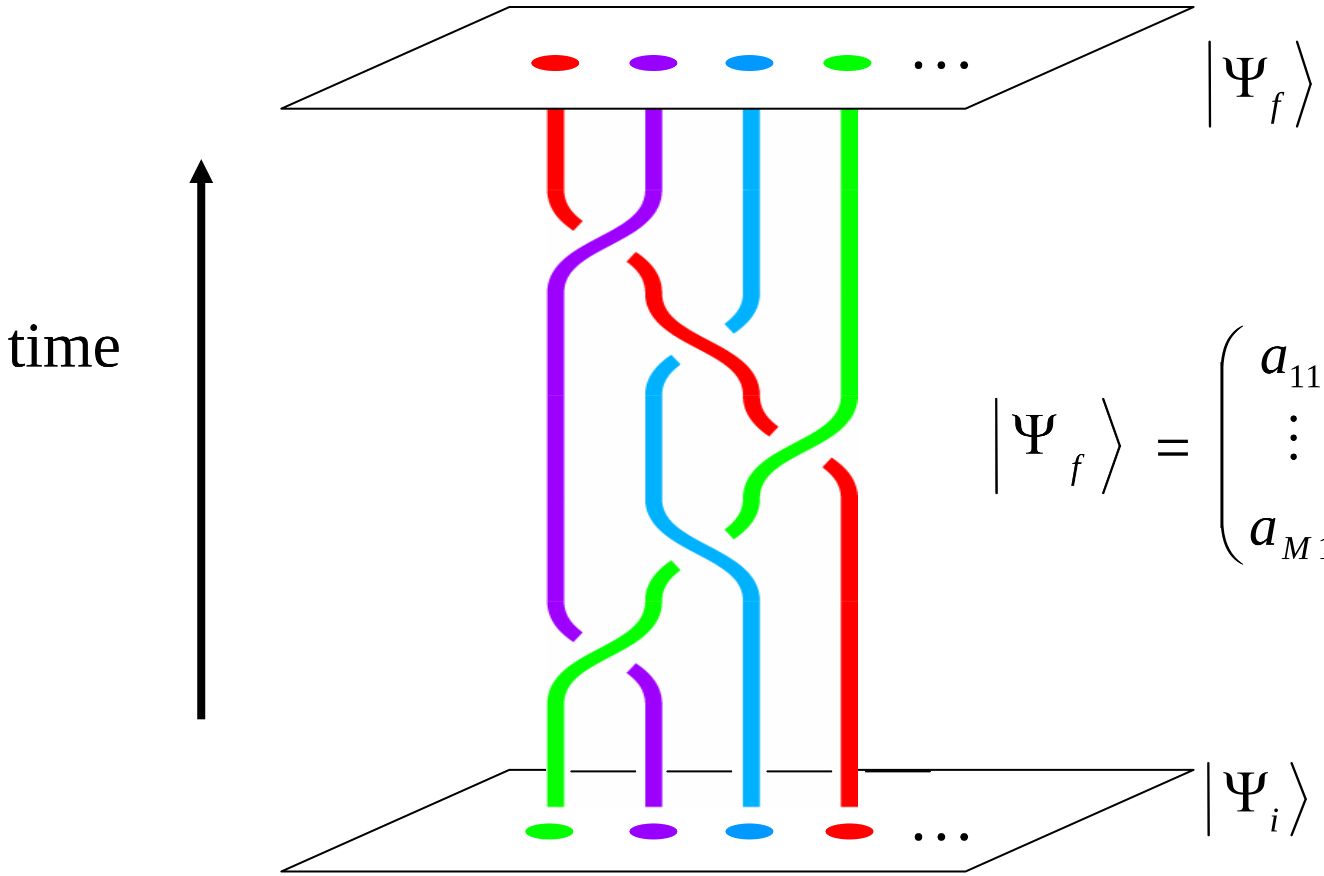
$$|\psi_i\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Matrix!

Matrices form a **non-Abelian** representation of the **braid group**.

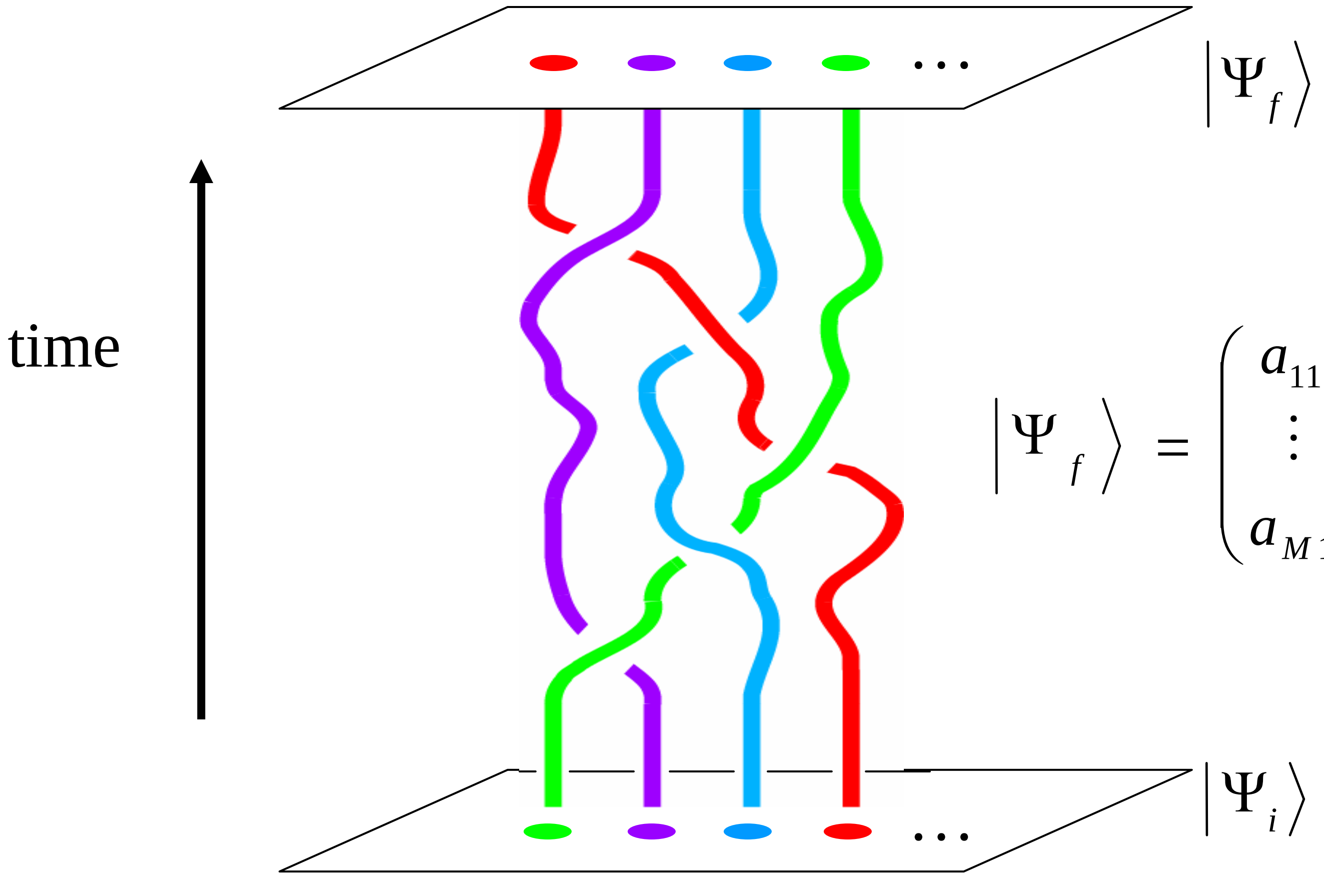
(Related to the Jones Polynomial, TQFT (Witten), Conformal Field Theory (Moore, Seiberg), etc.)

Many Non-Abelian Anyons



$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Many Non-Abelian Anyons

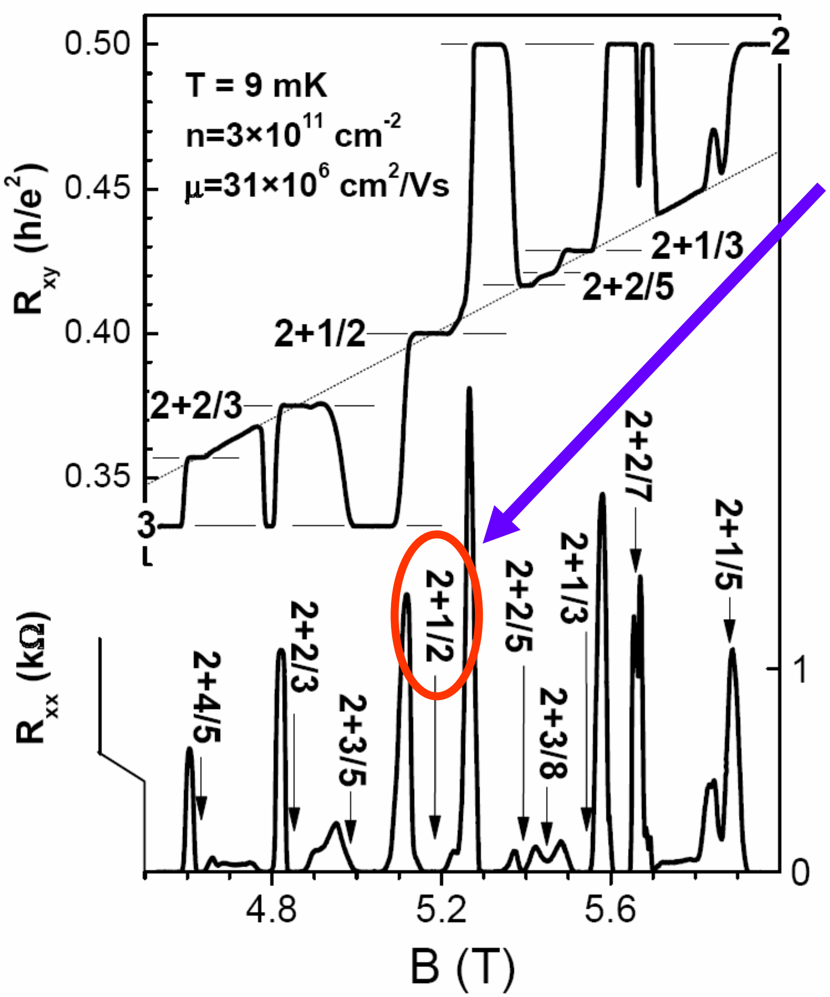


$$|\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle$$

Matrix depends only on the topology of the braid swept out by anyon world lines!

Robust quantum computation?

Possible Non-Abelian FQH States



J.S. Xia et al., PRL (2004).

$\nu = 5/2$

Very likely a Moore-Read "Pfaffian" state.

Moore and Read, 1991
Morf, 1998

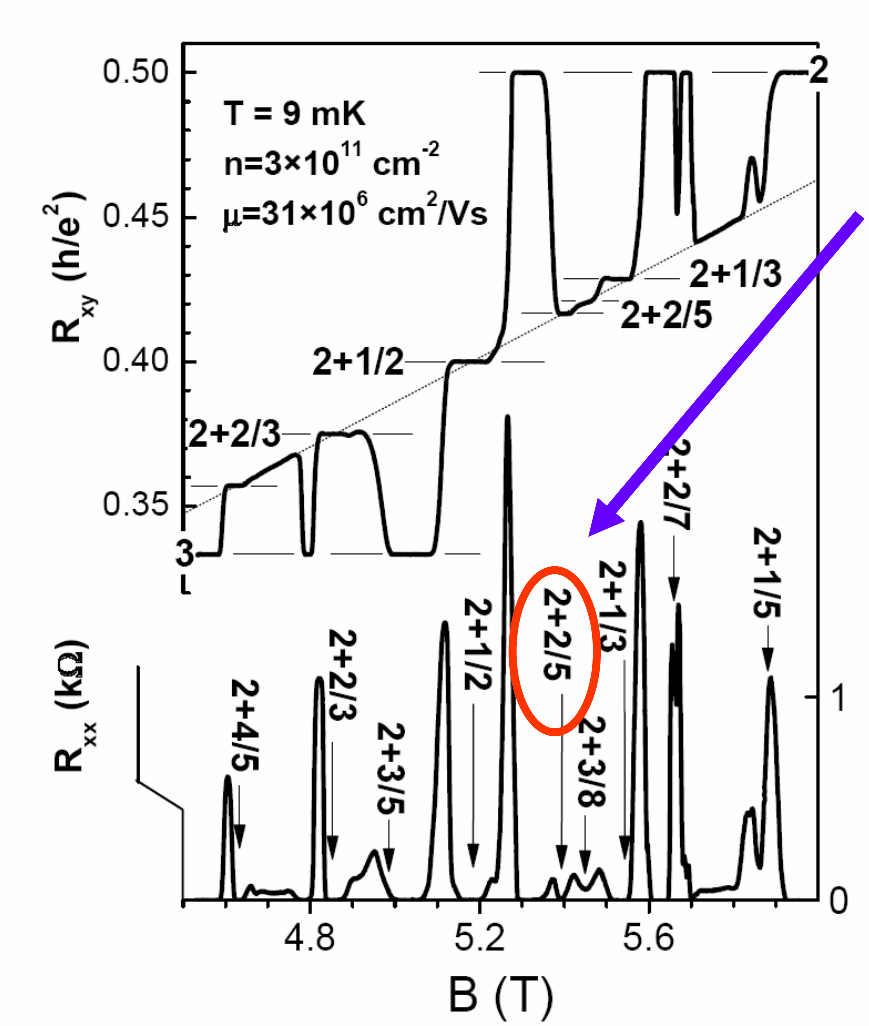
Charge *e/4* quasiparticles with braiding properties described by $SU(2)_2$ Chern-Simons Theory.

Nayak and Wilczek, 1996

Not sufficiently "rich" nonabelian statistics to do universal quantum computation.

But see, S. Bravyi, quant-ph/0511178 and M. Freedman, C. Nayak and K. Walker, cond-mat/0512066.

Possible Non-Abelian FQH States



J.S. Xia et al., PRL (2004).

$\nu = 12/5$

Possibly a Read-Rezayi k = 3 “Parafermion” state.

Read and Rezyai, 1999

Charge *e/5* quasiparticles with braiding properties described by $SU(2)_3$ Chern-Simons Theory.

Slingerland and Bais, 2001

$SU(2)_3$ ***is*** sufficiently “rich” to do universal quantum computation.

Freedman, Larsen, and Wang, 2001

Fibonacci Anyons (Kuperberg, Preskill)

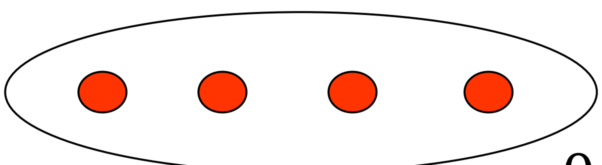
A Fibonacci Anyon → ●   Fibonacci → 

The laws of Fibonacci anyons:

1. Fibonacci anyons have a quantum attribute known as q-spin:

   ● ← q-spin = 1

2. A collection of Fibonacci anyons can have a total q-spin of either 0 or 1:



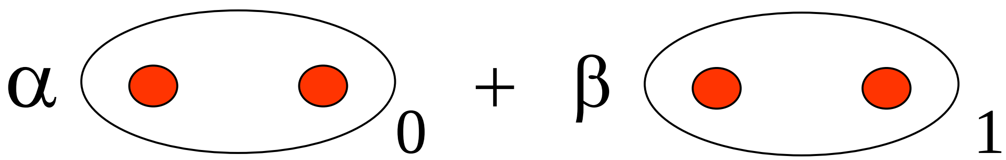
0,1

Notation: Ovals are labeled by total q-spin of enclosed particles.

Fibonacci Anyons

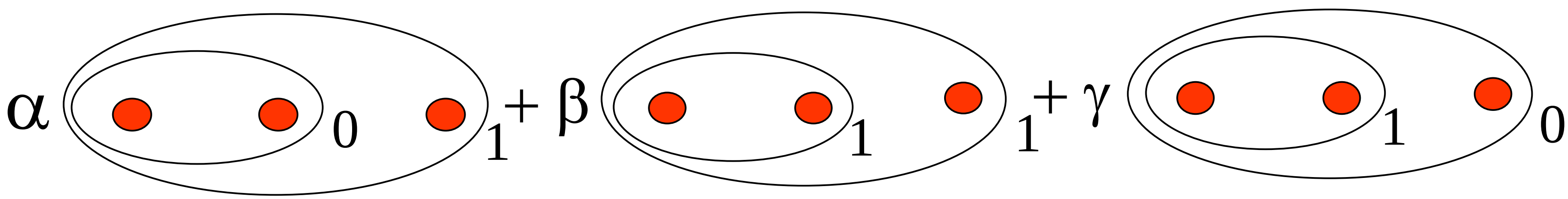
3. The “fusion” rule for combining q-spin is: 1 x 1 = 0 + 1

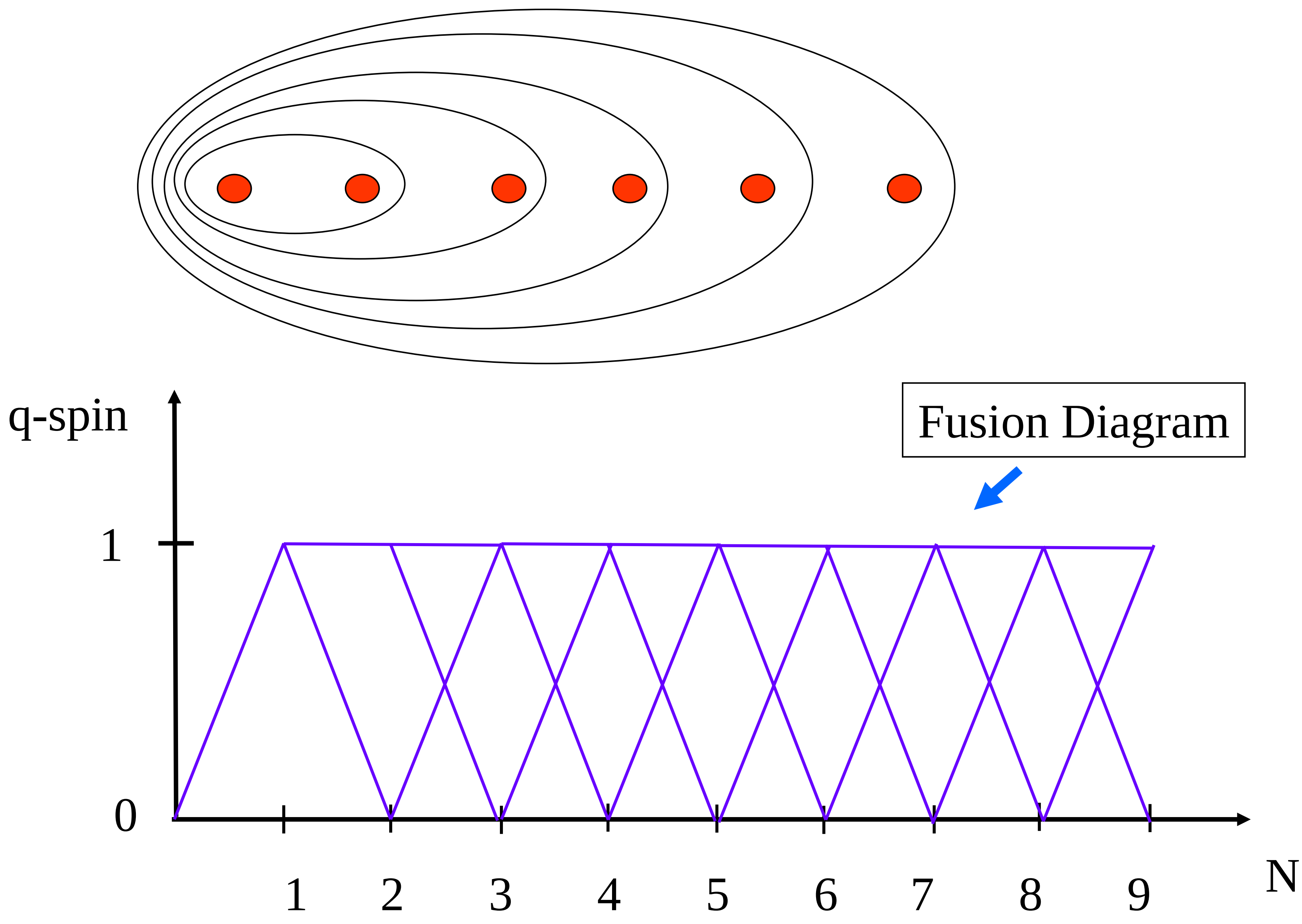
This means that two Fibonacci anyons can have total q-spin 0 or 1, or be in any quantum superposition of the two.



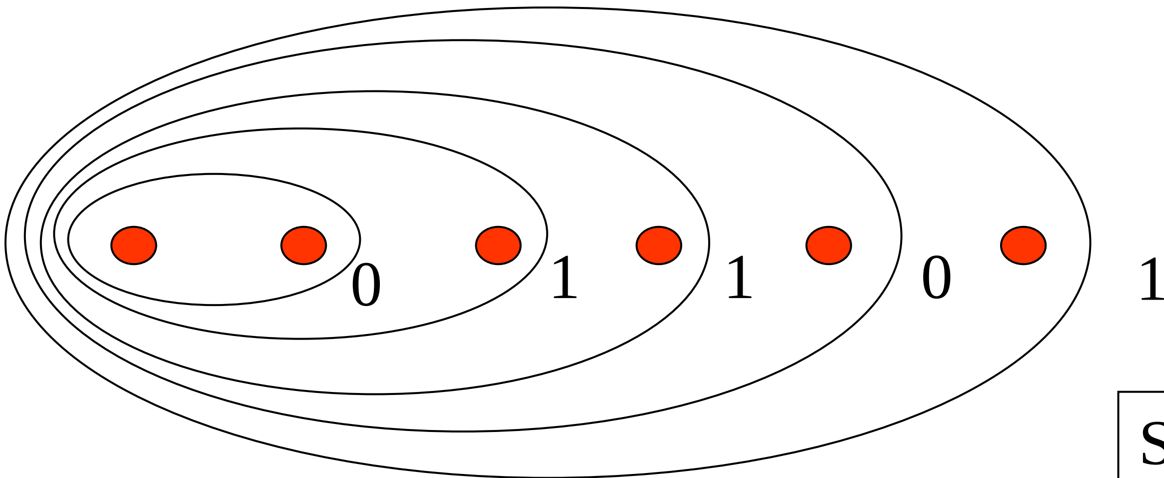
Two dimensional Hilbert space

Three Fibonacci anyons → Three dimensional Hilbert space

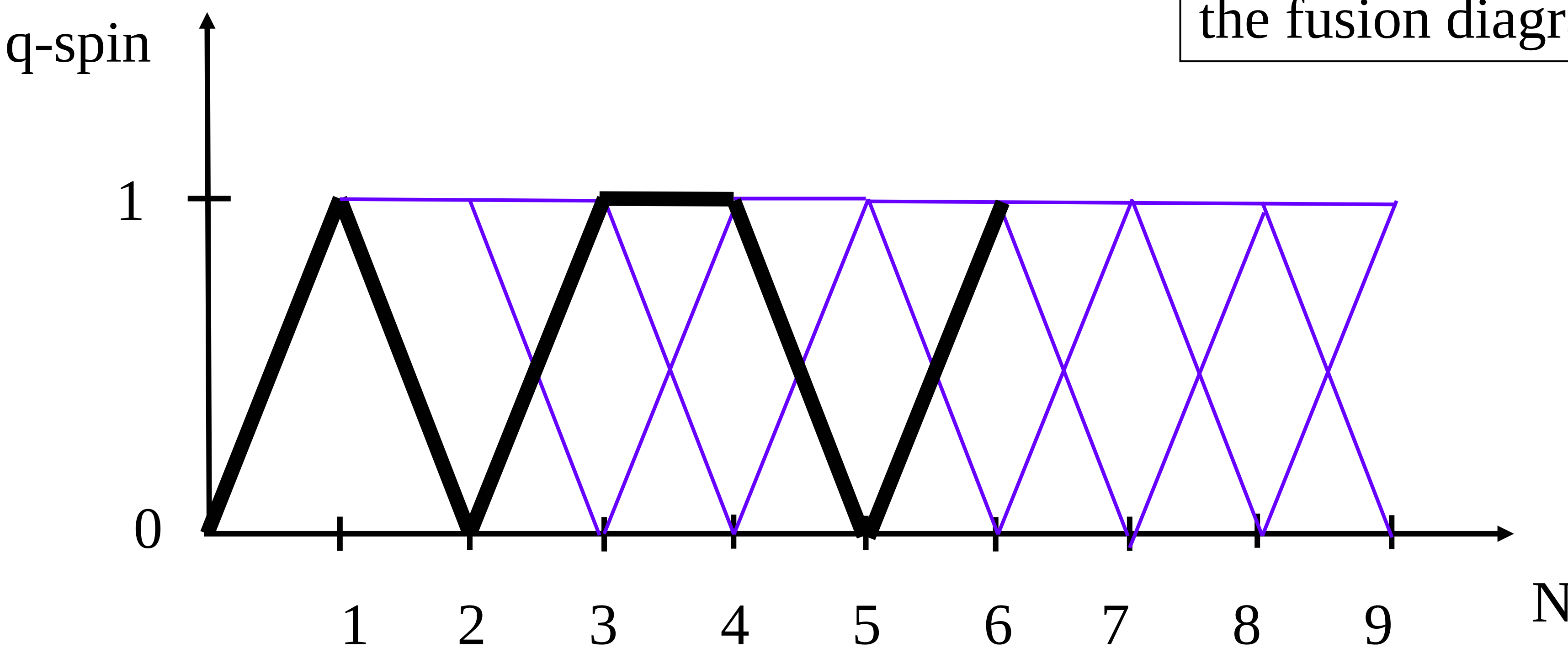


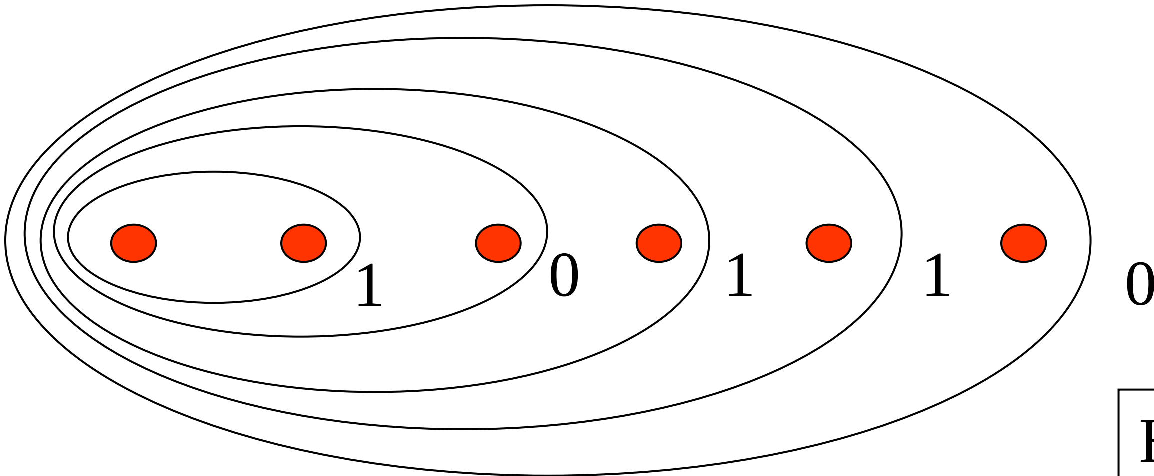


q-spin
Fusion Diagram
1
0
1
2
3
4
5
6
7
8
9
N

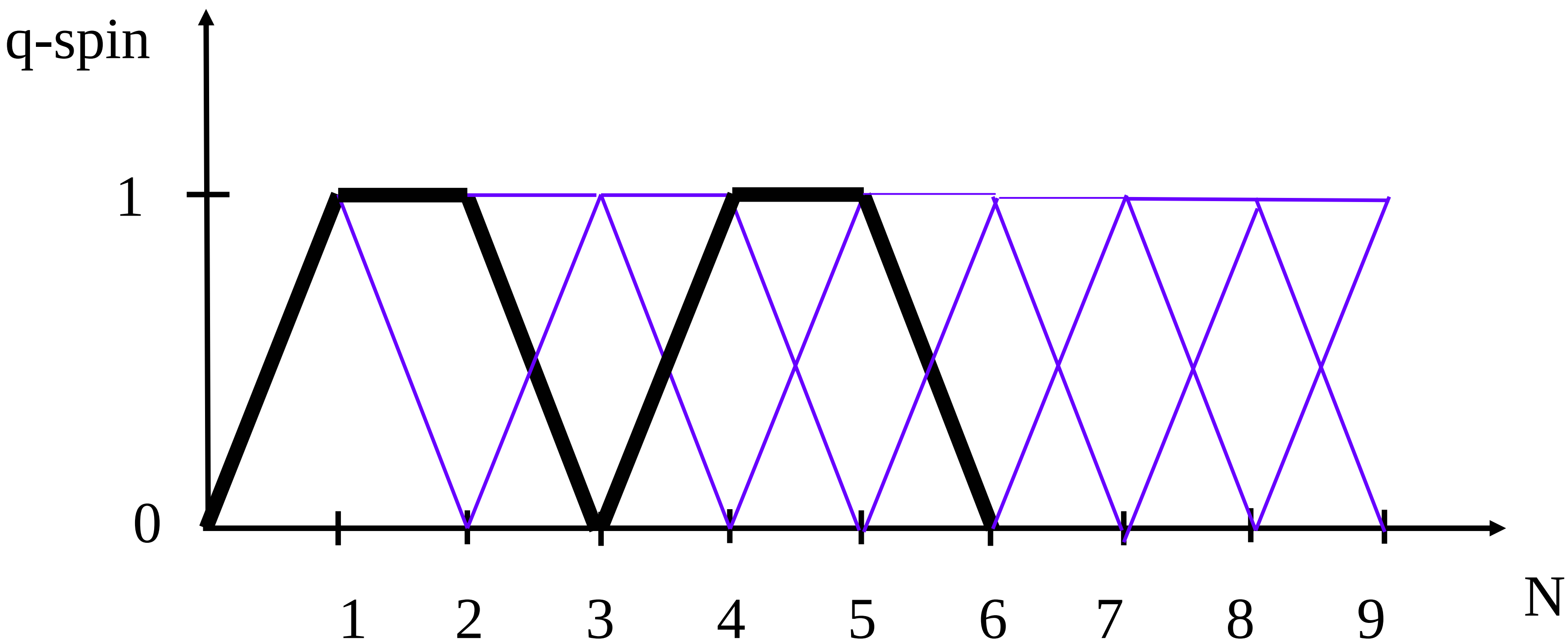


States are paths in the fusion diagram



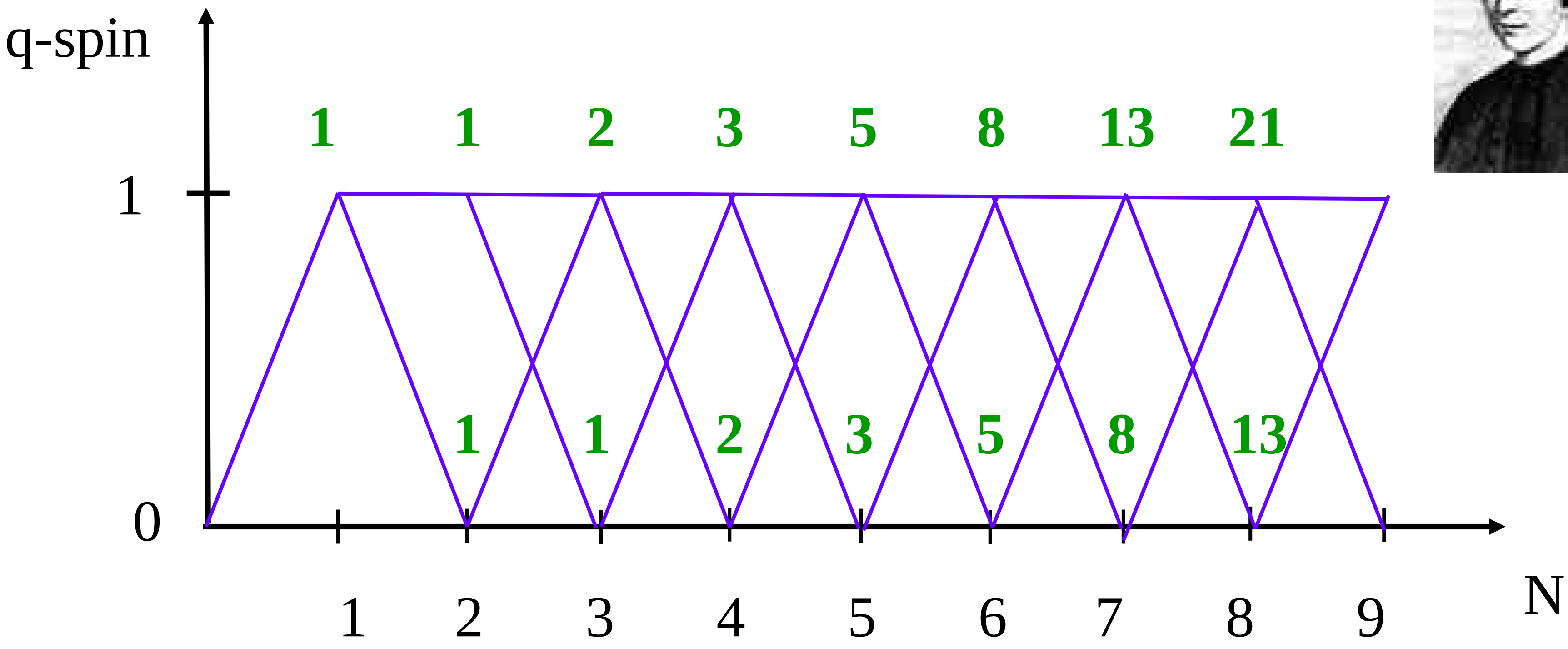


Here's another one



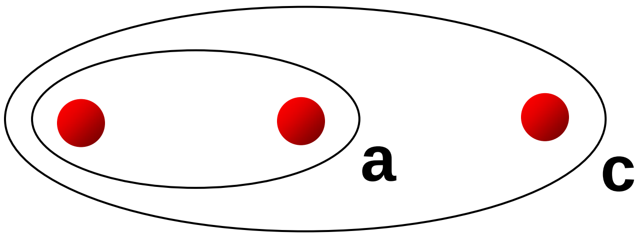
Count states by counting paths

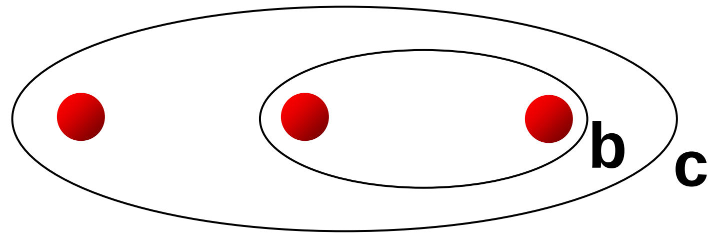
- Hilbert space dimensionality grows as the **Fibonacci sequence**!
- Exponentially large in the number of quasiparticles, so big enough for quantum computing.



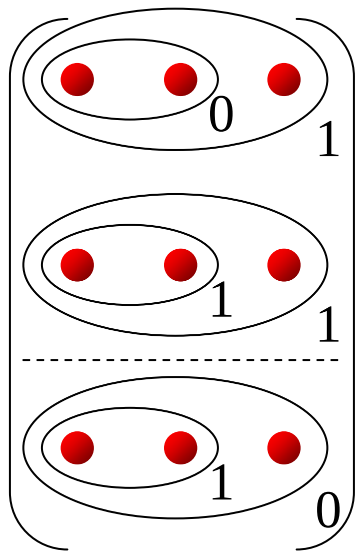
The F Matrix

Changing fusion bases:

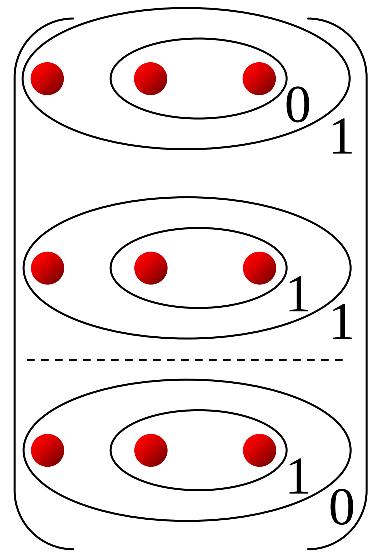
$$\sum_a F_{ab}^c$$ 

$=$ 

$$\underbrace{\begin{pmatrix} -\tau & \sqrt{\tau} & 0 \\ \sqrt{\tau} & \tau & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F_{ab}^c}$$



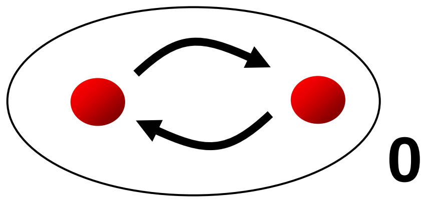
$=$



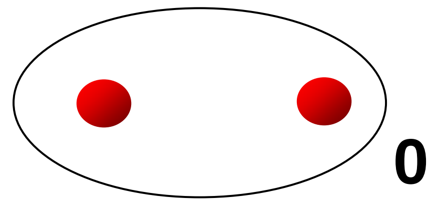
$$\tau = \frac{\sqrt{5}-1}{2}$$

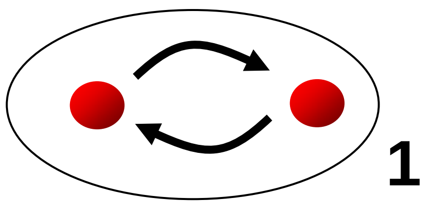
The R Matrix

Exchanging particles:

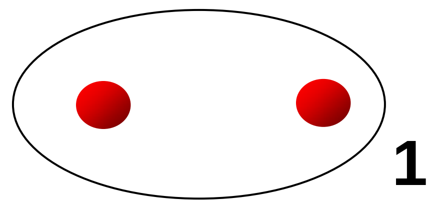


$$= \; e^{-i4\pi/5}$$





$$= \; e^{i3\pi/5}$$



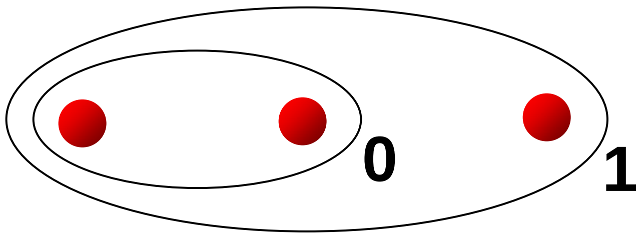
$$R = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$

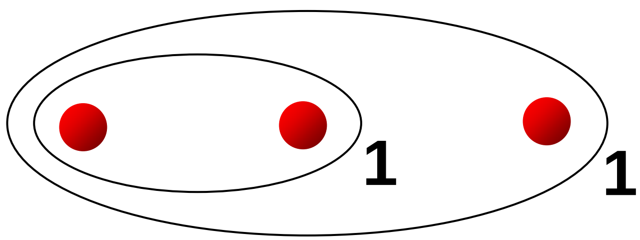
F and *R* must satisfy certain consistency conditions (the “pentagon” and “hexagon” equations). For Fibonacci anyons these equations *uniquely determine* *F* and *R*.

Encoding a Qubit

(Freedman, Larsen, and Wang, 2001)

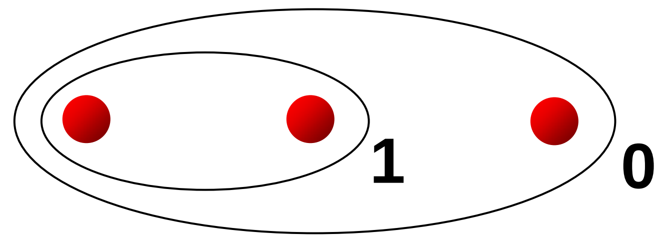
Qubit States

$|0\rangle =$ 

$|1\rangle =$ 

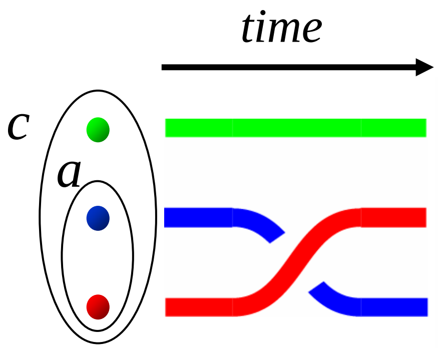
State of qubit is determined by q-spin of two leftmost particles

Non-Computational State



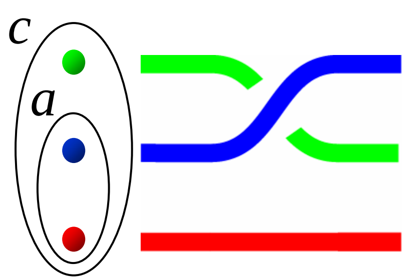
Transitions to this state are **leakage errors**

Braiding Matrices for 3 Fibonacci Anyons



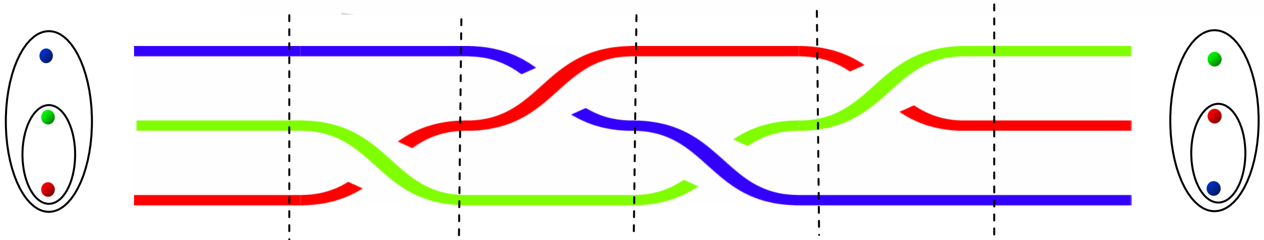
$c = 1$ $c = 0$

$$\sigma_1 = \left[\begin{array}{cc|c} e^{-i4\pi/5} & 0 & 0 \\ 0 & -e^{-i2\pi/5} & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right]$$



$$\sigma_2 = \left[\begin{array}{cc|c} -\tau\, e^{-i\pi/5} & -i\sqrt{\tau}\, e^{-i\pi/10} & 0 \\ -i\sqrt{\tau}\, e^{-i\pi/10} & -\tau & 0 \\ \hline 0 & 0 & -e^{-i2\pi/5} \end{array}\right] \qquad \tau = \frac{\sqrt{5}-1}{2}$$

$|\Psi_i\rangle$



$|\Psi_f\rangle$

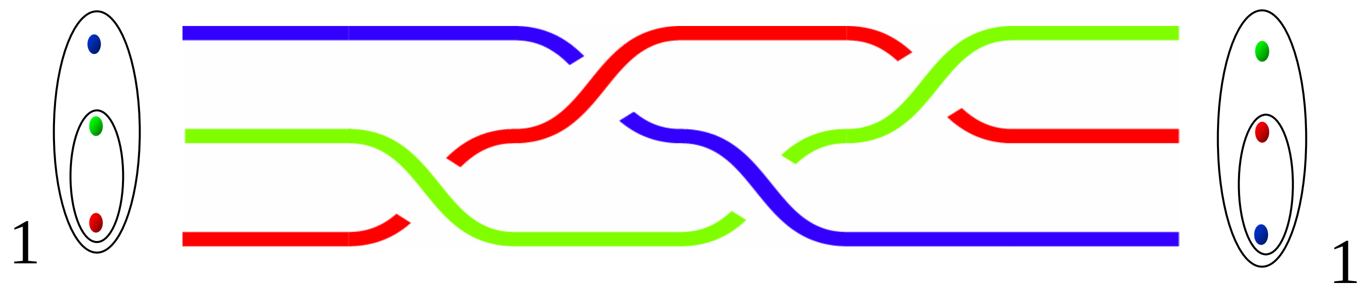
$$\sigma_1^{-1} \quad \sigma_2 \quad \sigma_1^{-1} \quad \sigma_2 \;=\; \boldsymbol{M}$$

$$|\Psi_f\rangle = \boldsymbol{M}^{-1}\, |\Psi_i\rangle$$

Single Qubit Operations

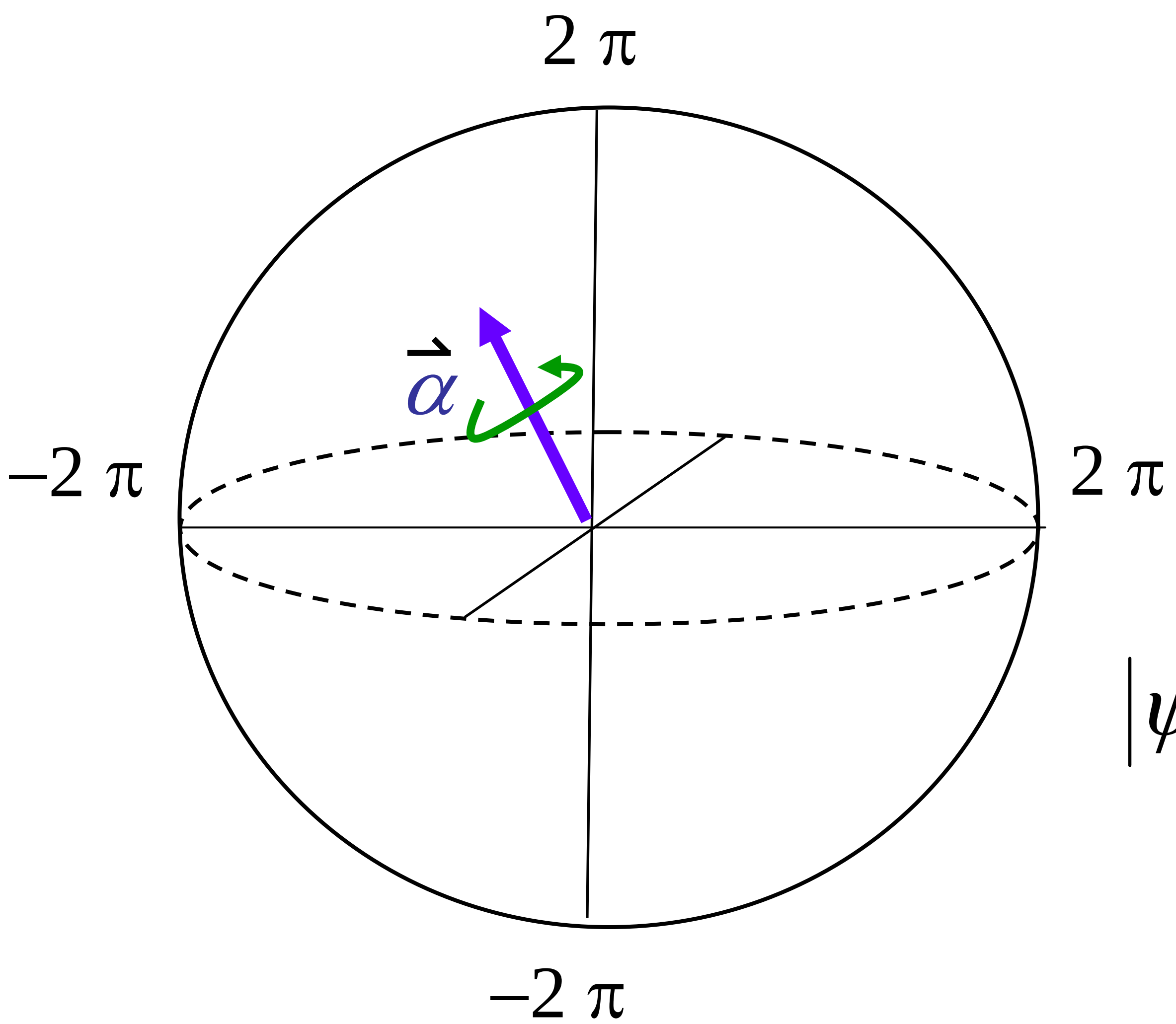
General rule: Braiding inside an oval does not change the total q-spin of the enclosed particles.

Important consequence: As long as we braid *within* a qubit, there is **no leakage error.**



Can we do arbitrary single qubit rotations this way?

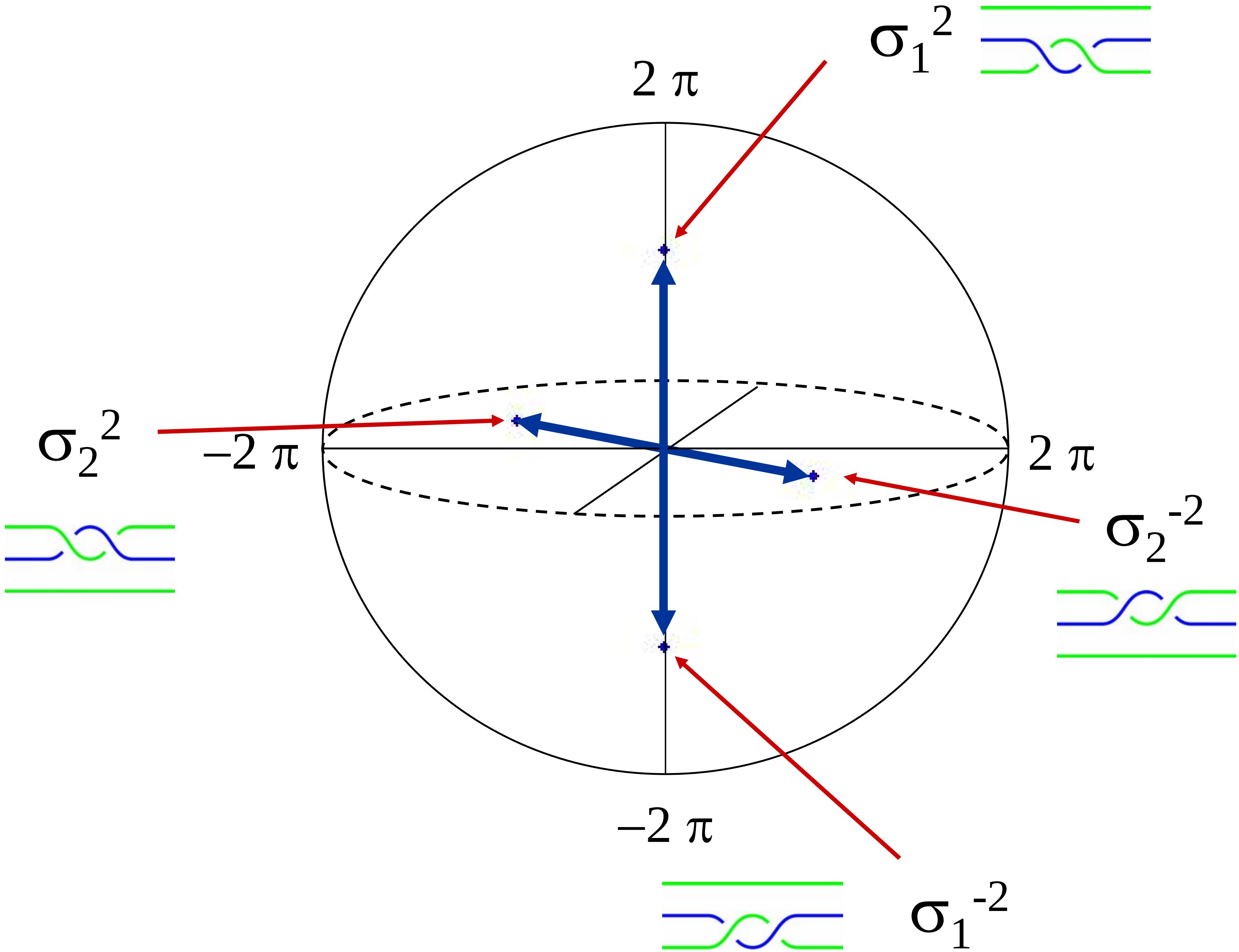
Single Qubit Operations are Rotations



The set of all single qubit rotations lives in a solid sphere of radius 2π.

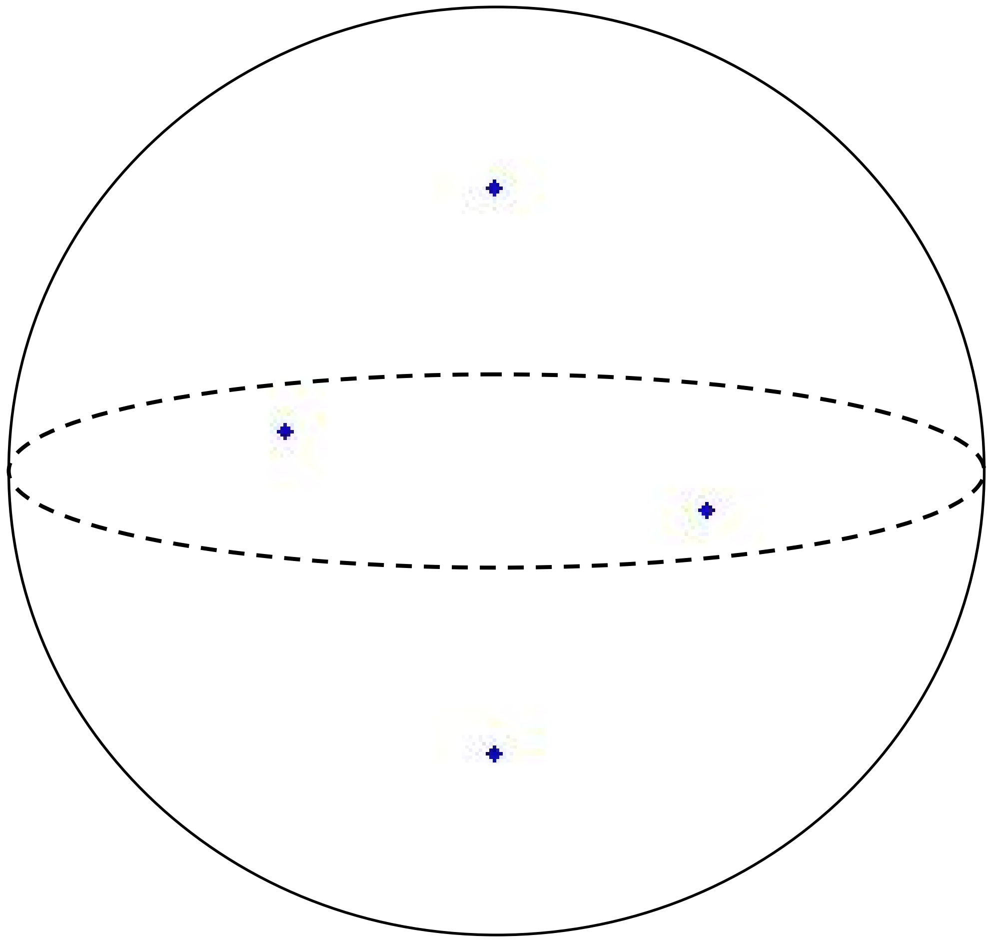
$$|\psi\rangle \;-\!\boxed{U_{\vec{\alpha}}}\!-\; U_{\vec{\alpha}}\,|\psi\rangle$$

$$U_{\vec{\alpha}} = \exp\frac{i\,\vec{\alpha}\cdot\vec{\sigma}}{2}$$

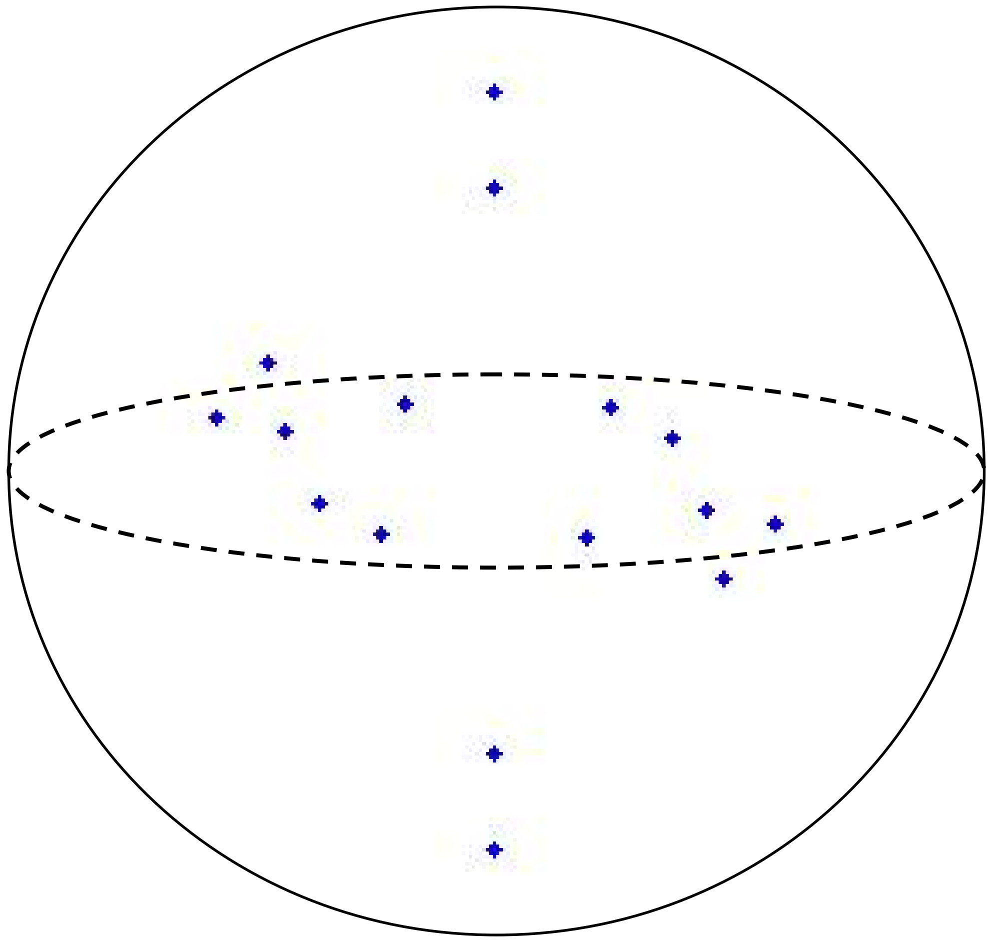


σ_1^{2}
2π
σ_2^{2}
-2π
2π
σ_2^{-2}
-2π
σ_1^{-2}

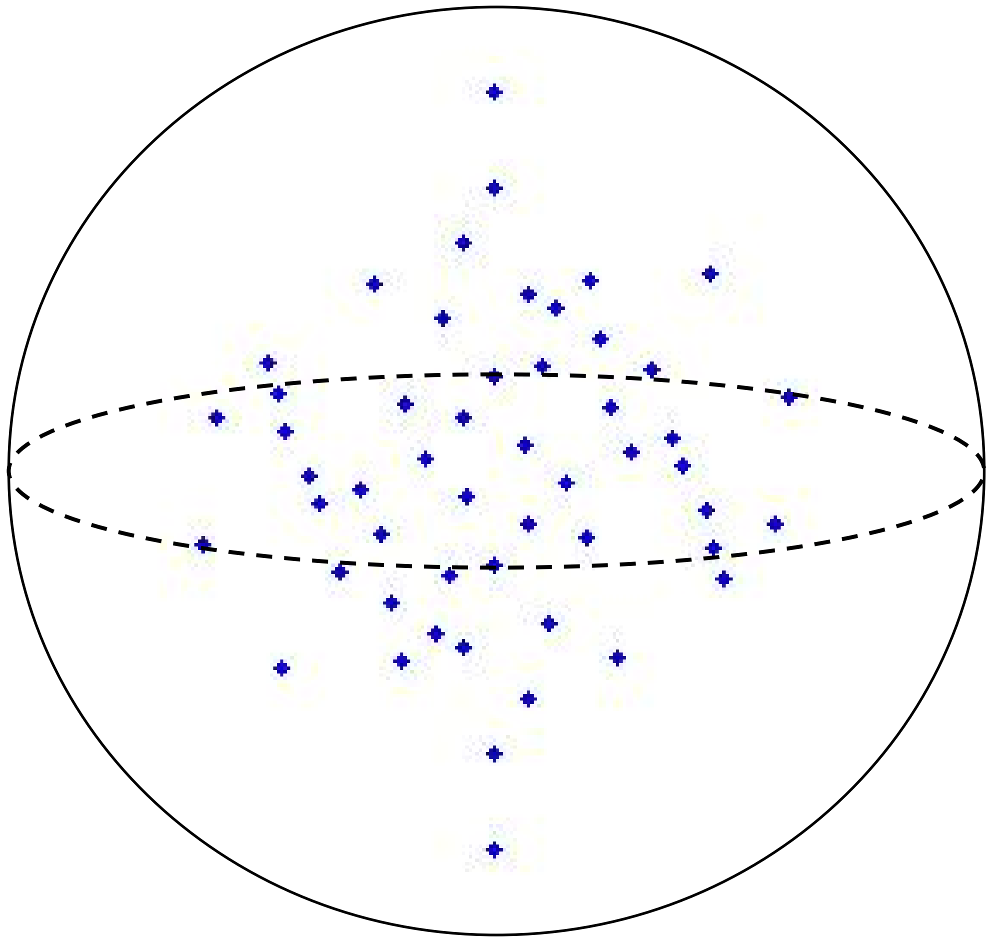
N = 1



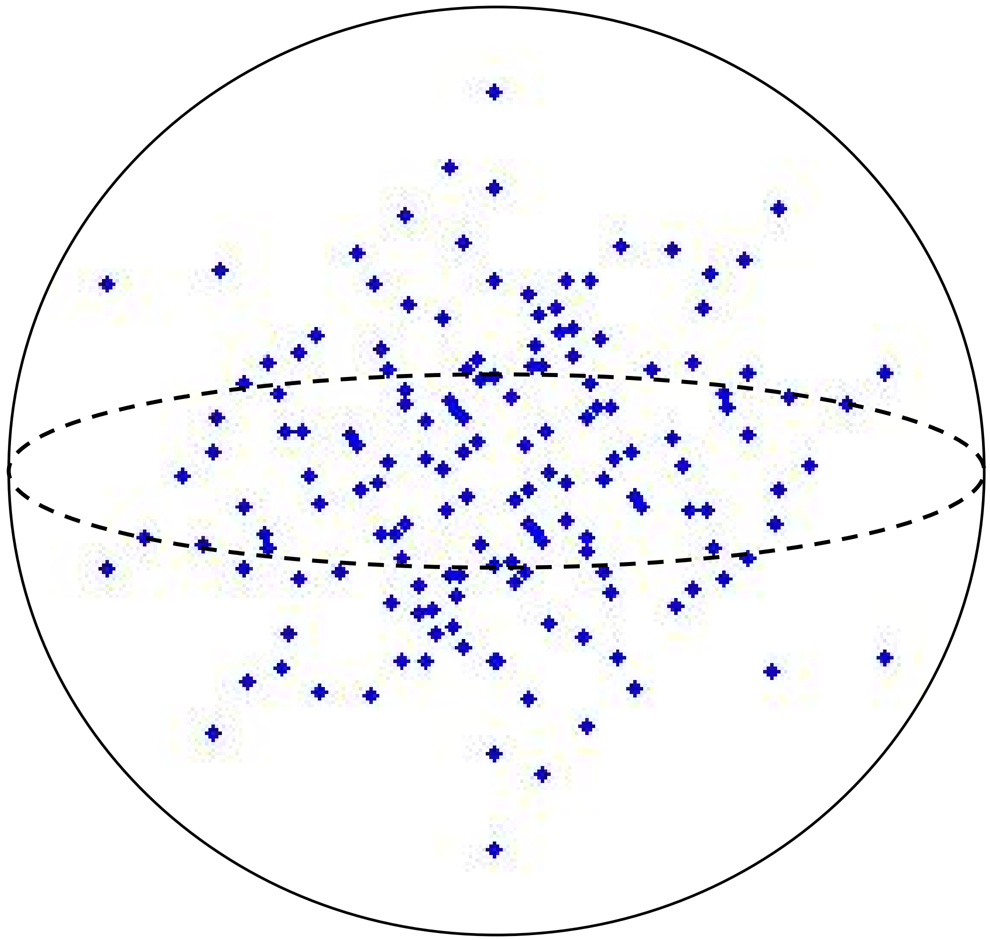
N = 2



N = 3



N = 4



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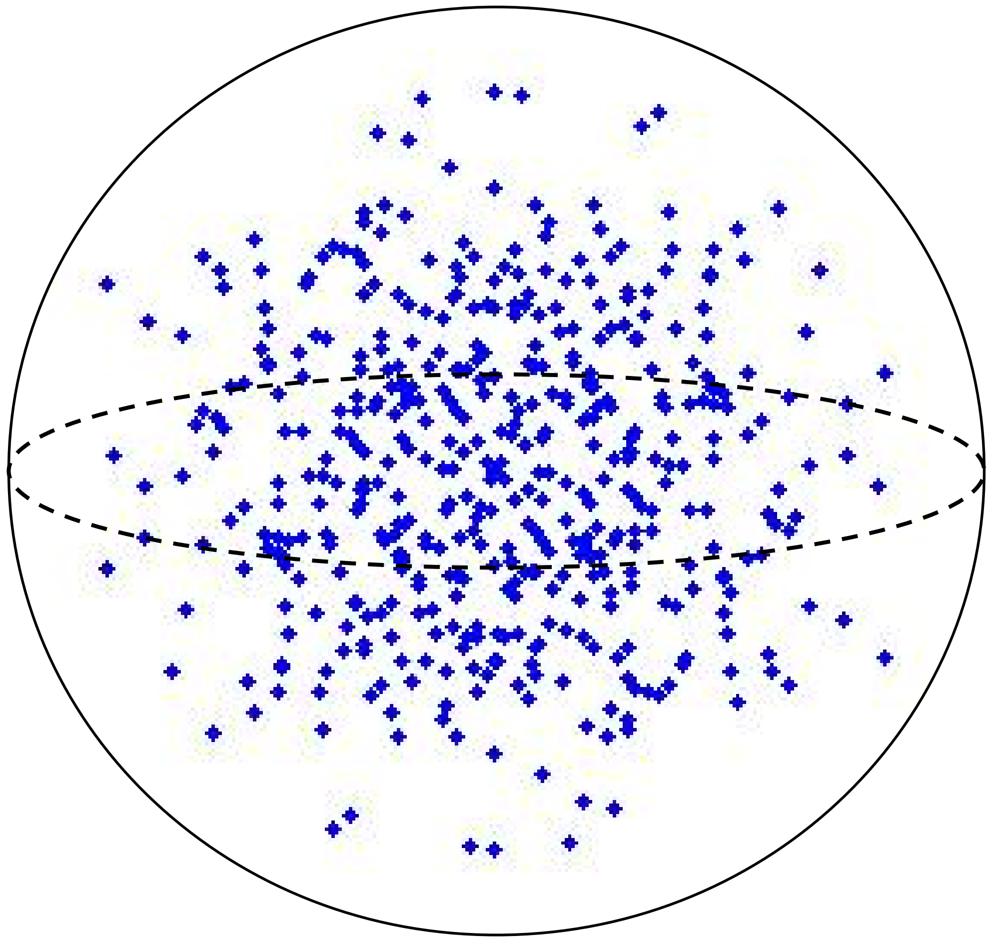
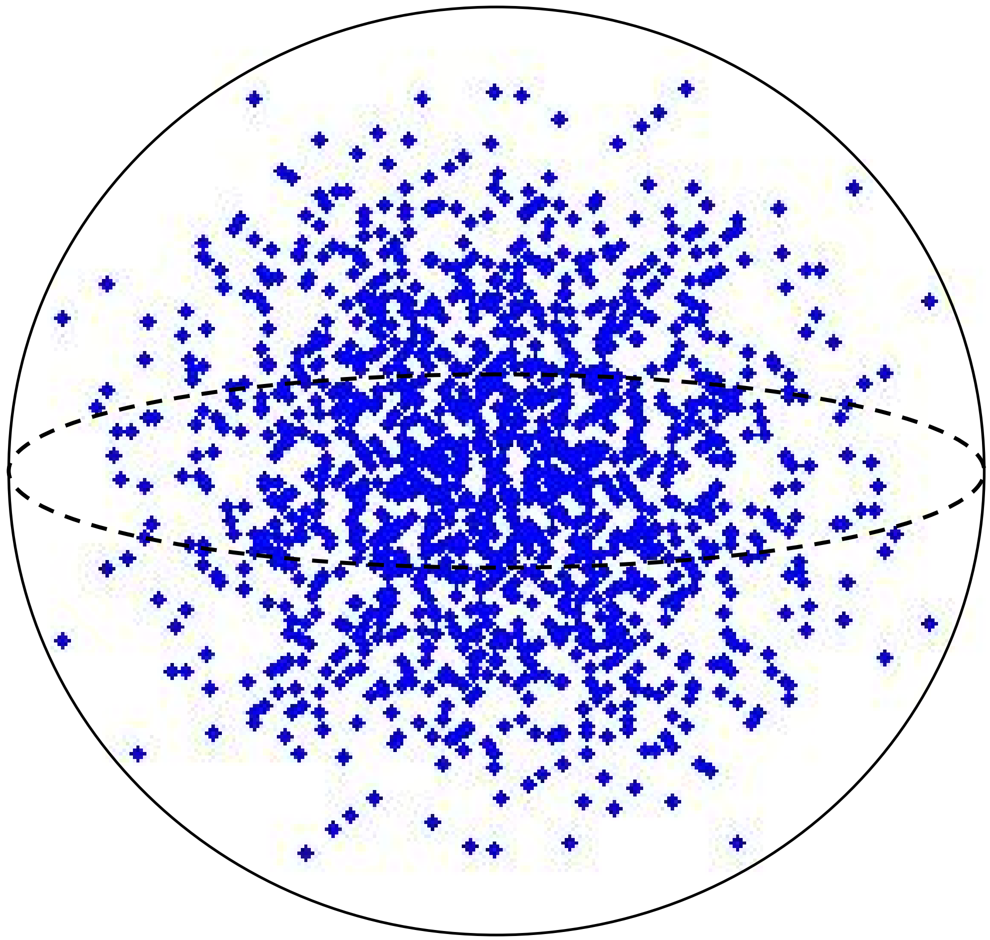
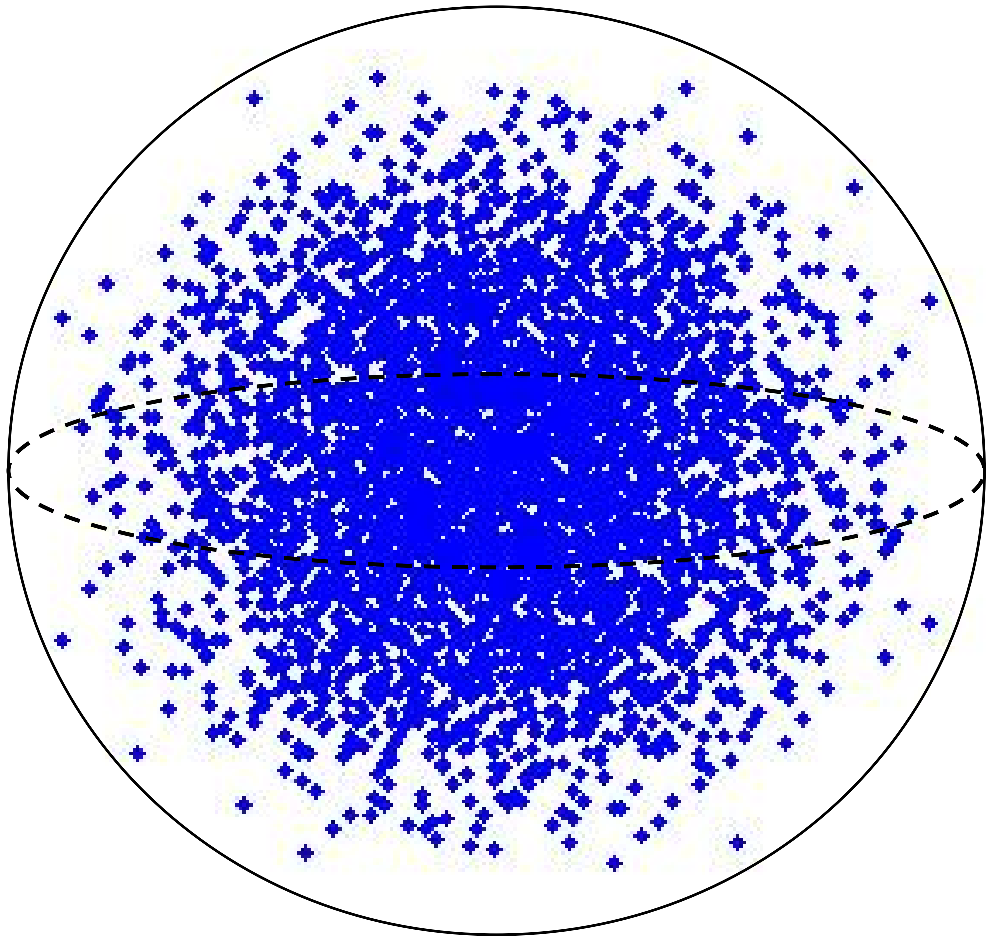


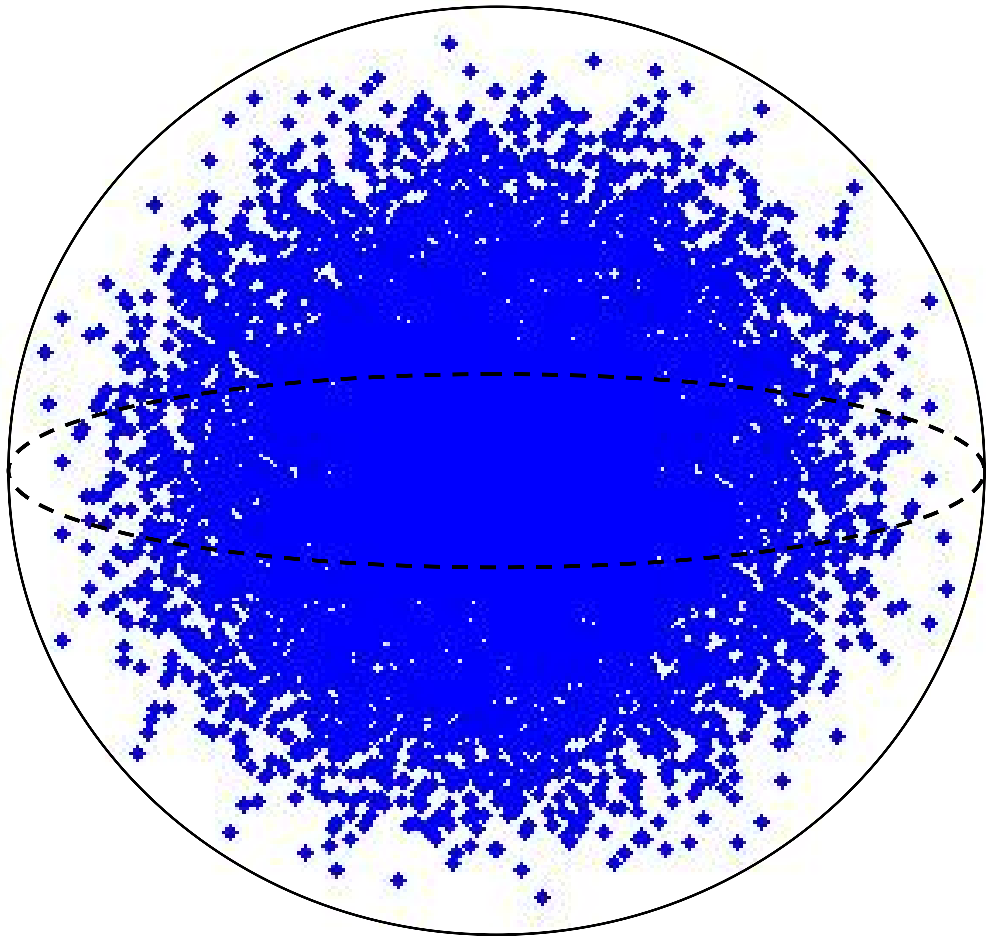
N = 6



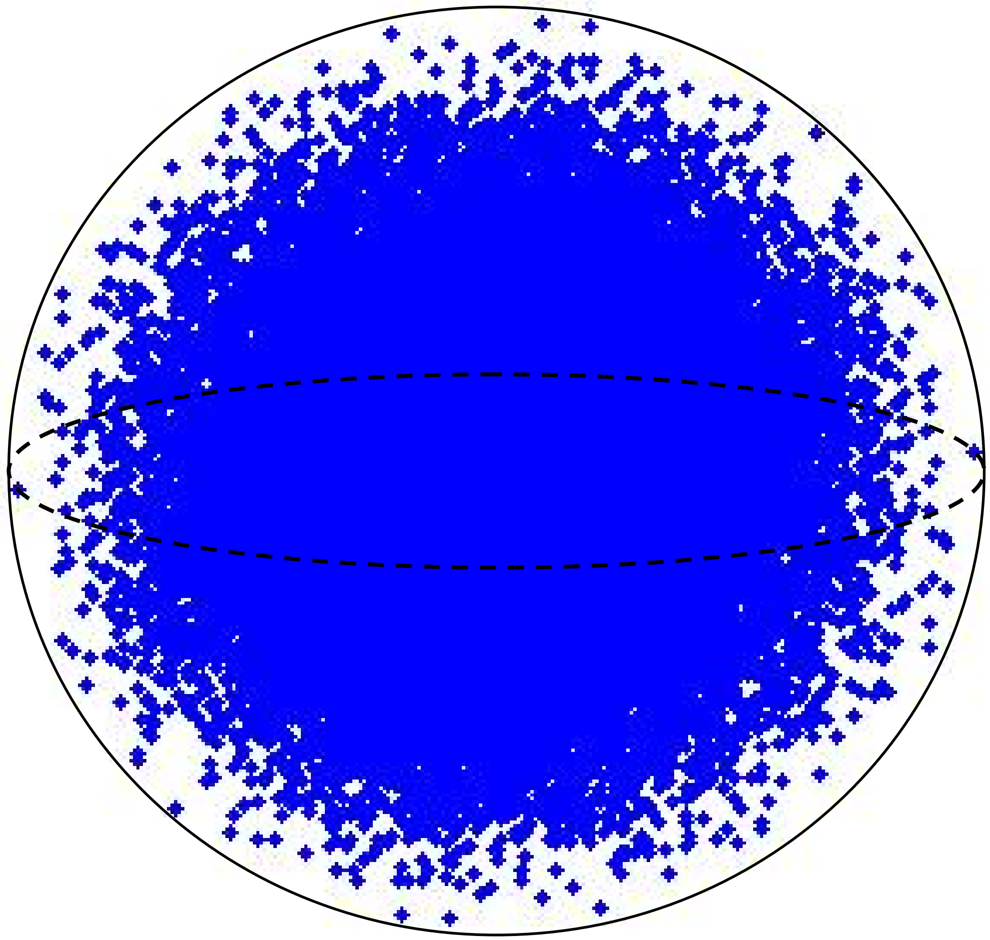
N = 7



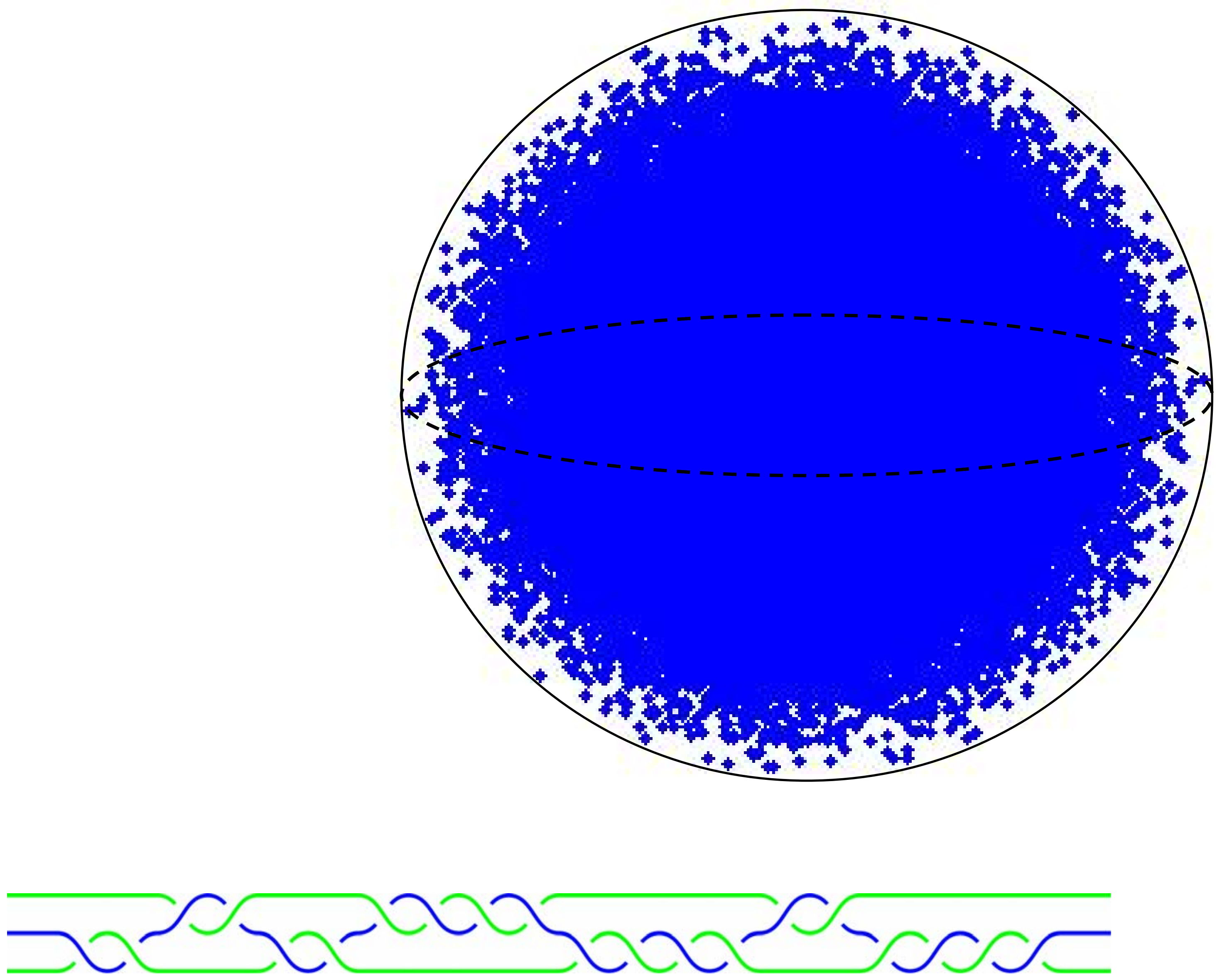
N = 8



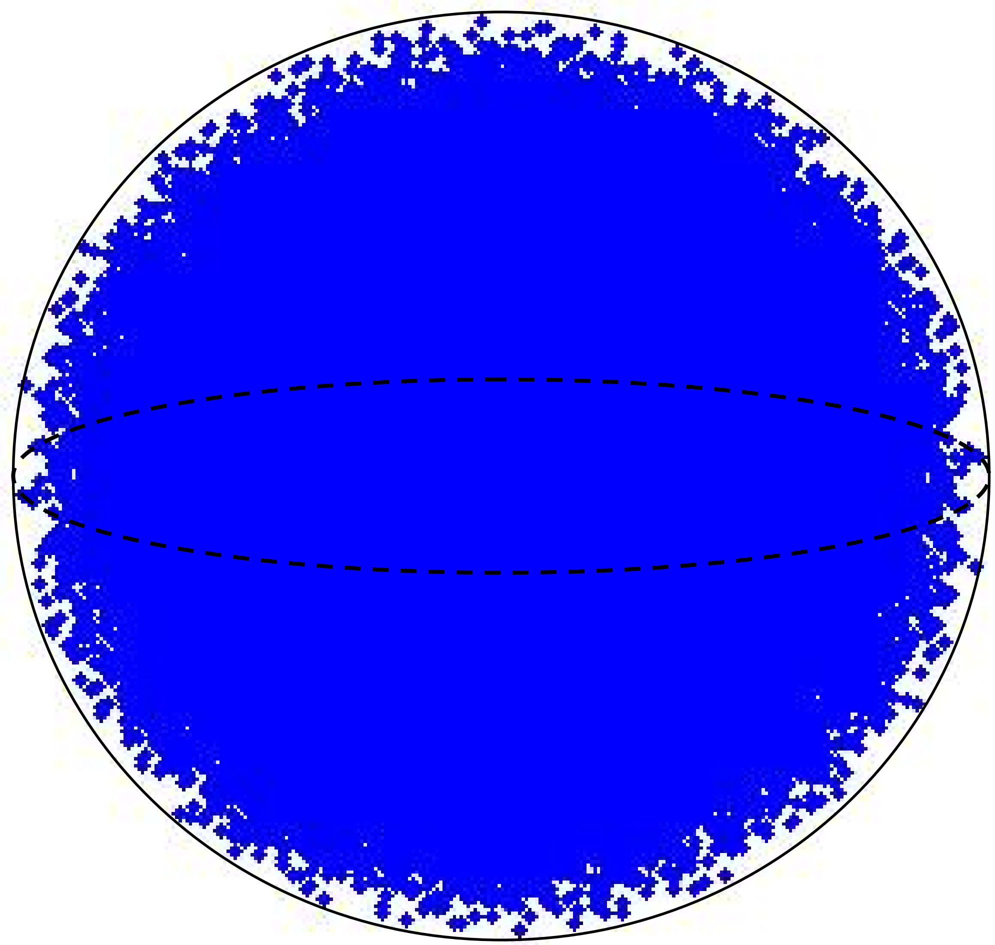
N = 9



N = 10



N = 11



Brute Force Search

qubit operation

$$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(10^{-3})$$

Brute force searching rapidly becomes infeasible as braids get longer.

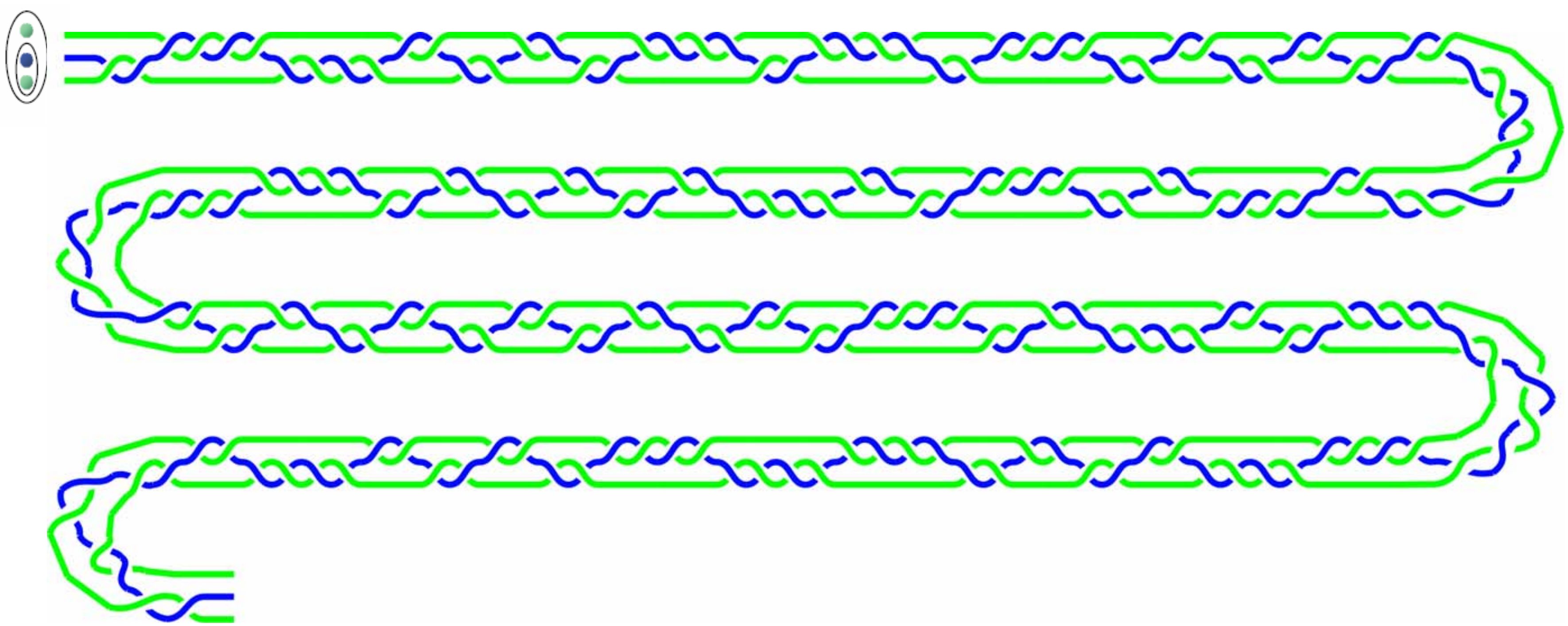
Fortunately, a clever algorithm due to Solovay and Kitaev allows for systematic improvement of the braid given a sufficiently dense covering of *SU(2)*.

Solovay-Kitaev Construction

(Actual calculation)

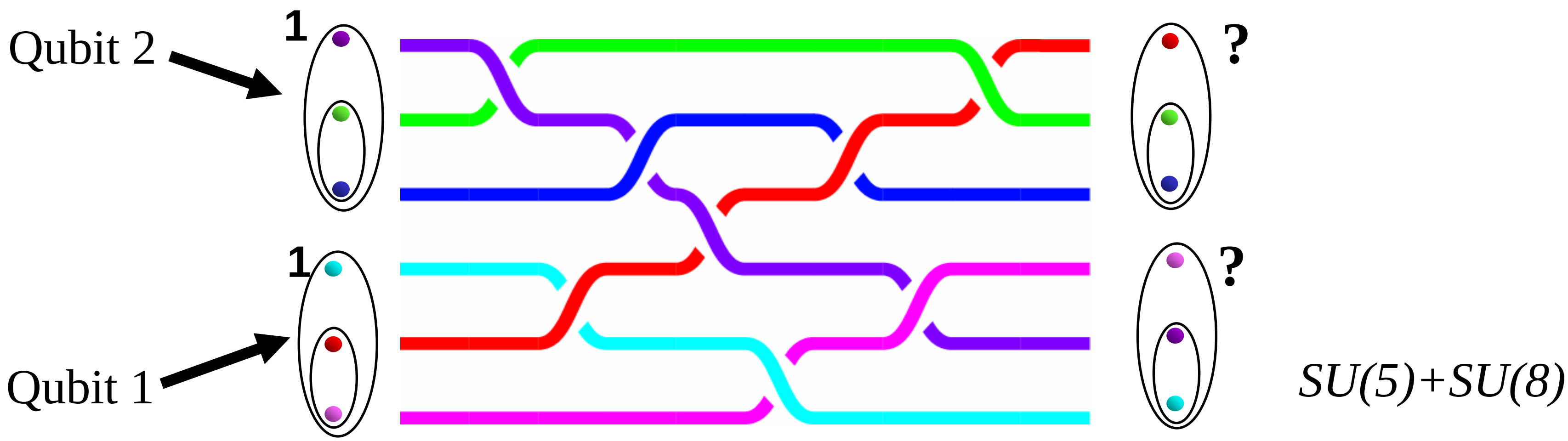
$$\left(\begin{array}{cc|c} 0 & i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) + O(10^{-4})$$

ε



Braid Length $\sim |\ln \varepsilon|^c, \quad c \approx 4$

What About Two Qubit Gates?

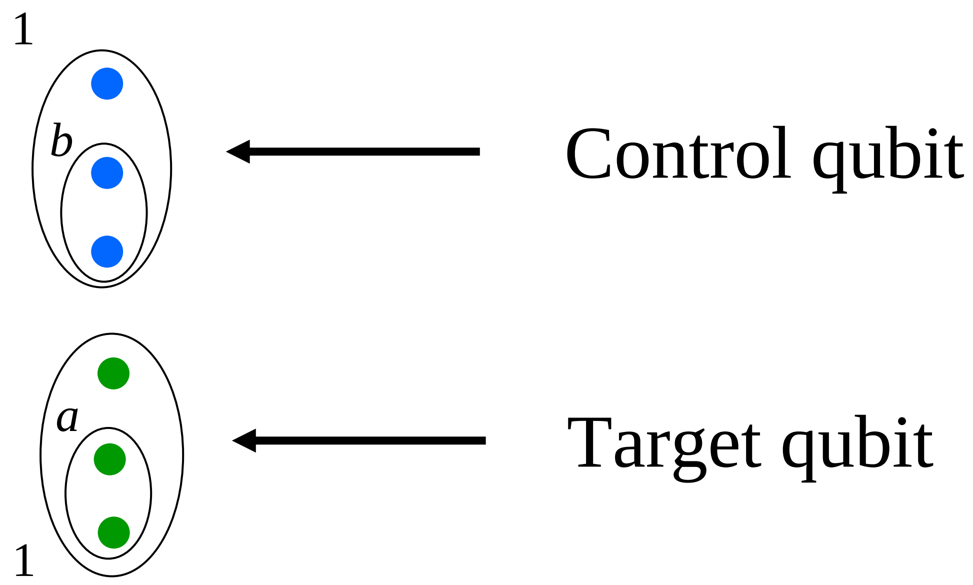


Problems:

1. We are pulling quasiparticles out of qubits: **Leakage error!**

2. **87 dimensional** search space (as opposed to **3** for three-particle braids). Straightforward "brute force" search is problematic.

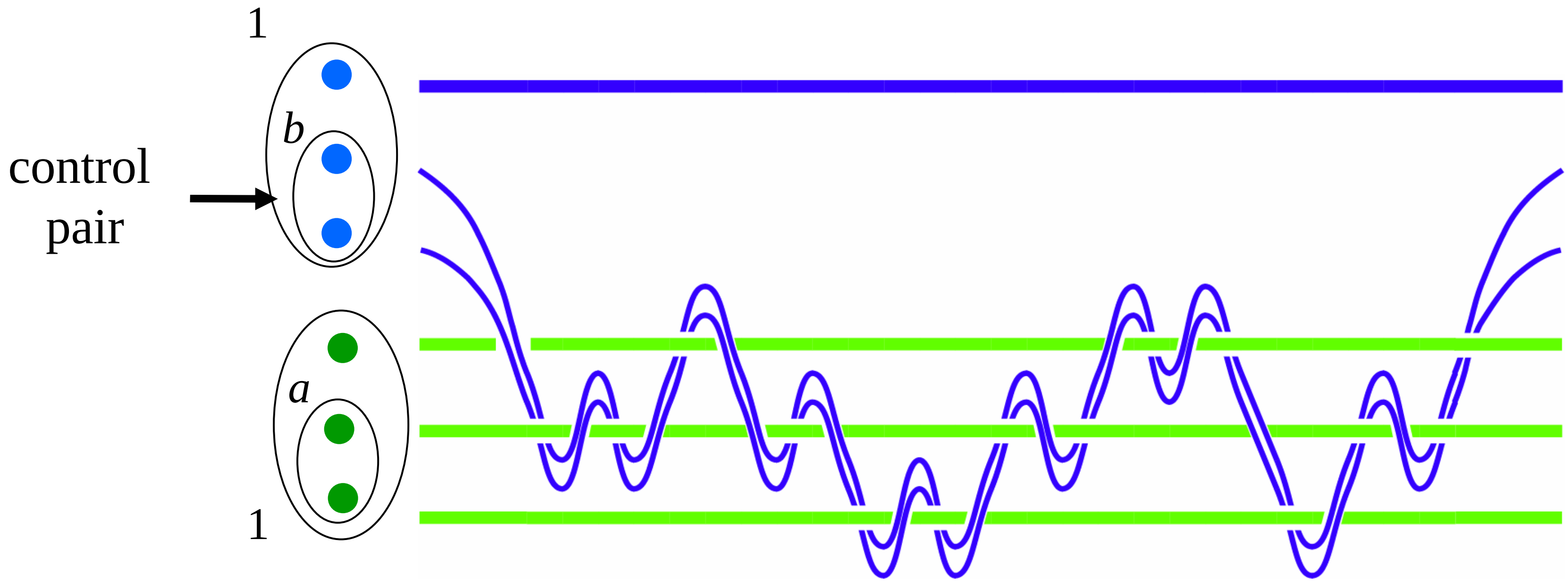
Two Qubit Controlled Gates



Goal: Find a braid in which some rotation is performed on the target qubit only if the control qubit is in the state 1. (***b*=*1***)

Constructing Two Qubit Gates by “Weaving”

Weave a *pair* of anyons from the control qubit between anyons in the target qubit.



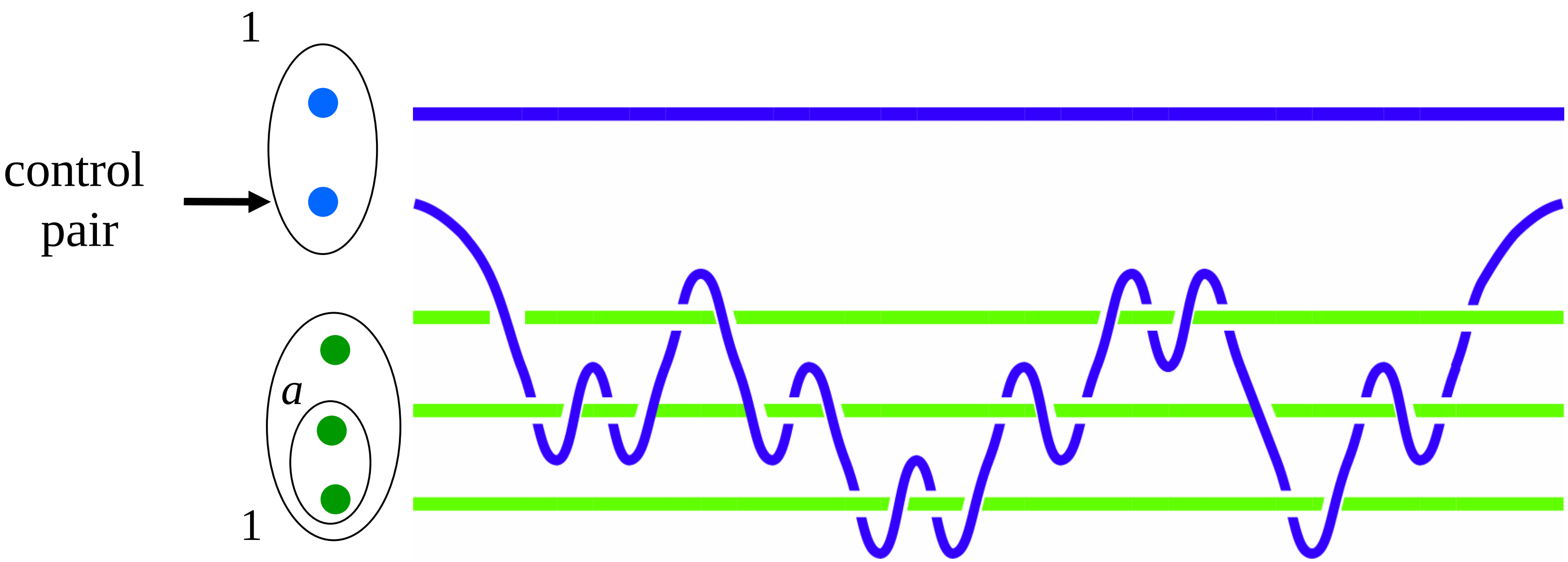
Important Rule: Braiding a q-spin 0 object does not induce transitions.

➡ Target qubit is only affected if control qubit is in state $|1\rangle$

(b = 1)

Constructing Two Qubit Gates by "Weaving"

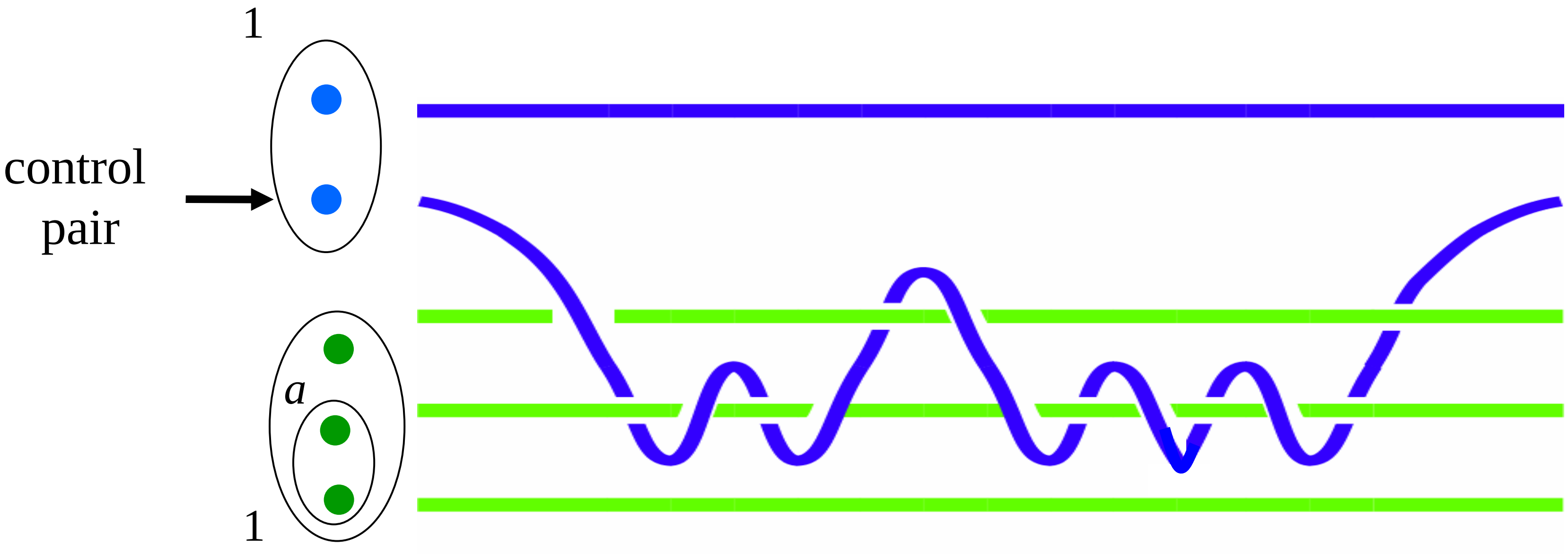
Only nontrivial case is when the control pair has q-spin 1.



We've reduced the problem to weaving one anyon around three others. **Still too hard for brute force approach!**

OK, Try Weaving Through Only Two Particles

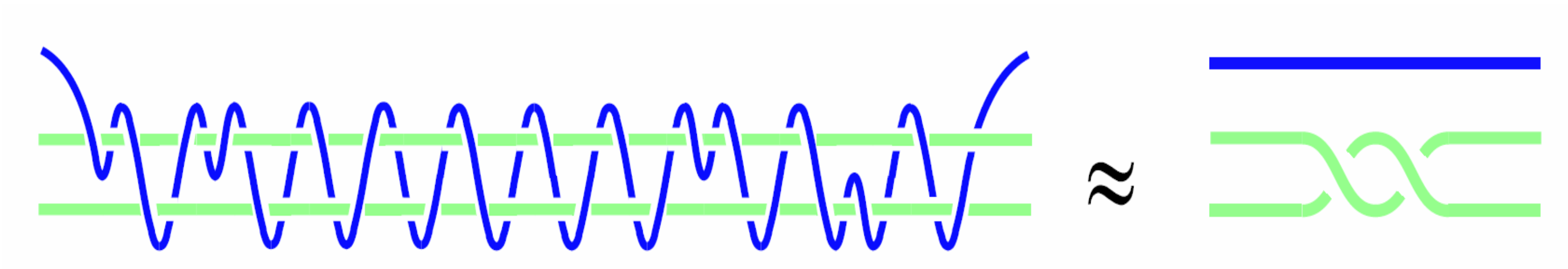
We're back to *SU(2)*, so this is numerically feasible.



Question: Can we find a weave which does not lead to **leakage errors**?

A Trick: Effective Braiding

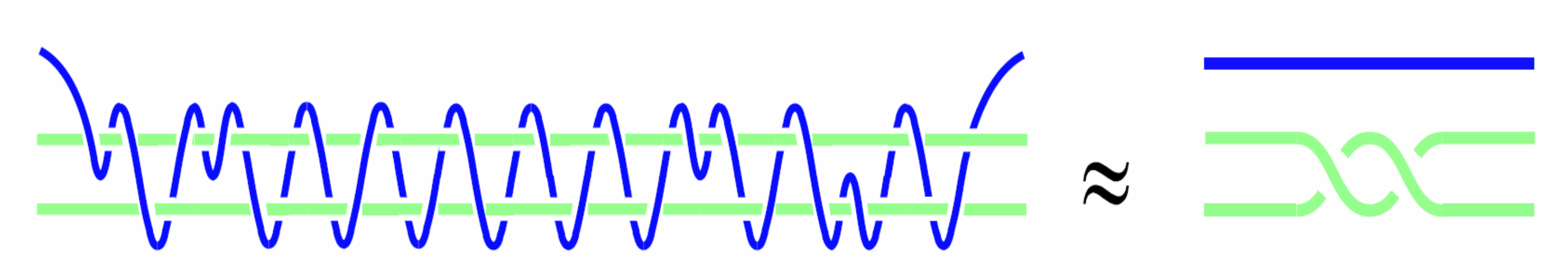
Actual Weaving *Effective* Braiding

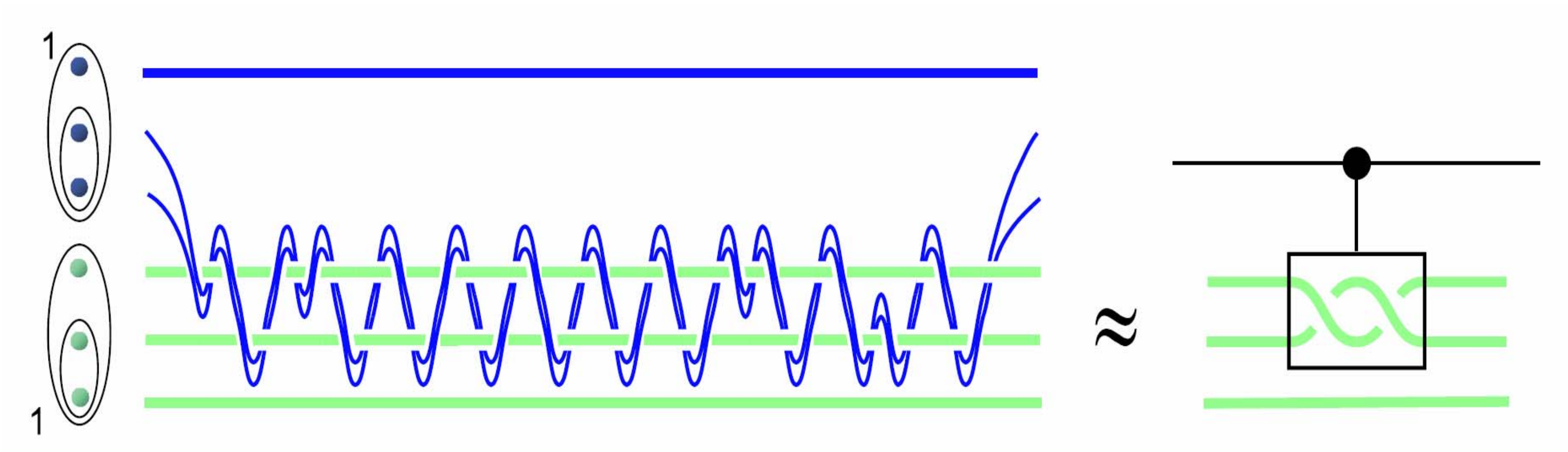


$$\sigma_2^{3}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{4}\sigma_2^{2}\sigma_1^{-2}\sigma_2 \approx \sigma_1^{2}$$

The effect of weaving the **blue anyon** through the two **green anyons** has approximately the same effect as braiding the two **green anyons** twice.

Controlled–"Knot" Gate

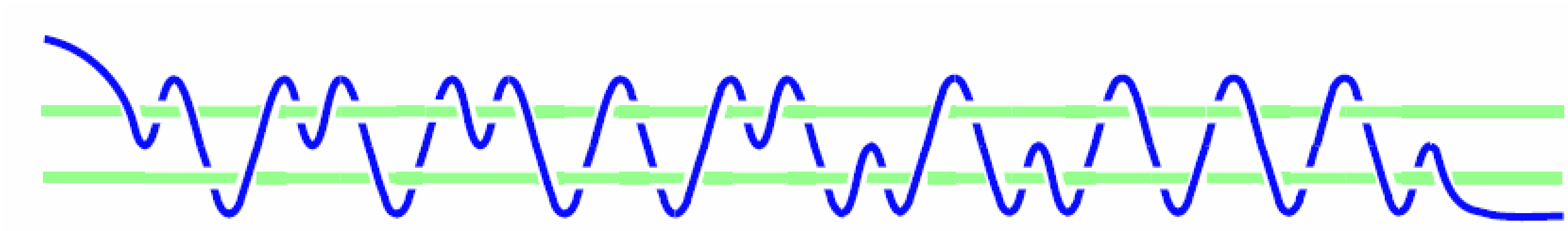




Effective braiding is all within the target qubit → No leakage!

Not a CNOT, but sufficient for universal quantum computation.

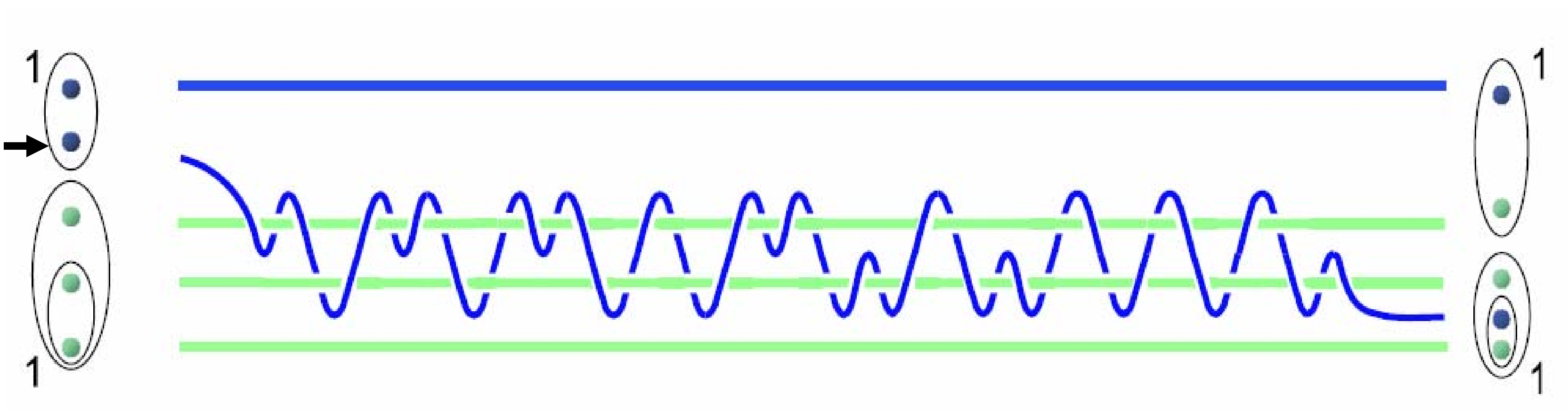
Another Trick: Injection Weaving

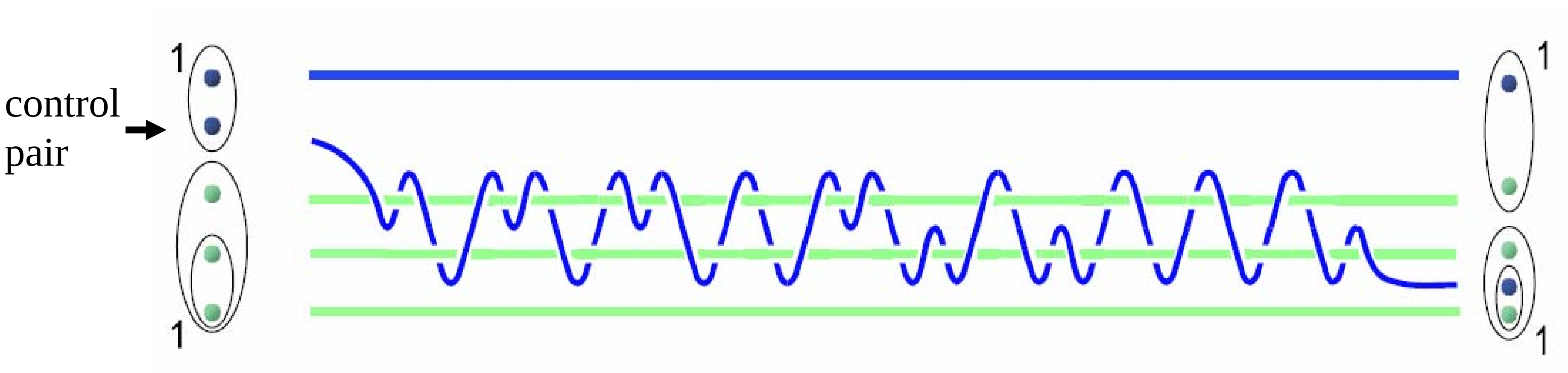


$$\sigma_2^{3}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{-4}\sigma_2^{-2}\sigma_1^{4}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{3} \approx \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

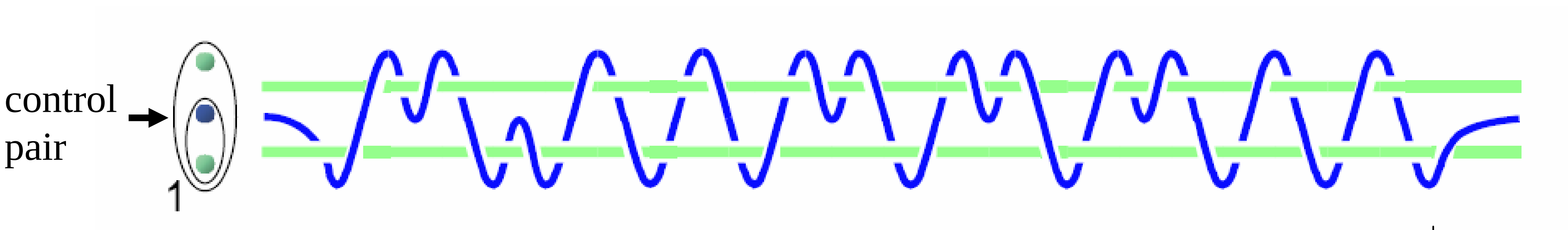
Step 1: Inject the control pair into the target qubit.

control pair



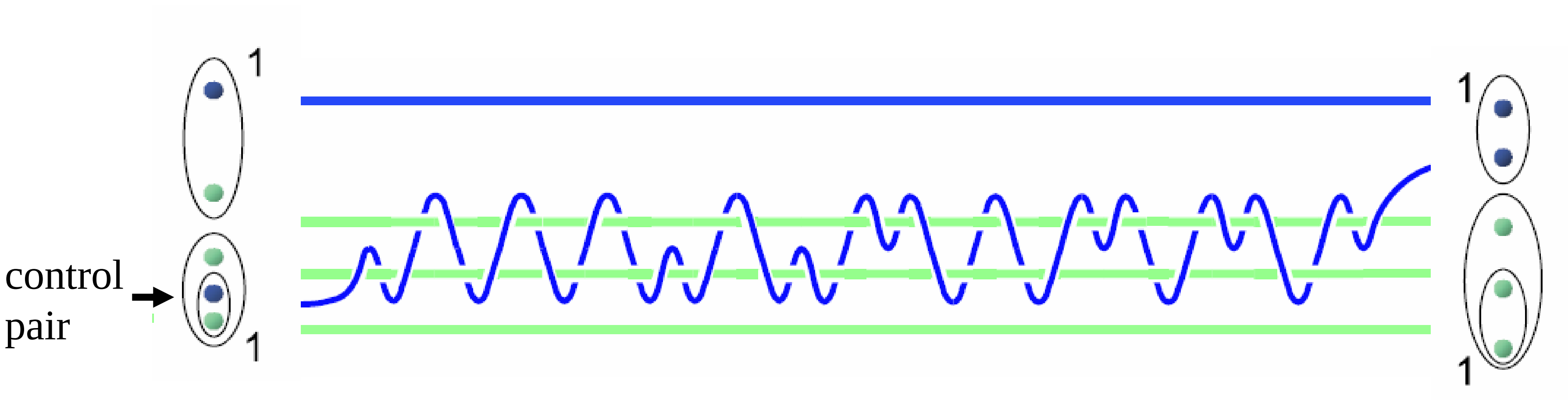


Step 2: Weave the control pair inside the injected target qubit.

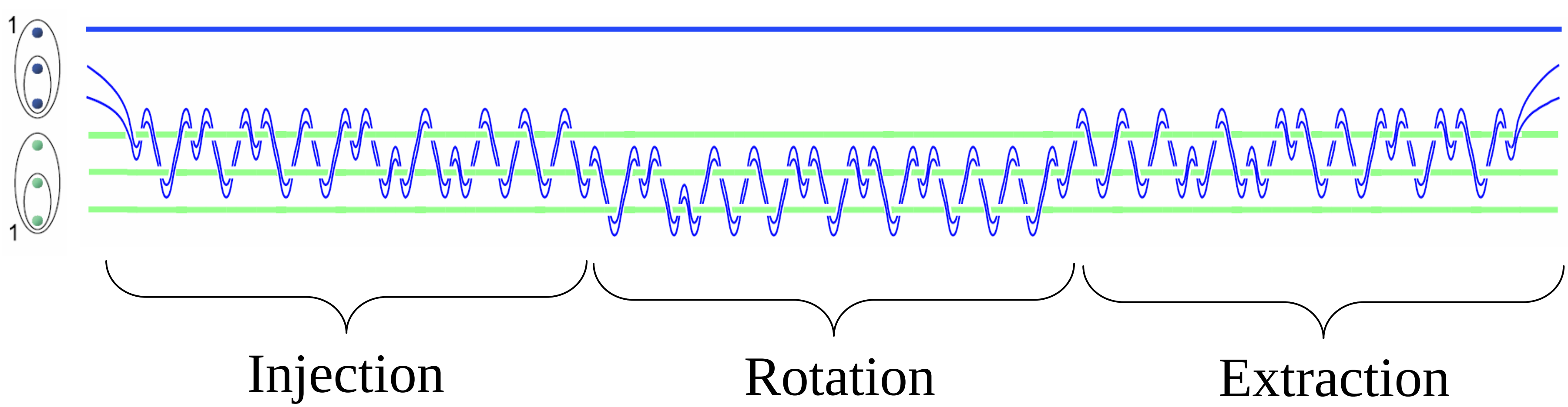


$$\sigma_1^{-2}\sigma_2^{-4}\sigma_1^{4}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{2}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{-2}\sigma_2^{4}\sigma_1^{2}\sigma_2^{-4}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{-2} \approx \left(\begin{array}{cc|c} 0 & i & 0 \\ i & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

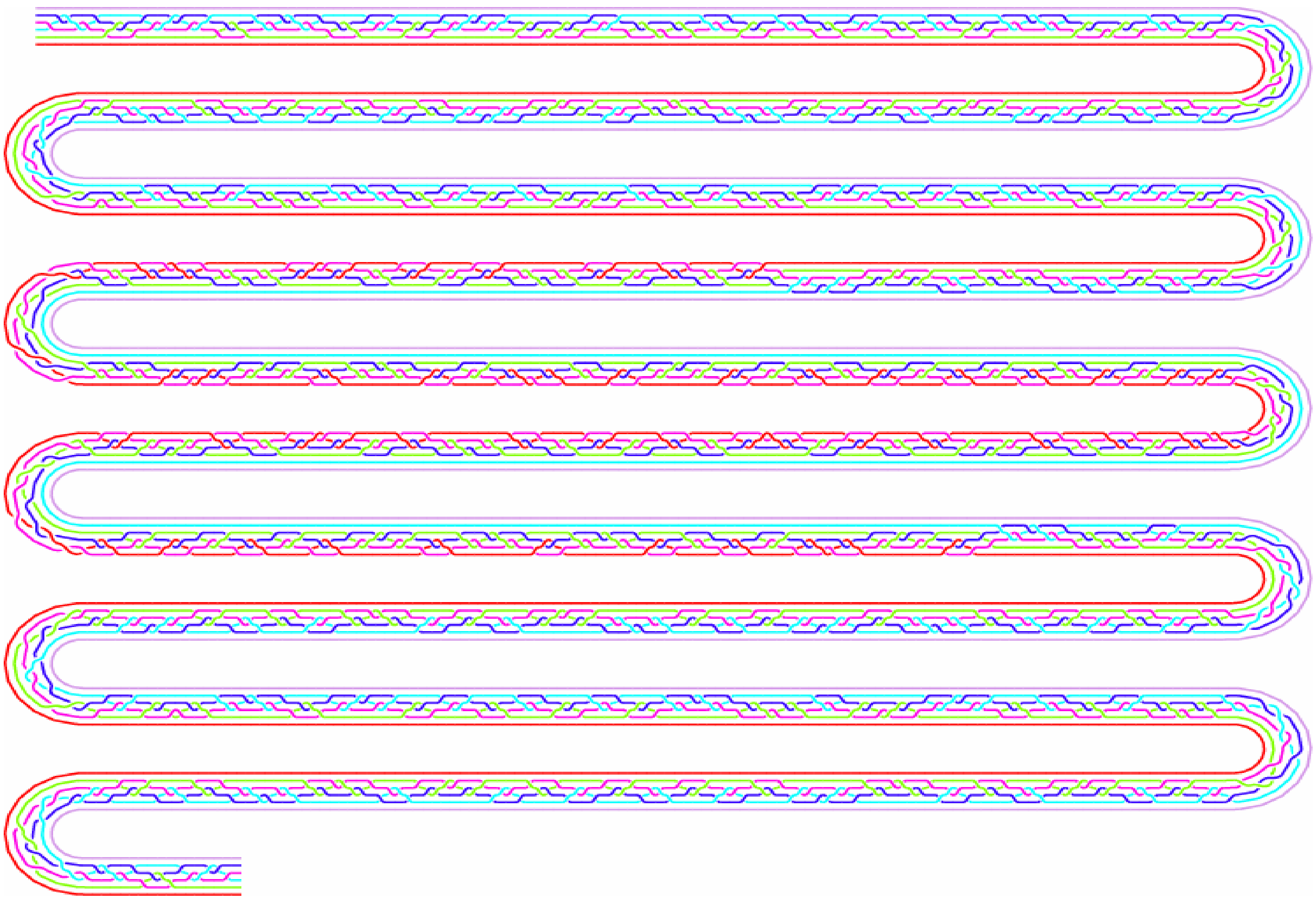
Step 3: Extract the control pair from the target using the inverse of the injection weave.



Putting it all together we have a CNOT gate:

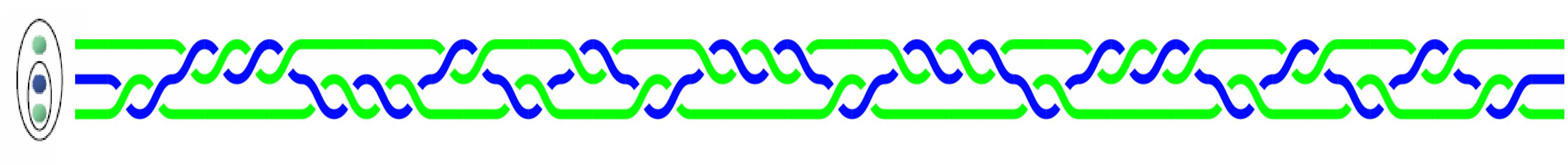


Solovay-Kitaev Improved CNOT

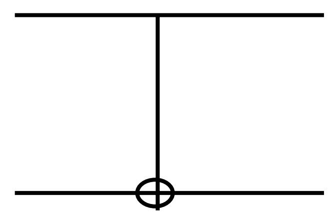


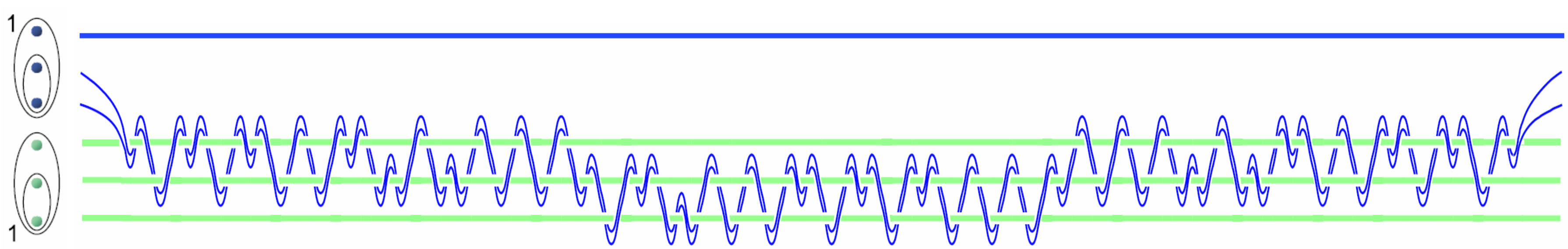
Universal Set of Fault Tolerant Gates

Single qubit rotations: $|\psi\rangle$ —$\boxed{U_{\vec{\phi}}}$— $U_{\vec{\phi}}\,|\psi\rangle$

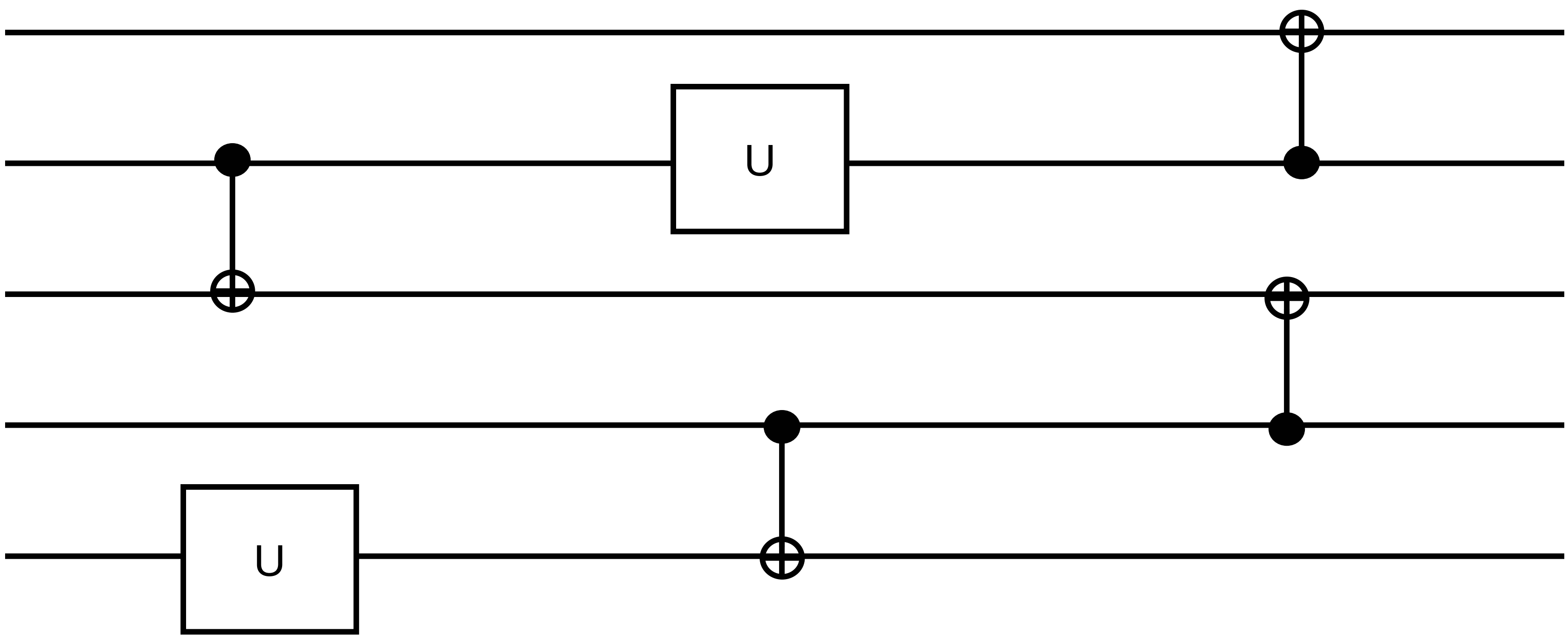


Controlled NOT:

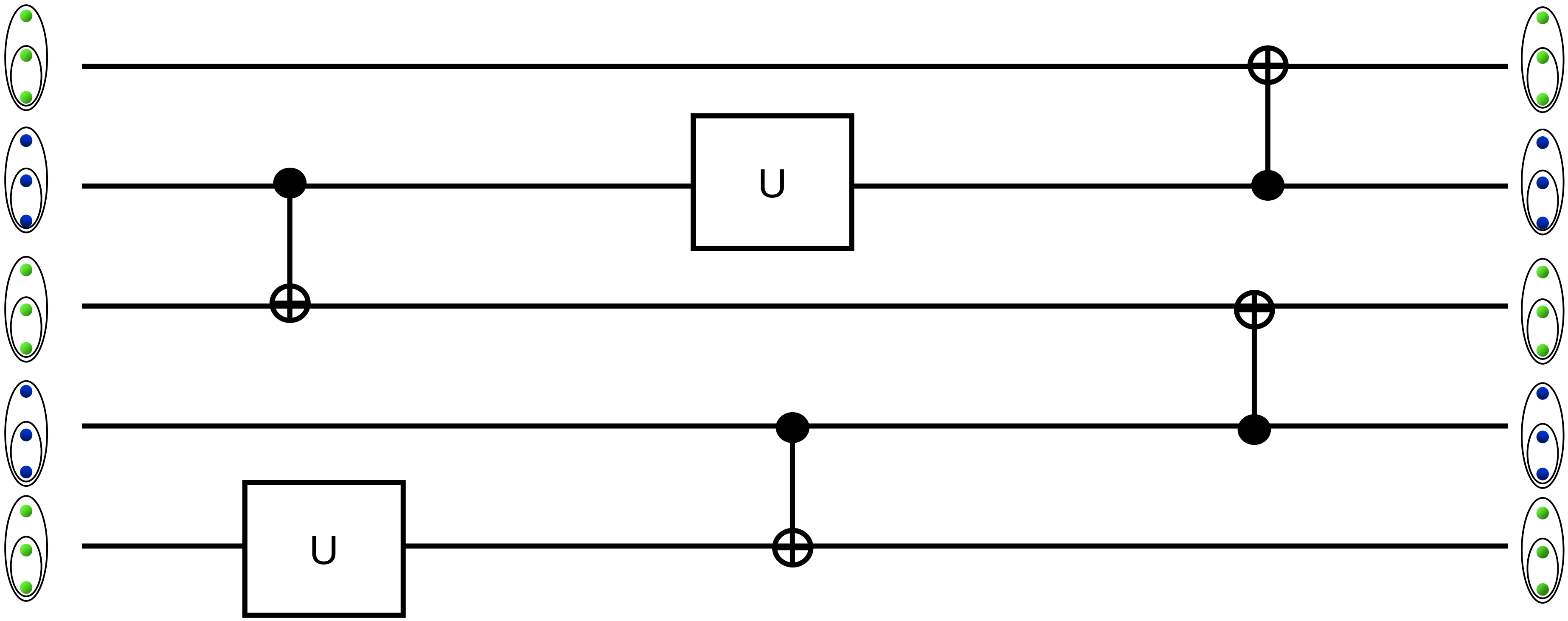




Quantum Circuit



Quantum Circuit



Braid