Topological order and quantum entanglement

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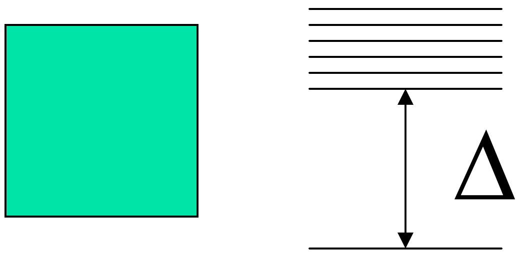
Topological phases

Topological phases

- Gapped



Topological phases

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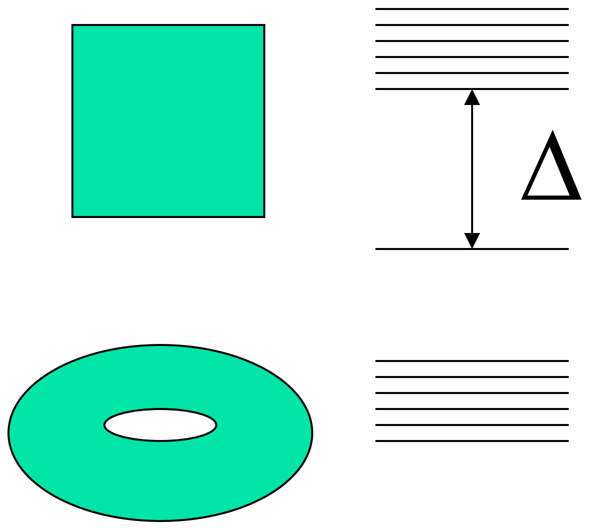
- Degenerate ground state on torus



Topological phases

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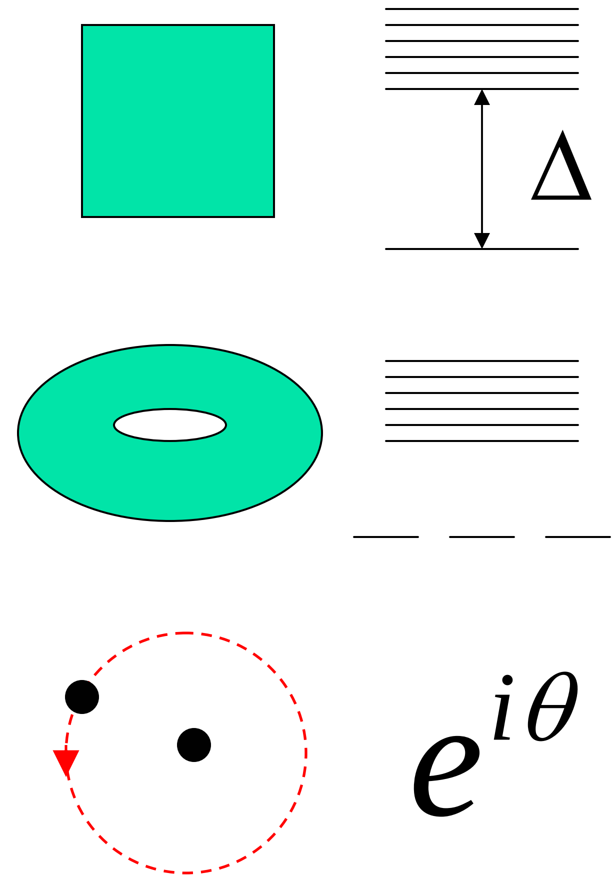
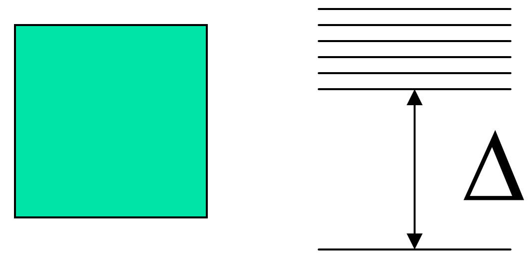
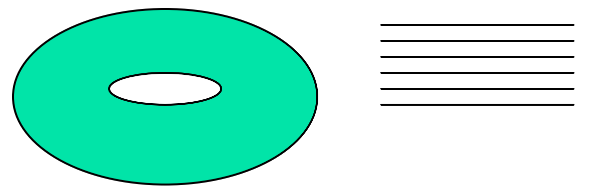
- Fractional statistics

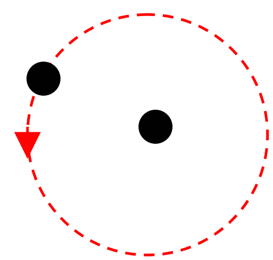


Topological phases

- Gapped
- Degenerate ground state on torus
- Fractional statistics
- “Topological order”







$$e^{i\theta}$$

Real life examples

- FQH liquids.

Real life examples

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- Hope: Frustrated magnets

  - Many theoretical models

  - A few candidate materials

    - Cs_2CuCl_4

    - κ-$(BEDT\text{-}TTF)_2Cu_2(CN)_3$

Theory of topological phases

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  - Low energy effective theory: TQFT

  - Mathematical framework: Tensor categories

  - Physical picture: String condensation, etc.

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  - Mathematical framework: Tensor categories

  - Physical picture: String condensation, etc.

- We're missing a lot

Physical characterization is incomplete

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- Symmetry breaking order can be detected in a wave function
- But we don’t know how to detect topological order in a wave function

Many wave functions

- Theoretical wave functions

  - Gutzwiller projected states: $\Psi_{spin} = P\ \Psi_{ferm}$

  - Quantum loop gases: $\Psi_d(X) = d^{N_{loops}(X)}$

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  \- J_1-J_3 spin-1/2 Heisenberg model:

$$H = J_1 \sum_{n.n} S_i\ \text{£} S_j + J_3 \sum_{3rd\ n.n} S_i\ \text{£} S_j$$

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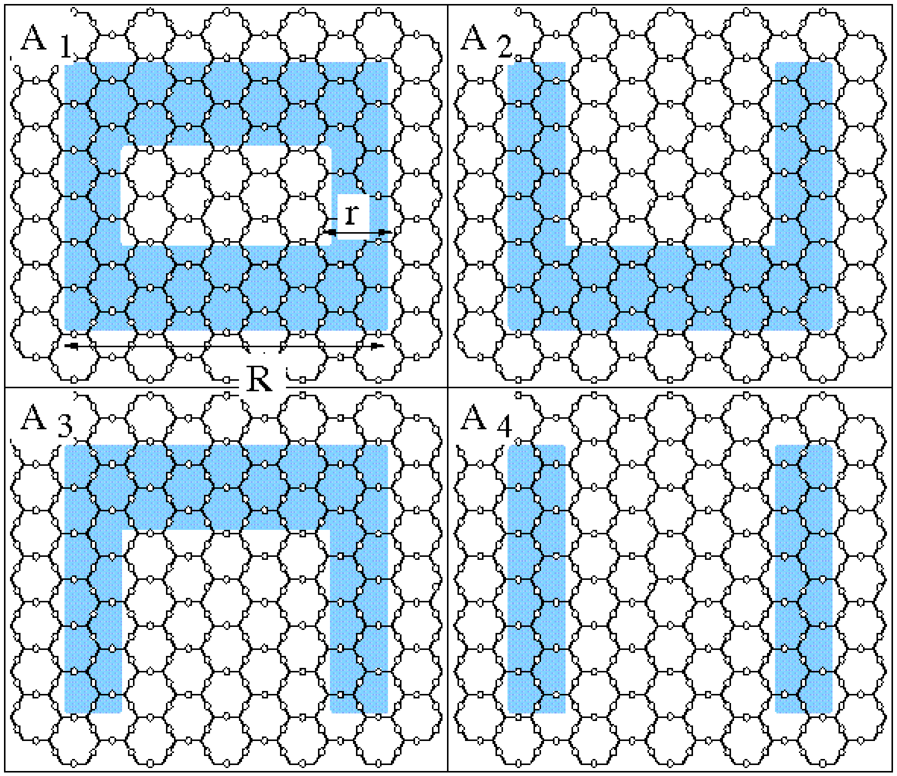
  - J_1-J_3 spin-1/2 Heisenberg model:

$$H = J_1 \sum_{n.n} S_i \pounds S_j + J_3 \sum_{3rd\ n.n} S_i \pounds S_j$$

- How do we know if they're topologically ordered?

Topological entropy

Define: $-S_{top} = (S_1 - S_2) - (S_3 - S_4)$



Main Result

- Then: $S_{top}=0$ for normal states, $S_{top} \neq 0$ for topologically ordered states

- S_{top} is universal for each topological phase:

$$S_{top} = \log(D^2)$$

where $D^2 = \sum_\alpha d_\alpha^2$

- cond-mat/0510613 (Kitaev/Preskill posted similar result on hep-th/0510092)

Physical picture

- Why does $S_{top} = 0$ for normal phases, $S_{top} \neq 0$ for topological phases?

Physical picture

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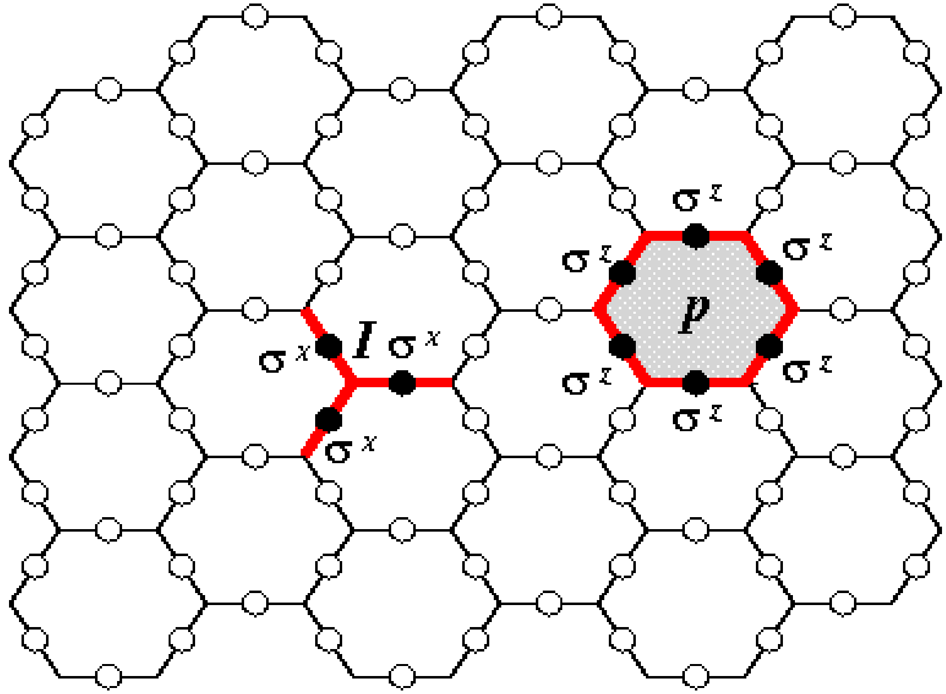
Nonlocal entanglement!

- Topologically ordered states have nonlocal entanglement
- S_{top} measures nonlocal entanglement

Topologically ordered states have nonlocal entanglement

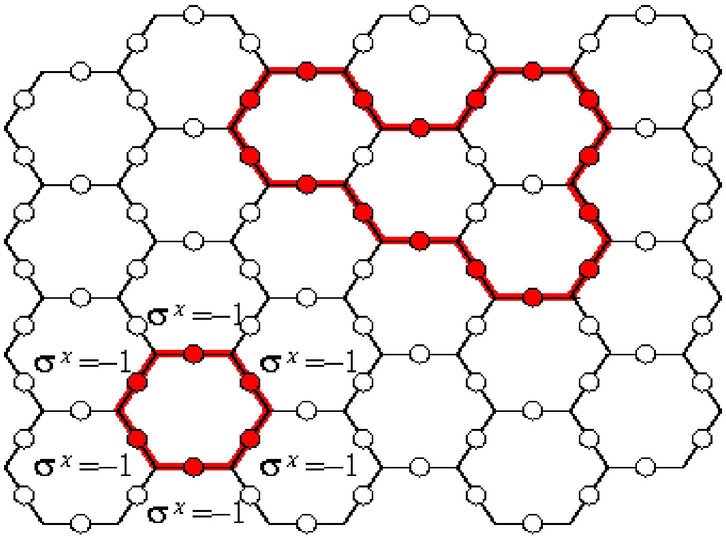
$$H = -V\sum_{I}\sigma^x_{I1}\sigma^x_{I2}\sigma^x_{I3} - t\sum_{p}\sigma^z_{p1}\sigma^z_{p2}\sigma^z_{p3}\sigma^z_{p4}\sigma^z_{p5}\sigma^z_{p6}$$



Z_2 topological order

Ground state wave function

- Use string picture:
  σ^x_i =-1 – string on link
  σ^x_i=+1 – no string on link
- Ψ is uniform superposition of closed string configurations



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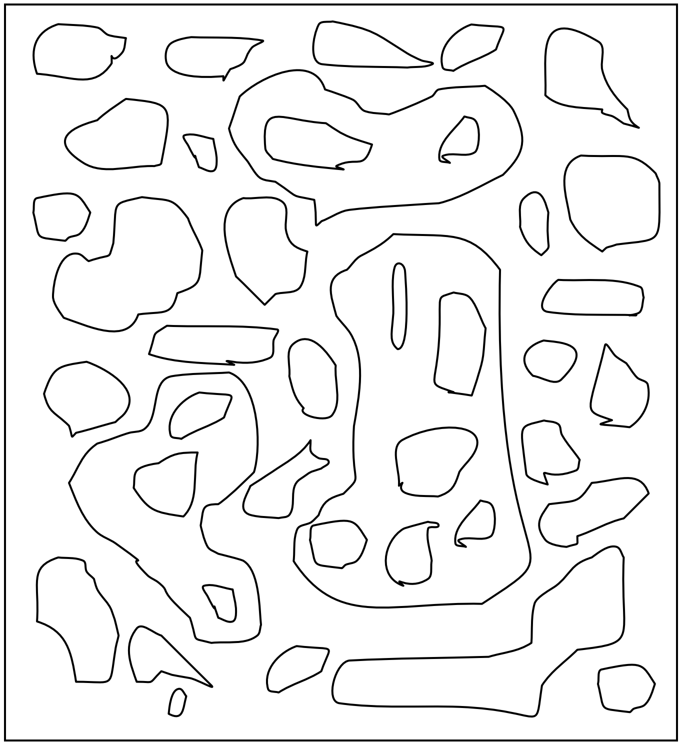

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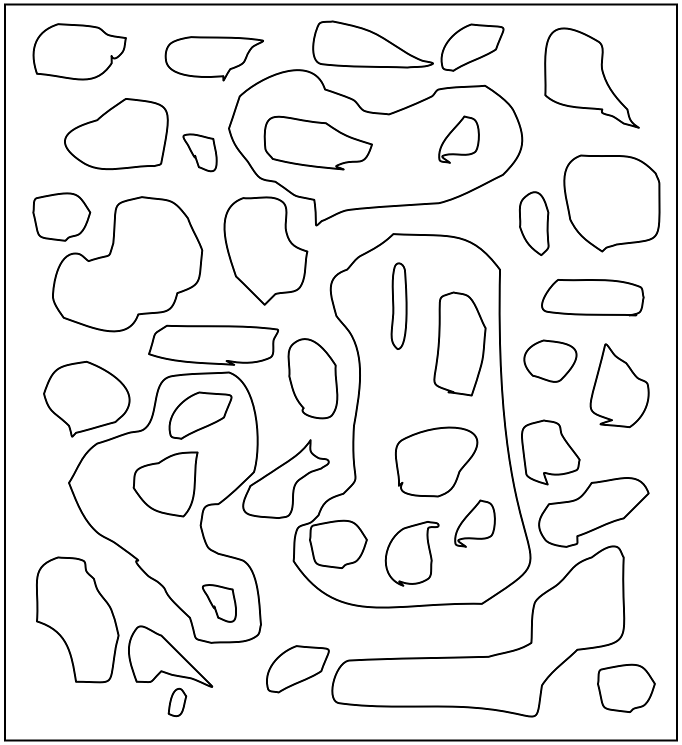

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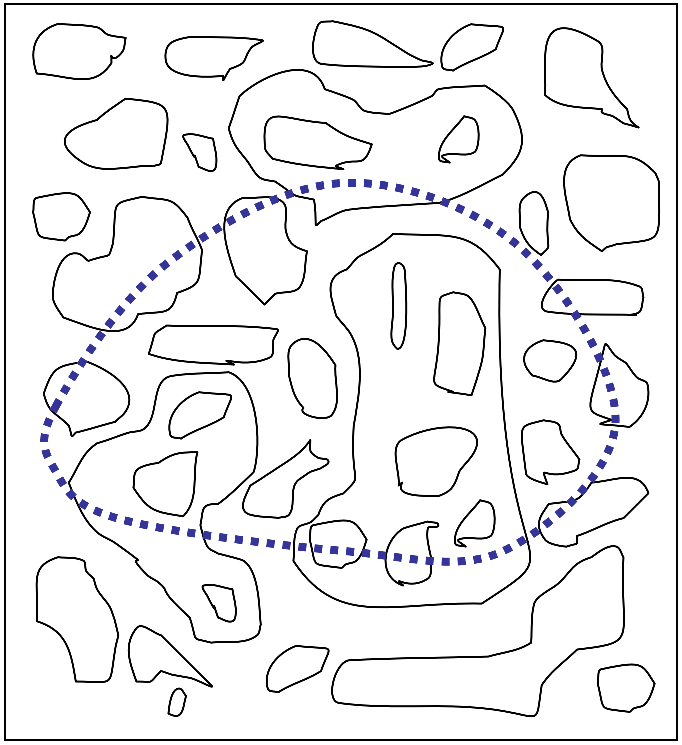
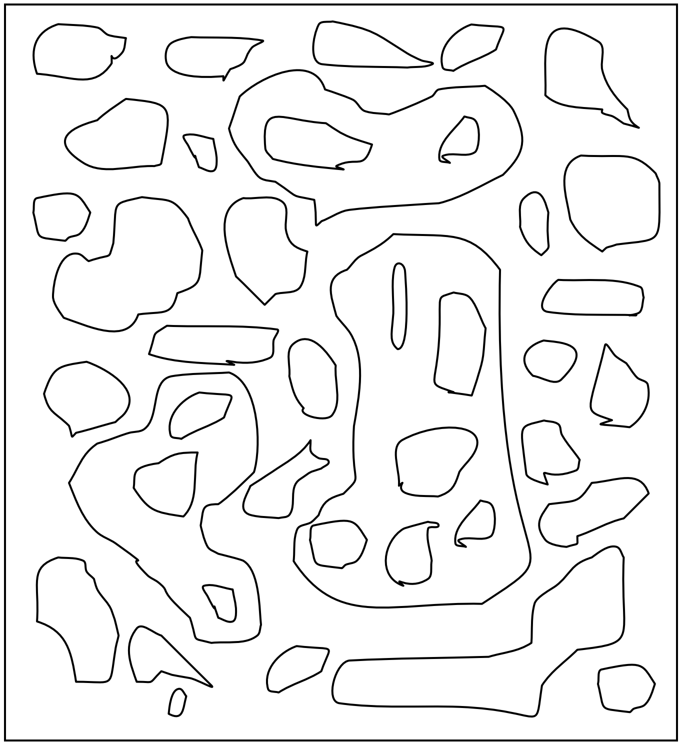
- There is a *nonlocal* correlation: $\langle \prod_{i \in C} \sigma^x_i \rangle = 1$



String correlations are very general

- Perturb Hamiltonian:

  $H \rightarrow H + \varepsilon \sum_i \sigma_i^z$



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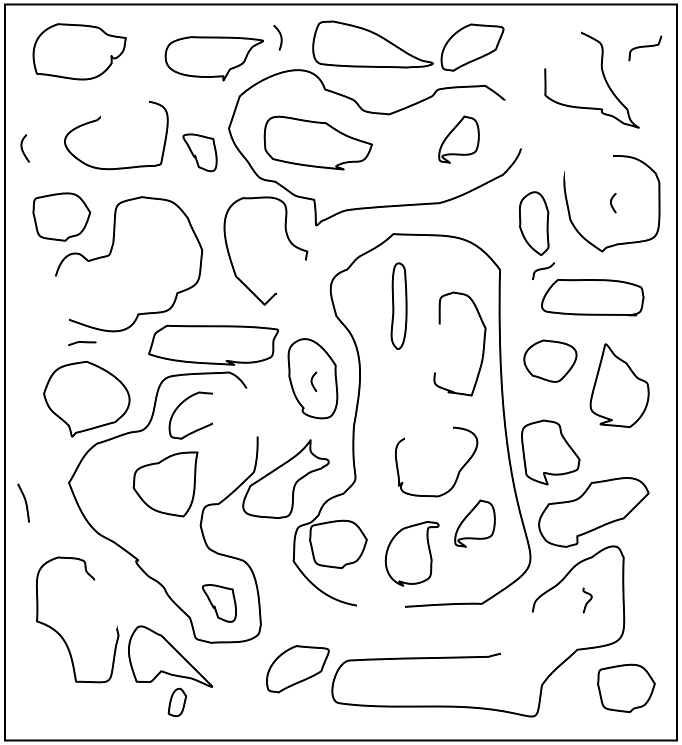

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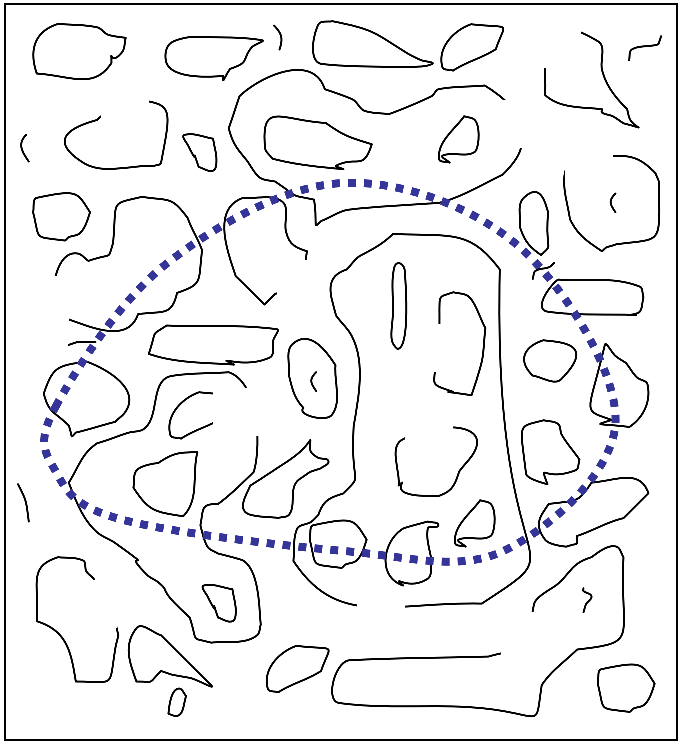
  $H \to H + \varepsilon \sum_i \sigma_i^z$

- Thin string operator fails:

  $\langle \prod_{i \in C} \sigma_i^x \rangle = 0$



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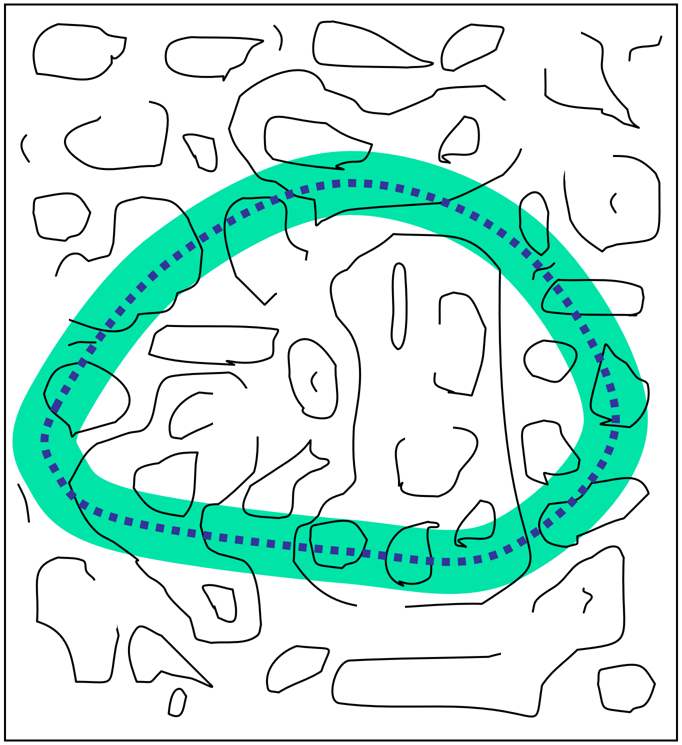
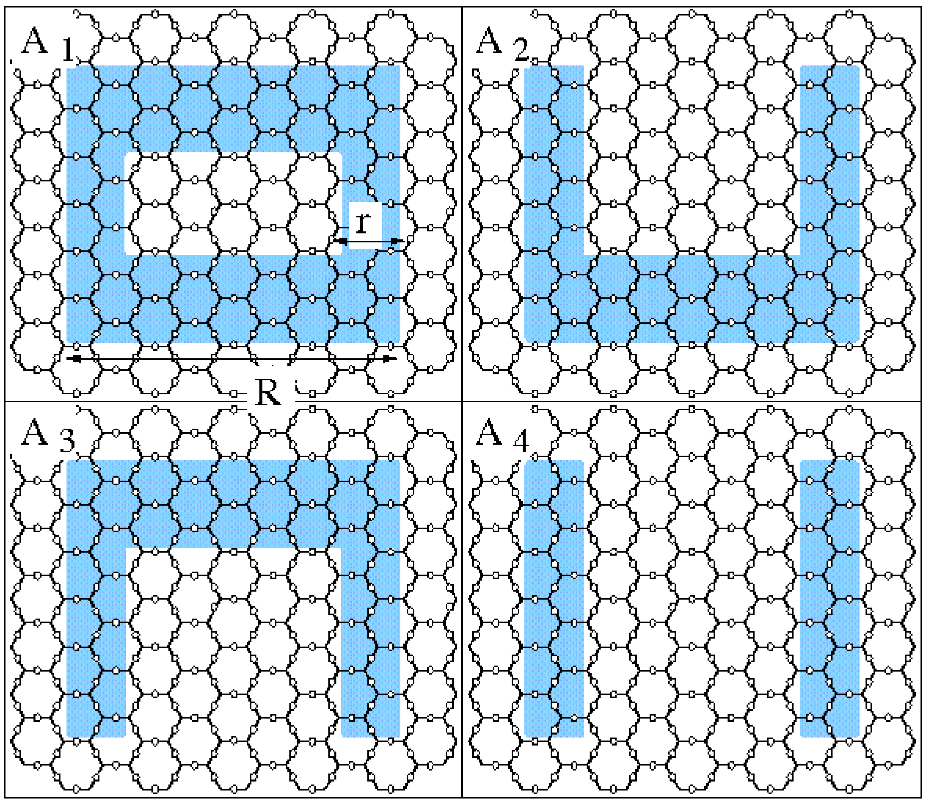
- But "fattened" string operator works



S_{top} measures string correlations

$$-S_{top} = (S_1 - S_2) - (S_3 - S_4)$$

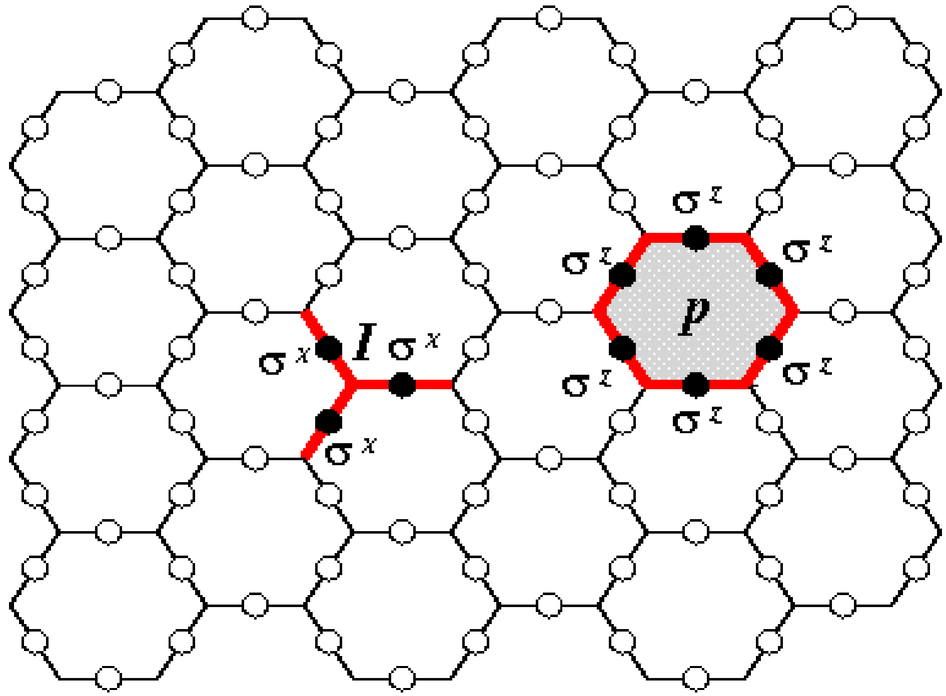


We have argued:

- $S_{top} = 0$ for normal phases
- $S_{top} \neq 0$ for topological phases
- S_{top} is universal
- But why does $S_{top} = \log(D^2)$?

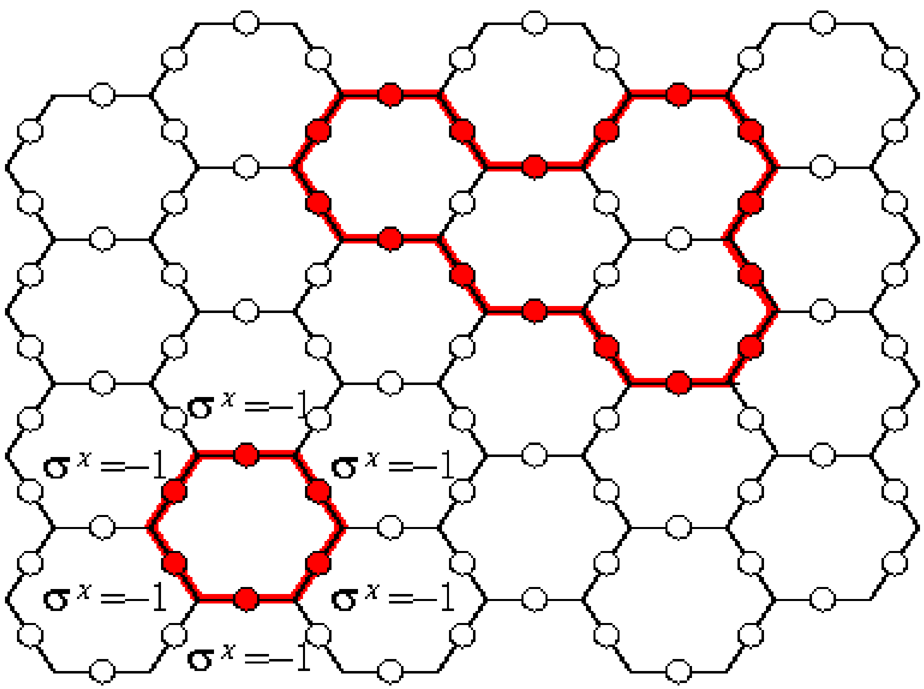
Exactly soluble example

$$H = -V\sum_{I}\sigma^x_{I1}\sigma^x_{I2}\sigma^x_{I3} - t\sum_{p}\sigma^z_{p1}\sigma^z_{p2}\sigma^z_{p3}\sigma^z_{p4}\sigma^z_{p5}\sigma^z_{p6}$$

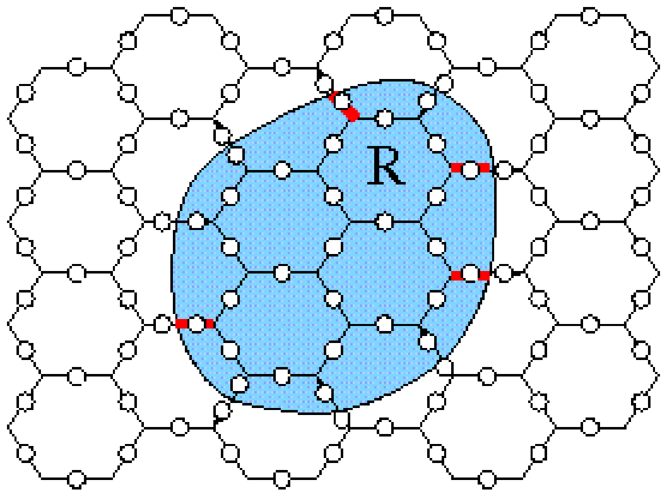


Ground state wave function

Ψ is uniform superposition of closed string configurations



Entanglement entropy



- For any $q_1,\ldots,q_n = 0,1$, $\sum_m q_m$ even, define $\Psi^{in}{}_{q_1,\ldots q_n}$
- Similarly define $\Psi^{out}{}_{q_1,\ldots q_n}$

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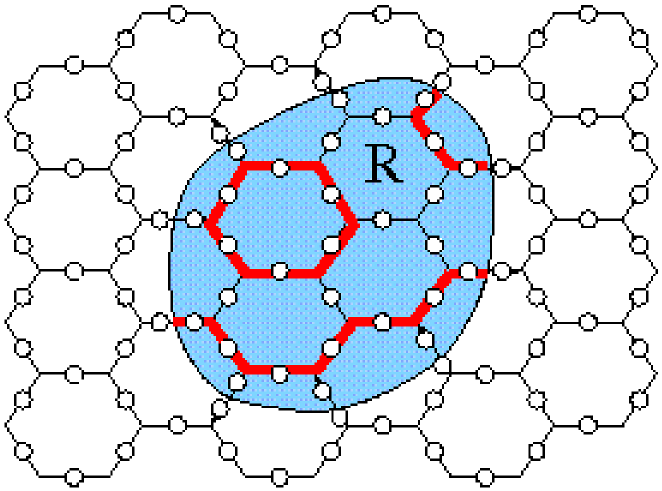

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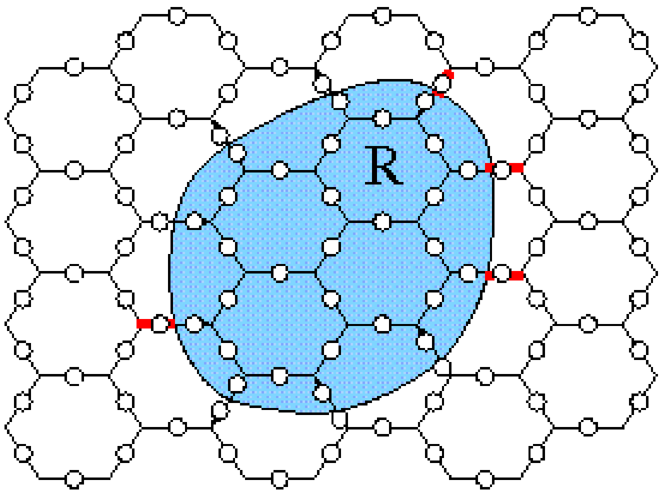

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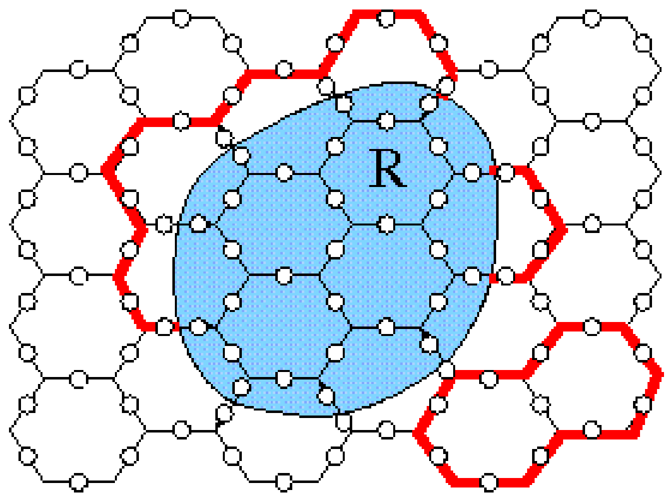
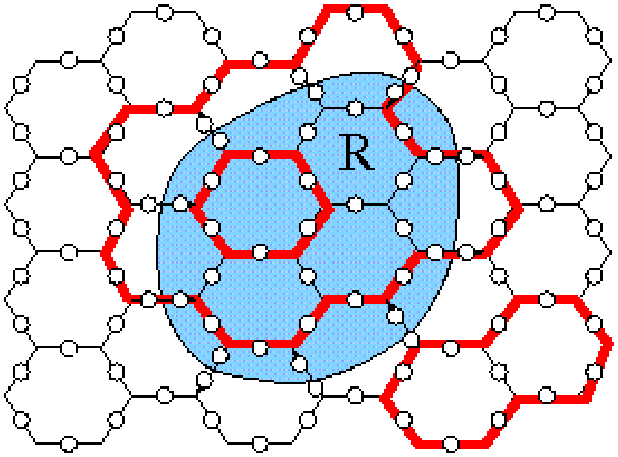


- For any $q_1,\ldots,q_n = 0,1$, $\sum_m q_m$ even, define $\Psi^{in}_{q_1,\ldots,q_n}$
- Similarly define $\Psi^{out}_{q_1,\ldots,q_n}$

Entanglement entropy

Then: $\Psi = \sum_q \Psi^{in}_q \Psi^{out}_q$



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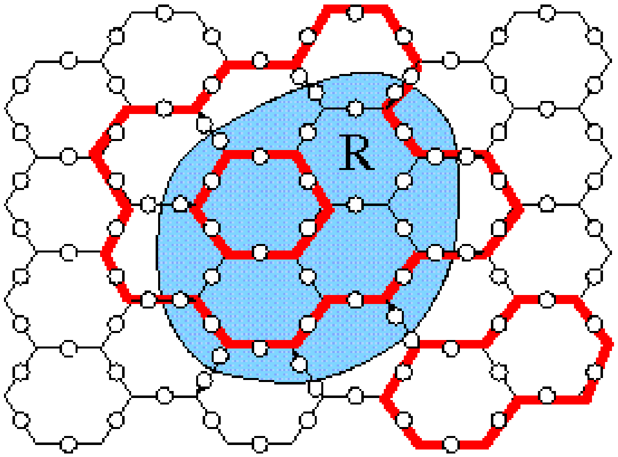

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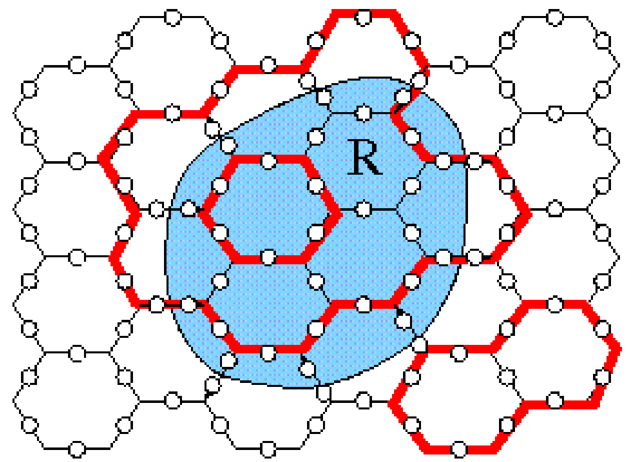
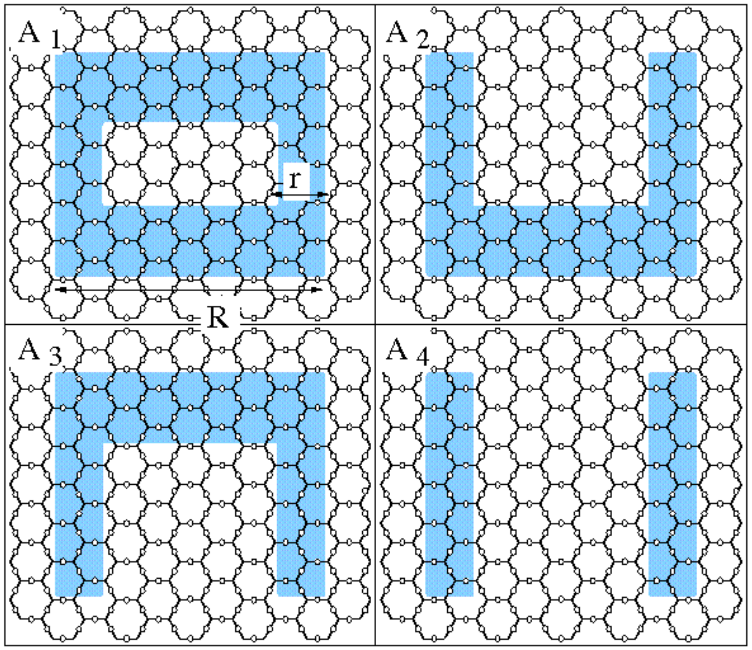


Therefore ρ_R is an equal mixture of all Ψ^{in}_q

There are 2^{n-1} different Ψ^{in}_q * $S_R = (n-1) \log 2$

Topological entropy

- S_R = (n-k) log 2 where
  k = # boundary curves



Topological entropy

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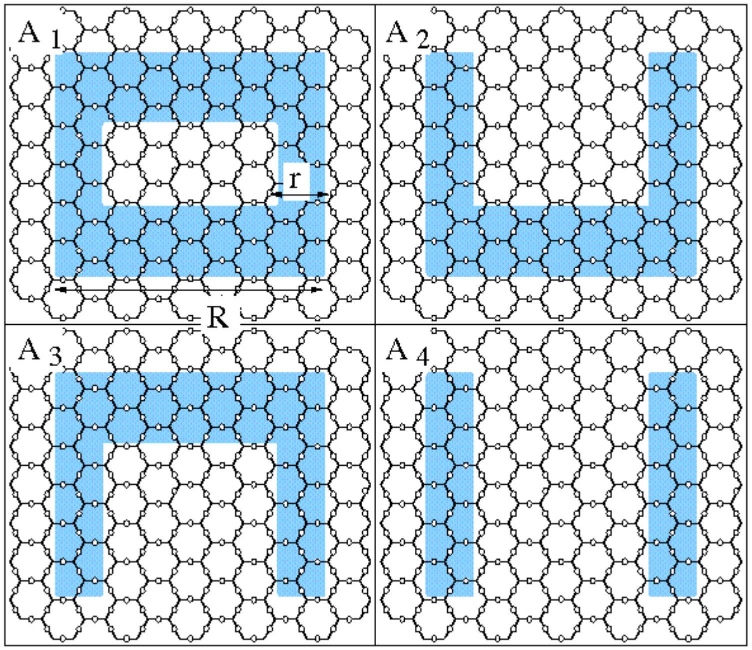

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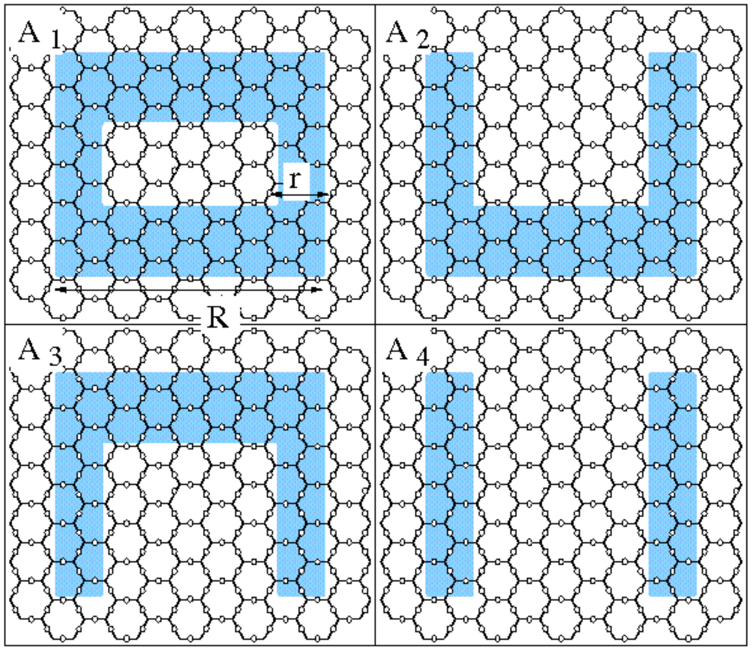

$$-S_{top} = (n_1-n_2-n_3+n_4-2) \log 2 = -2 \log 2 = -\log(2^2)$$

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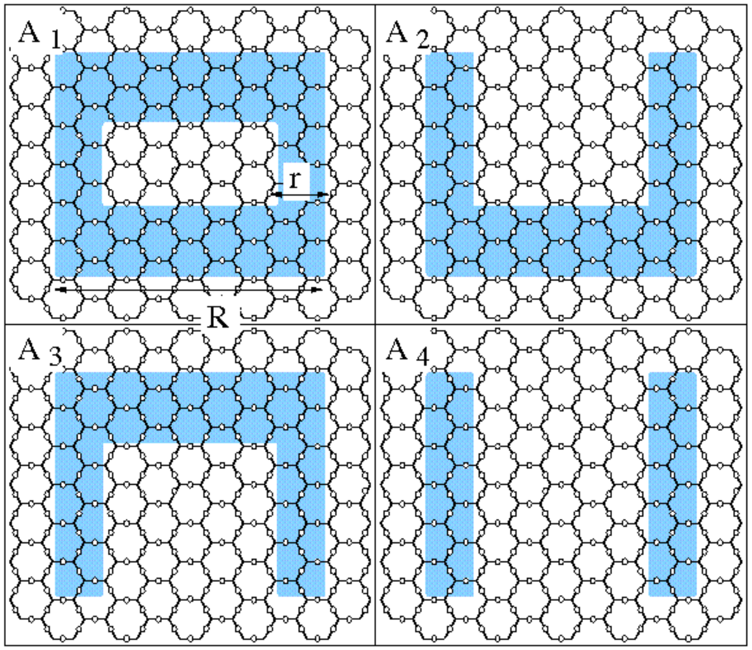

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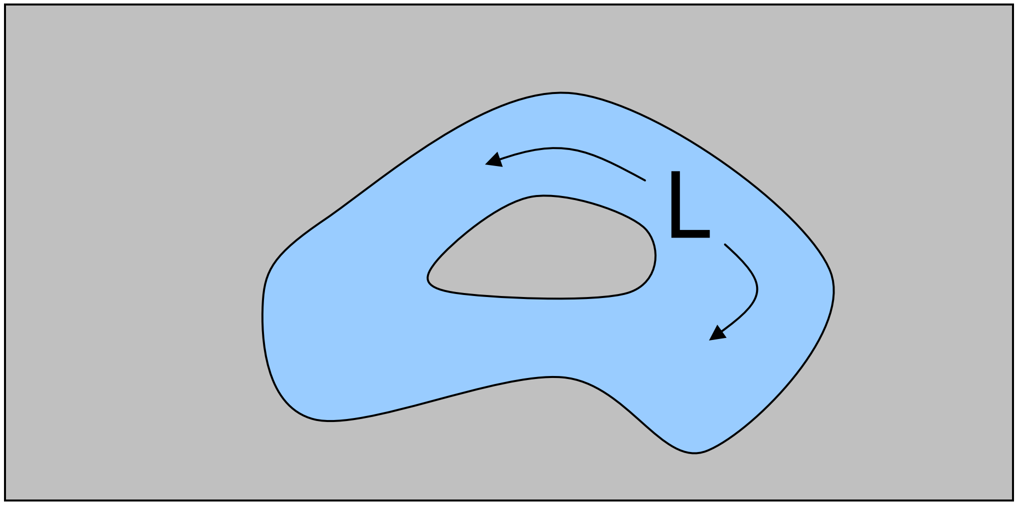
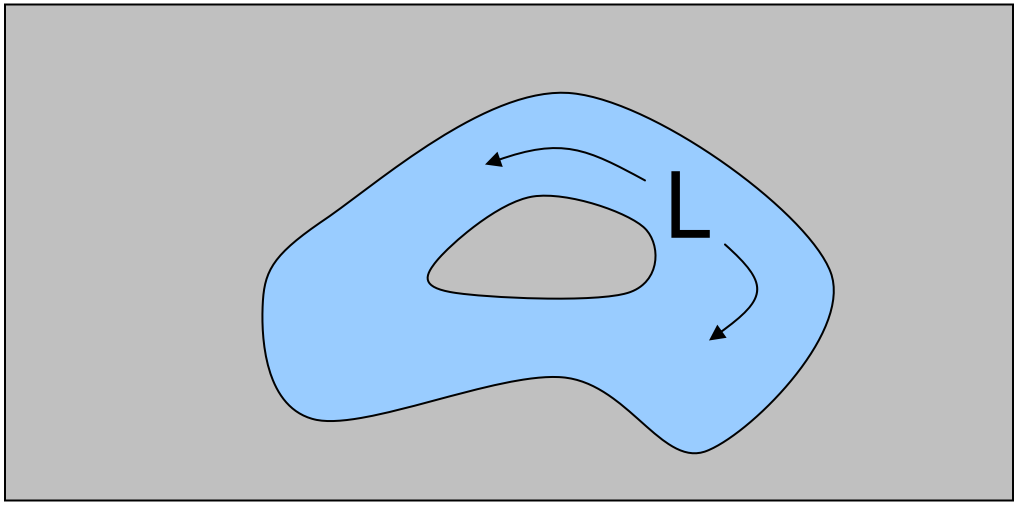
Right result: D=2 for Z_2 topological order!

Topological entropy in the continuum



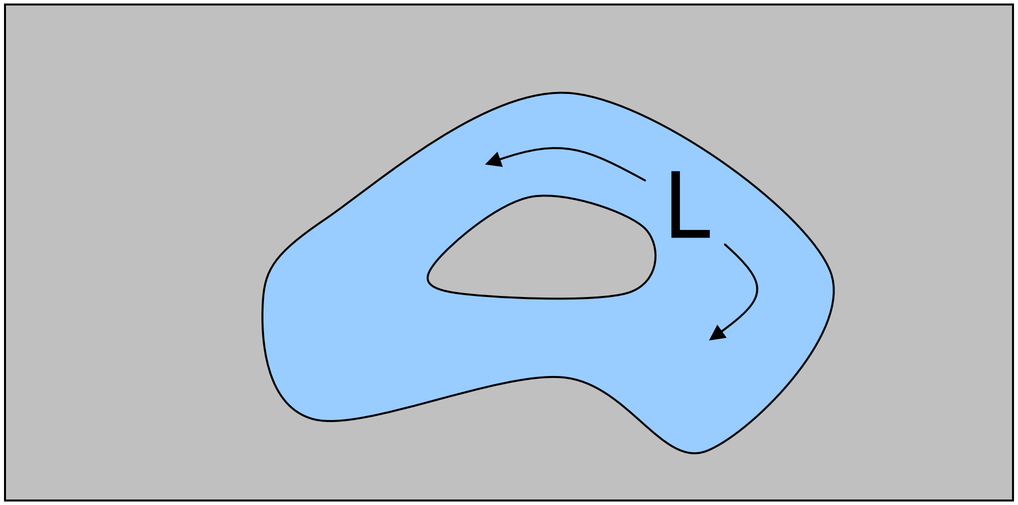
S = c L + ...

Topological entropy in the continuum



$$S = c\,L - S_{top}$$

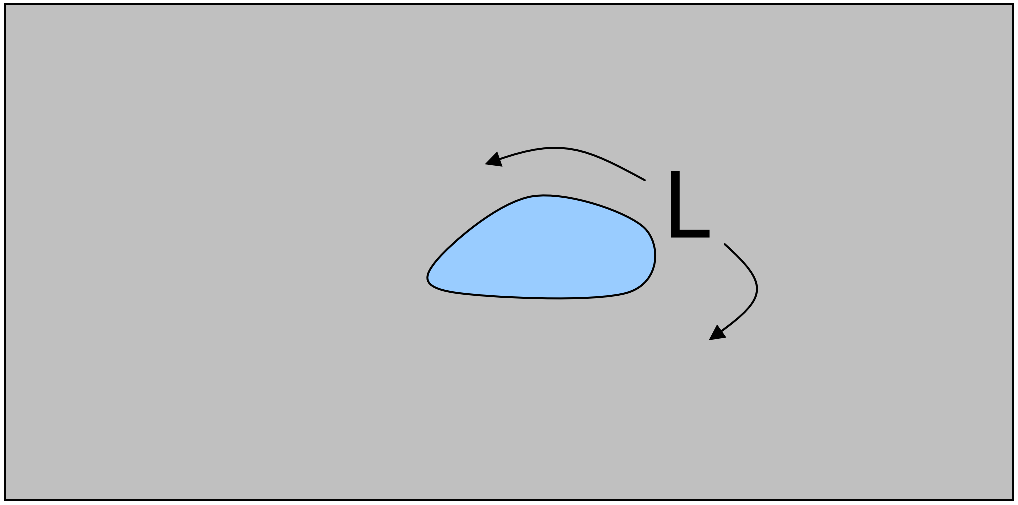
Topological entropy in the continuum



$$S = c\,L - S_{top}$$

Universal finite size correction!

Topological entropy for disk



$$S = c\,L - S_{top}/2$$

Universal finite size correction!

Conclusions/New directions

- Compute S_{top} for J_1-J_3 model, quantum loop gas, etc.
- S_{top} and critical theories
- Can we get more information from Ψ – e.g. statistics of quasiparticles?